Journal of Nonlinear Mathematical Physics Volume 9, Number 4 (2002), 517–529 ARTICLE

Tau Functions Associated to Pseudodifferential Operators of Several Variables

Min Ho LEE

Department of Mathematics, University of Northern Iowa, Cedar Falls, IA 50614, U.S.A.
E-mail: lee@math.uni.edu

Received April 26, 2002; Accepted June 3, 2002

Abstract

Pseudodifferential operators of several variables are formal Laurent series in the formal inverses of $\partial_1, \ldots, \partial_n$ with $\partial_i = d/dx_i$ for $1 \le i \le n$. As in the single variable case, Lax equations can be constructed using such pseudodifferential operators, whose solutions can be provided by Baker functions. We extend the usual notion of tau functions to the case of pseudodifferential operators of several variables so that each Baker function can be expressed in terms of the corresponding tau function.

1 Introduction

One of the most actively studied areas in mathematics for the past few decades is the theory of integrable nonlinear partial differential equations (see e.g. [3, 4, 6, 8]). Such equations are also known as soliton equations because they possess localized nonlinear waves called solitons as solutions. Examples of soliton equations include many well-known equations in mathematical physics such as the nonlinear Schrödinger equation, the Sine-Gordon equation, the Korteweg-de Vries (KdV) equation, and the Katomtsev–Petviashvili (KP) equation.

The main tool used in a systematic study of soliton equations is the notion of Lax equations, which describe certain compatibility conditions for pairs of differential operators. A system of soliton equations called a KP hierarchy is produced by a set of Lax equations, and as a result, solutions of Lax equations can be used to construct solutions of the associated soliton equations. The interpretation of soliton equations in terms of Lax equations leads to the derivation of the integrability as well as other interesting properties of soliton equations.

A few decades ago, Krichever (see e.g. [7]) introduced the method of constructing an infinite dimensional subspace of $\mathbb{C}((z))$ associated to some algebro-geometric data, where $\mathbb{C}((z))$ is the space of Laurent series. This construction is nowadays called the Krichever map, and it has been used successfully in the soliton theory and is closely linked to the theory of moduli of algebraic curves (cf. [1, 7, 13]). Thus the Krichever map provides a connection of soliton theory with algebraic geometry, which is one of the most intriguing features of the theory of soliton equations. More specifically, to each subspace of $\mathbb{C}((z))$

produced by the Krichever map there corresponds a so-called Baker–Akhiezer function, which determines an algebro-geometric solution of a soliton equation (see [2, 4, 7, 13]). Baker functions are a generalized version of Baker–Akhiezer functions, and they supply formal solutions of Lax equations. Tau functions also play an important role in algebro-geometric theory of solitons, and in particular, each Baker function can be expressed in terms of the associated tau function. Such an expression of a Baker function in terms of a tau function is an important contribution of the Japanese school (see e.g. [5]). Tau functions can be used to construct soliton solutions of soliton equations, and they are essential in linking soliton theory to quantum field theory as well as to the theory of Virasoro algebras or vertex operators.

Pseudodifferential operators are formal Laurent series in the formal inverse ∂^{-1} of the differentiation operator $\partial = d/dx$ with respect to the single variable x, and they are essential ingredients in the construction of Lax equations. For this reason pseudodifferential operators have played a major role in the theory of soliton equations. In a recent paper, Parshin [12] studied pseudodifferential operators of several variables by considering formal Laurent series in the formal inverses of $\partial_1, \dots, \partial_n$ with $\partial_i = d/dx_i$ for $1 \le i \le n$. Among other things, he constructed Lax equations associated to such pseudodifferential operators and studied some of their properties. Since then, algebro-geometric connections of those pseudodifferential operators have been studied in [11] and [10], where the possibility of extending the Krichever map to the case of higher dimensional varieties was discussed. Baker functions which provide solutions to Lax equations of Parshin type have also been investigated in [9], where some of the properties of the usual Baker functions were extended to the case of pseudodifferential operators of several variables. The goal of this paper is to prove the existence of tau functions associated to Baker functions constructed in [9].

2 Pseudodifferential operators

In this section we review pseudodifferential operators of several variables studied by Parshin [12] as well as the associated Lax equations. We also describe an example of a system of partial differential equations determined by such a Lax equation.

We fix a positive integer n and consider the variables $x_1, \dots, x_n$. We denote by

$$\mathbb{C}((x_1)) \cdots ((x_n))$$

the associated field of iterated Laurent series over $\mathbb{C}$, and let P be the space of iterated formal Laurent series of the form

$$P = \mathbb{C}((x_1)) \cdots ((x_n)) \left((\partial_1^{-1}) \right) \cdots \left((\partial_n^{-1}) \right)$$

in the formal inverses of the differential operators

$$\partial_1 = \frac{\partial}{\partial x_1}, \quad \dots, \quad \partial_n = \frac{\partial}{\partial x_n}.$$

Throughout this paper we shall often use the usual multi-index notation. Thus, given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we may write

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

with $\partial = (\partial_1, \dots, \partial_n)$. We also write $\alpha \geq \beta$ for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ if $\alpha_i \geq \beta_i$ for each i, and use $\mathbf{0}$ and $\mathbf{1}$ to denote the elements $(0, \dots, 0)$ and $(1, \dots, 1)$ in $\mathbb{Z}^n$, respectively. Thus, for example, an element $\psi \in P$ can be written in the form

$$\psi = \sum_{\alpha \leq \nu} f_\alpha(x) \partial^\alpha \tag{2.1}$$

for some $\nu \in \mathbb{Z}^n$. We introduce a multiplication operation on P defined by the Leibniz rule, which means that

$$\left(\sum_\alpha f_\alpha(x) \partial^\alpha \right) \left(\sum_\beta h_\beta(x) \partial^\beta \right) = \sum_{\alpha, \beta} \sum_{\gamma \geq \mathbf{0}} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} f_\alpha(x) (\partial^\gamma h_\beta(x)) \partial^{\alpha + \beta - \gamma},$$

where $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \gamma_1 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ \gamma_n \end{pmatrix}$ for elements $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$ of $\mathbb{Z}^n$ with $\gamma \geq \mathbf{0}$. We now set

$$\mathbb{Z}^n_+ = \{ \alpha \in \mathbb{Z}^n \mid \alpha \geq \mathbf{0},\ |\alpha| \geq 1 \},$$

and assume that each coefficient $f_\alpha(x)$ in (2.1) is a function of the infinitely many variables $\{ t_\alpha \mid \alpha \in \mathbb{Z}^n_+ \}$. Let $\mathbf{e}_1 = (1, 0, \dots, 0)$, $\dots$, $\mathbf{e}_n = (0, \dots, 0, 1)$ be the standard basis for the $\mathbb{Z}$-module $\mathbb{Z}^n$, and assume that

$$t_{\mathbf{e}_1} = x_1, \quad \dots, \quad t_{\mathbf{e}_n} = x_n. \tag{2.2}$$

Thus we may write $\psi \in P$ in (2.1) in the form

$$\psi = \sum_{\alpha \leq \nu} f_\alpha(t) \partial^\alpha$$

with $t = (t_\alpha)_{\alpha \in \mathbb{Z}^n_+}$.

If ψ is an element of P which can be written in the form

$$\psi = \sum_{i=-\infty}^{\nu_n} a_i \partial_n^i = \sum_{i=-\infty}^{\nu_n} a_i(t; \partial_1, \dots, \partial_{n-1}) \partial_n^i$$

with $\nu_n \geq 0$, we set

$$\psi_+ = \sum_{i=0}^{\nu_n} a_i \partial_n^i, \qquad \psi_- = \psi - \psi_+ = \sum_{i=-\infty}^{-1} a_i \partial_n^i;$$

if $\nu_n < 0$, we set $\psi_+ = 0$ and $\psi_- = \psi$. Thus we have $\psi = \psi_+ + \psi_-$ for all $\psi \in P$, and therefore P can be decomposed as

$$P = P_+ + P_-,$$

where P_+ is the set of elements of P of the form $\sum\limits_{i=0}^{m} a_i \partial_n^i$ for some nonnegative integer m, and P_- is the set of elements of the form $\sum\limits_{j=0}^{k} b_j \partial_n^j$ with $k < 0$. Let P^n be the Cartesian product of n copies of P, and consider an element $L = (L_1, \ldots, L_n) \in P^n$ which satisfies the generalized Lax equation

$$\partial_{t_\alpha} L = [L_+^\alpha, L] = L_+^\alpha L - L L_+^\alpha \tag{2.3}$$

for all $\alpha \in \mathbb{Z}_+^n$, where $L_+^\alpha = (L^\alpha)_+ = (L_1^{\alpha_1} \cdots L_n^{\alpha_n})_+ \in P_+$ and

$$\partial_{t_\alpha} L = \frac{\partial L}{\partial t_\alpha} = \left(\frac{\partial L_1}{\partial t_\alpha}, \ldots, \frac{\partial L_n}{\partial t_\alpha} \right).$$

Thus (2.3) is equivalent to the system of equations

$$\frac{\partial L_i}{\partial t_\alpha} = [L_+^\alpha, L_i]$$

for $1 \leq i \leq n$.

We now consider an element $\phi \in 1 + P_-$ satisfying the relation

$$\partial_{t_\alpha} \phi = -\left(\phi \partial^\alpha \phi^{-1} \right)_- \phi \tag{2.4}$$

for each $\alpha \in \mathbb{Z}_+^n$, and set

$$L = \phi \partial \phi^{-1} = \left(\phi \partial_1 \phi^{-1}, \ldots, \phi \partial_n \phi^{-1} \right) \in P^n. \tag{2.5}$$

Thus, if $L = (L_1, \ldots, L_n)$, then each L_i is of the form

$$L_i = \phi \partial_i \phi^{-1} = \partial_i + u_i$$

for some $u_i \in P_-$. Then it can be shown that the pseudodifferential operator L given by (2.5) satisfies the Lax equation (2.3) for each $\alpha \in \mathbb{Z}_+^n$. The Lax equation (2.3) also implies the relation

$$\frac{\partial L_+^\beta}{\partial t_\alpha} - \frac{\partial L_+^\alpha}{\partial t_\beta} = [L_+^\alpha, L_+^\beta] \tag{2.6}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$ (see [12, Proposition 4]). For each pair (α, β) of elements of $\mathbb{Z}_+^n$ the relation (2.6) determines a system of partial differential equations as can be seen in the following example.

Example. We shall derive partial differential equations which are determined by the Lax equation for $n = 2$ associated to the pseudodifferential operators $L_1, L_2 \in P$ given by

$$L_1 = \partial_2 + a \partial_1 \partial_2^{-1} + b \partial_2^{-2} + O\left(\partial_2^{-3}\right), \tag{2.7}$$

$$L_2 = \partial_2 + c \partial_2^{-1} + d \partial_1 \partial_2^{-2} + O\left(\partial_2^{-3}\right) \tag{2.8}$$

for some functions $a = a(t)$, $b = b(t)$, $c = b(t)$ and $d = d(t)$ with $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$. We also consider the indices

$$\alpha = (1, 1), \qquad \beta = (1, 2),$$

so that $L^\alpha = L_1 L_2$ and $L^\beta = L_1 L_2^2$, where $L = (L_1, L_2) \in P^2$. Then by (2.6) the differential operators L_+^α and L_+^β satisfy

$$\frac{\partial L_+^\beta}{\partial t_\alpha} - \frac{\partial L_+^\alpha}{\partial t_\beta} = L_+^\alpha L_+^\beta - L_+^\beta L_+^\alpha. \tag{2.9}$$

Using (2.7) and (2.8), we obtain

$$\begin{aligned}
&L_2^2 = \partial_2^2 + c_y \partial_2^{-1} + 2c + 2d\partial_1 \partial_2^{-1} + d_y \partial_1 \partial_2^{-2} + c^2 \partial_2^{-2} + O\left(\partial_2^{-3}\right), \\
&L_1 L_2 = \partial_2^2 + a\partial_1 + c + O\left(\partial_2^{-1}\right), \\
&L_1 L_2^2 = \partial_2^3 + a\partial_1\partial_2 + 2c\partial_2 + 2d\partial_1 + 3c_y + b + O\left(\partial_2^{-1}\right),
\end{aligned}$$

where the subscripts x and y denote the partial derivatives with respect to $x = x_1$ and $y = x_2$, respectively. Hence we have

$$L_+^\alpha = (L_1 L_2)_+ = \partial_2^2 + a\partial_1 + c, \tag{2.10}$$
$$L_+^\beta = \left(L_1 L_2^2\right)_+ = \partial_2^3 + a\partial_1\partial_2 + 2c\partial_2 + 2d\partial_1 + 3c_y + b. \tag{2.11}$$

Using (2.10) and (2.11), we obtain

$$\begin{aligned}
L_+^\alpha L_+^\beta = {}& \partial_2^5 + 2a\partial_1\partial_2^3 + 3c\partial_2^3 + (2a_y + 2d)a\partial_1\partial_2^2 + (7c_y + b)\partial_2^2 + a^2\partial_1^2\partial_2 \\
&+ (a_{yy} + 4d_y + aa_x + 3ac)\partial_1\partial_2 + \left(8c_{yy} + 2b_y + 2ac_x + 2c^2\right)\partial_2 \\
&+ 2ad\partial_1^2 + (2d_{yy} + 2ad_x + 3ac_y + ab + cd)\partial_1 \\
&+ 3c_{yyy} + b_{yy} + 3ac_{yx} + ab_x + 3cc_y + bc, \\
L_+^\beta L_+^\alpha = {}& \partial_2^5 + 2a\partial_1\partial_2^3 + 3c\partial_2^3 + (3a_y + 2d)a\partial_1\partial_2^2 + (6c_y + b)\partial_2^2 + a^2\partial_1^2\partial_2 \\
&+ (3a_{yy} + aa_x + 3ac)\partial_1\partial_2 + \left(3c_{yy} + ac_x + 2c^2\right)\partial_2 + (aa_y + 2ad)\partial_1^2 \\
&+ (a_{yyy} + aa_{yx} + 4ac_y + 2a_y c + 2a_x d + ab + 2cd)\partial_1 \\
&+ c_{yyy} + ac_{yx} + 5cc_y + 2c_x d + bc.
\end{aligned}$$

If we set $t_\alpha = s$ and $t_\beta = t$, then by (2.10) and (2.11) the left hand side of (2.9) becomes

$$\frac{\partial L_+^\beta}{\partial s} - \frac{\partial L_+^\alpha}{\partial t} = a_s\partial_1\partial_2 + 2c_s\partial_2 + (2d_s - a_t)\partial_1 + 3c_{ys} + b_s - c_t.$$

Thus by comparing the coefficients we see that (2.9) determines the system of partial differential equations given by

$$\begin{aligned}
&a_y = c_y = 0, \qquad 2c_s = 2b_y + ac_x, \\
&a_t + 2ad_x + 2d_{yy} = 2a_x d + 2d_s + cd, \qquad b_s + 2c_x d = ab_x + b_{yy} + c_t.
\end{aligned}$$

3 Baker functions

Baker functions associated to pseudodifferential operators of several variables discussed in Section 2 were introduced in [9]. As in the single variable case, these Baker functions provide solutions of Lax equations of the from (2.3). In this section we review the construction of such Baker functions.

First, we need to introduce an additional set of complex variables $z_1, \dots, z_n$. We then consider the formal series given by

$$\xi(t,z) = \sum_{\alpha\in\mathbb{Z}_+^n} t_\alpha z^\alpha, \tag{3.1}$$

where $z = (z_1, \dots, z_n)$ so that $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$. If $\phi \in 1 + P_-$ is as in Section 2 satisfying (2.4), we define the associated *Baker function* w by

$$w = w(t,z) = \phi e^{\xi(t,z)}. \tag{3.2}$$

Since $x_i = t_{\mathbf{e}_i}$ for $1 \le i \le n$ by (2.2), we see that

$$\partial_i e^{\xi(t,z)} = \frac{\partial}{\partial x_i} e^{\xi(t,z)} = \frac{\partial}{\partial t_{\mathbf{e}_i}} e^{\xi(t,z)} = z^{\mathbf{e}_i} e^{\xi(t,z)} = z_i e^{\xi(t,z)}.$$

Thus, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, we have

$$\begin{aligned} \partial^\alpha e^{\xi(t,z)} &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} e^{\xi(t,z)} = \partial_{\mathbf{e}_1}^{\alpha_1} \cdots \partial_{\mathbf{e}_n}^{\alpha_n} e^{\xi(t,z)} \\ &= z_1^{\alpha_1} \cdots z_n^{\alpha_n} e^{\xi(t,z)} = z^\alpha e^{\xi(t,z)}. \end{aligned}$$

Hence, if $\phi = 1 + \sum_\alpha a_\alpha(t)\partial^\alpha \in 1 + P_-$, then the Baker function in (3.2) can be written in the form

$$w(t,z) = \widehat{w}(t,z) e^{\xi(t,z)}, \tag{3.3}$$

where $\widehat{w}(t,z)$ is a formal power series in $z_1, \dots, z_n$ of the form

$$\widehat{w}(t,z) = 1 + \sum_\alpha a_\alpha(t) z^\alpha.$$

If $L = (L_1, \dots, L_n) = \phi\partial\phi^{-1} \in P^n$ is an element associated to $\phi \in 1 + P_-$ satisfying (2.4) as in (2.5), then the Baker function w given by (3.2) satisfies $Lw = zw$, that is, $L_i w = z_i w$ for each i (see [9, Lemma 3.1]). In addition, it can also be shown that $\partial_{t_\alpha} w = L_+^\alpha w$ for each $\alpha \in \mathbb{Z}_+^n$ (cf. [9, Lemma 3.2]).

Given an element $\psi = \sum_{\alpha\le\nu} f_\alpha(t)\partial^\alpha \in P$, we define its adjoint $\psi^* \in P$ by

$$\psi^* = \sum_{\alpha\le\nu} (-1)^{|\alpha|} \partial^\alpha f_\alpha(t), \tag{3.4}$$

and its residue with respect to ∂ by

$$\mathrm{Res}_\partial \psi = f_{-\mathbf{1}}(t) = f_{(-1,\dots,-1)}(t).$$

On the other hand, if $h(z) = h(z_1, \dots, z_n)$ is a Laurent series in $z_1, \dots, z_n$ which can be written in the form $h(z) = \sum_\alpha b_\alpha z^\alpha$, then its residue with respect to z is given by

$$\mathrm{Res}_z h(z) = b_{-\mathbf{1}} = b_{(-1,\dots,-1)}. \tag{3.5}$$

If $\psi = \sum_{\alpha} a_\alpha \partial^\alpha \in P$ and $\eta = \sum_{\beta} b_\beta \partial^\beta \in 1 + P_-$, then we have

$$\mathrm{Res}_z \left(\psi e^{\xi(t,z)}\right) \left(\eta e^{-\xi(t,z)}\right) = \mathrm{Res}_\partial \psi \eta^*,$$

where η^* is the adjoint of η given by (3.4) (see [9, Lemma 3.3]).

We define the adjoint w^* of the Baker function w in (3.2) by

$$w^*(t,z) = (\phi^*)^{-1} e^{-\xi(t,z)}, \tag{3.6}$$

where ϕ^* is the adjoint of ϕ given by (3.4). Then it can be shown that the Baker function w in (3.2) satisfies

$$\mathrm{Res}_z w(t', z) w^*(t,z) = 0 \tag{3.7}$$

for all t, t' (see [9]).

We now consider the subset $\widehat{P}_-$ of P_- defined by

$$\widehat{P}_- = \left\{ \sum_\alpha f_\alpha(t) \partial^\alpha \;\middle|\; \alpha \le -\mathbf{1} = (-1, \ldots, -1) \text{ whenever } f_\alpha(t) \neq 0 \right\}. \tag{3.8}$$

Then the following theorem extends the result in [6, Proposition 7.3.5] to the case of several variables.

Theorem 1. *Let w and $w^\#$ be formal power series of the form*

$$w = \phi e^{\xi(t,z)}, \qquad w^\# = \psi e^{-\xi(t,z)}$$

with $\phi, \psi \in 1 + \widehat{P}_-$ satisfying the condition

$$\mathrm{Res}_z \left(\partial^\alpha w w^\#\right) = 0.$$

Then there exists an operator $L = (L_1, \ldots, L_n) \in P^n$ with $L_i = \partial_i + u_i$ and $u_i \in P_-$ for $1 \le i \le n$ which satisfies the Lax equation (2.3) with w and $w^\#$ being the associated Baker function and adjoint Baker function, respectively.

Proof. See [9, Theorem 3.6]. ■

4 Tau functions

In this section we extend the notion of tau functions associated to the usual pseudodifferential operators to the case of pseudodifferential operators of several variables. As in the single variable case, a Baker function given by (3.2) can be expressed in terms of such a tau function.

Let $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ and $z = (z_1, \ldots, z_n)$ be the complex variables considered in Section 3. Given a vector $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, we define the operator $G(s)$ on functions of the form $f(t,z) = f((t_\alpha)_{\alpha \in \mathbb{Z}_+^n}, (z_1, \ldots, z_n))$ by

$$G(s) f(t,z) = f\left(\left(t_\alpha - \alpha^{-1} s^{-\alpha}\right)_{\alpha \in \mathbb{Z}_+^n}, z \right), \tag{4.1}$$

where $\alpha^{-1}s^{-\alpha} = \alpha_1^{-1}\cdots\alpha_n^{-1}s_1^{-\alpha_1}\cdots s_n^{-\alpha_n}$ according to the multi-index notation. Thus, if $\xi(t,z)$ is as in (3.1), we have

$$G(s)\xi(t,z) = \sum_{\alpha\in\mathbb{Z}_+^n} \left(t_\alpha - \alpha^{-1}s^{-\alpha}\right) z^\alpha = \xi(t,z) - \sum_{\alpha\in\mathbb{Z}_+^n} \alpha^{-1}s^{-\alpha}z^\alpha.$$

Hence it follows that $G(s)$ operates on the Baker function $w(t,z)$ in (3.2) associated to an element $\phi \in 1 + P_-$ and on the adjoint Baker function $w(t,z)$ in (3.6) by

$$G(s)w(t,z) = w(t,z)\exp\left(-\sum_{\alpha\in\mathbb{Z}_+^n} \alpha^{-1}s^{-\alpha}z^\alpha\right),$$

$$G(s)w^*(t,z) = w^*(t,z)\exp\left(\sum_{\alpha\in\mathbb{Z}_+^n} \alpha^{-1}s^{-\alpha}z^\alpha\right).$$

Using the relation

$$\ln\left(1 - \sum_{r=1}^n \frac{z_r}{s_r}\right) = -\sum_{\alpha\in\mathbb{Z}_+^n} \frac{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}{\alpha_1\cdots\alpha_n s_1^{\alpha_1}\cdots s_n^{\alpha_n}} = -\sum_{\alpha\in\mathbb{Z}_+^n} \alpha^{-1}s^{-\alpha}z^\alpha,$$

we see that the operation of $G(s)$ on $w^*(t,z)$ can be written in the form

$$G(s)w^*(t,z) = w^*(t,z)\left(1 - \sum_{r=1}^n \frac{z_r}{s_r}\right)^{-1}. \tag{4.2}$$

We now consider some calculations involving the residue operator Res_z given by (3.5).

Lemma 1. *Let $s = (s_1,\ldots,s_n) \in \mathbb{C}^n$, and consider a formal power series of the form $\eta(z) = 1 + \sum_{\alpha\leq-\mathbf{1}} f_\alpha z^\alpha$. Then we have*

$$\mathrm{Res}_z\eta(z)\left(1 - \sum_{r=1}^n \frac{z_r}{s_r}\right)^{-1} = s_1\cdots s_n(\eta(s) - 1)$$

for all $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$.

Proof. First, we write the formal power series $\eta(z)$ in the form

$$\eta(z) = 1 + \sum_{\alpha_1=-\infty}^{-1} \cdots \sum_{\alpha_n=-\infty}^{-1} f_{(\alpha_1,\ldots,\alpha_n)} z_1^{\alpha_1}\cdots z_n^{\alpha_n}.$$

Using this and the power series expansion

$$\left(1 - \sum_{r=1}^n \frac{z_r}{s_r}\right)^{-1} = \sum_{r=0}^\infty \left(s_1^{-1}z_1 + \cdots + s_n^{-1}z_n\right)^r = \sum_{\beta\geq\mathbf{0}} s^{-\beta}z^\beta = \sum_{\beta\geq\mathbf{0}} \frac{z_1^{\beta_1}\cdots z_n^{\beta_n}}{s_1^{\beta_1}\cdots s_n^{\beta_n}}$$

with $\beta = (\beta_1, \ldots, \beta_n)$, we see that

$$\begin{aligned}
\operatorname{Res}_z \eta(z) \left(1 - \sum_{r=1}^{n} \frac{z_r}{s_r}\right)^{-1} &= \sum_{\alpha_1=-\infty}^{-1} \cdots \sum_{\alpha_1=-\infty}^{-1} \frac{f_{(\alpha_1,\ldots,\alpha_n)}}{s_1^{-\alpha_1-1} \cdots s_n^{-\alpha_n-1}} \\
&= s_1 \cdots s_n \sum_{\alpha_1=-\infty}^{-1} \cdots \sum_{\alpha_n=-\infty}^{-1} f_{(\alpha_1,\ldots,\alpha_n)} s_1^{\alpha_1} \cdots s_n^{\alpha_n} \\
&= s_1 \cdots s_n \sum_{\alpha \leq -\mathbf{1}} f_\alpha s^\alpha = s_1 \cdots s_n (\eta(s) - 1);
\end{aligned}$$

hence the lemma follows. ■

Lemma 2. *Let $s = (s_1, \ldots, s_n)$ and $s' = (s'_1, \ldots, s'_n)$ be elements of $\mathbb{C}^n$, and consider a formal power series of the form $\eta(z) = 1 + \sum_{\alpha \leq -\mathbf{1}} f_\alpha z^\alpha$. Then we have*

$$\begin{aligned}
&\operatorname{Res}_z \eta(z) \left(1 - \sum_{r=1}^{n} \frac{z_r}{s_r}\right)^{-1} \left(1 - \sum_{r=1}^{n} \frac{z_r}{s'_r}\right)^{-1} \\
&\quad = (\eta(s) - 1) \sum_{\alpha \geq \mathbf{0}} \frac{s^{\alpha+1}}{s'^{\alpha}} = (\eta(s') - 1) \sum_{\alpha \geq \mathbf{0}} \frac{s'^{\alpha+1}}{s^{\alpha}}
\end{aligned}$$

for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Proof. Using power series expansions and the formal relation

$$\sum_{r=0}^{\infty} (u_1 + \cdots + u_n)^r = \sum_{\alpha \geq \mathbf{0}} u^\alpha$$

for each n-tuple $u = (u_1, \ldots, u_n)$, we see that

$$\begin{aligned}
&\operatorname{Res}_z \eta(z) \left(1 - \sum_{r=1}^{n} \frac{z_r}{s_r}\right)^{-1} \left(1 - \sum_{r=1}^{n} \frac{z_r}{s'_r}\right)^{-1} \\
&\quad = \operatorname{Res}_z \eta(z) \left(\sum_{r=0}^{\infty} \left(\sum_{r=1}^{n} \frac{z_r}{s_r}\right)^r\right) \left(\sum_{r=0}^{\infty} \left(\sum_{r=1}^{n} \frac{z_r}{s'_r}\right)^r\right) \\
&\quad = \operatorname{Res}_z \eta(z) \left(\sum_{\beta \geq \mathbf{0}} \frac{z^\beta}{s^\beta}\right) \left(\sum_{\gamma \geq \mathbf{0}} \frac{z^\gamma}{s'^\gamma}\right) = \sum_{\alpha \leq -\mathbf{1}} f_\alpha \sum_{\beta+\gamma=-\alpha-\mathbf{1}} \frac{1}{s^\beta s'^\gamma},
\end{aligned}$$

where the second summation in the previous line is over multi-indices $\beta, \gamma \geq \mathbf{0}$ such that $\beta + \gamma = -\alpha - \mathbf{1}$. Using $\beta = -\gamma - \alpha - 1$, we obtain

$$\begin{aligned}
&\operatorname{Res}_z \eta(z) \left(1 - \sum_{r=1}^{n} \frac{z_r}{s_r}\right)^{-1} \left(1 - \sum_{r=1}^{n} \frac{z_r}{s'_r}\right)^{-1} \\
&\quad = \sum_{\alpha \leq -\mathbf{1}} f_\alpha \sum_{\gamma \geq \mathbf{0}} \frac{1}{s^{-\gamma-\alpha-1} s'^\gamma} = \sum_{\alpha \leq -\mathbf{1}} f_\alpha s^\alpha \sum_{\gamma \geq \mathbf{0}} \frac{s^{\gamma+1}}{s'^\gamma} = (\eta(s) - 1) \sum_{\gamma \geq \mathbf{0}} \frac{s^{\gamma+1}}{s'^\gamma}.
\end{aligned}$$

Similarly, by using $\gamma = -\beta - \alpha - 1$ we have

$$\begin{aligned}&\mathrm{Res}_z \eta(z) \left(1 - \sum_{r=1}^{n} \frac{z_r}{s_r}\right)^{-1} \left(1 - \sum_{r=1}^{n} \frac{z_r}{s'_r}\right)^{-1} \\ &\quad = \sum_{\alpha \leq -\mathbf{1}} f_\alpha \sum_{\beta \geq \mathbf{0}} \frac{1}{s^\beta s'^{-\beta-\alpha-1}} = \sum_{\alpha \leq -\mathbf{1}} f_\alpha s'^\alpha \sum_{\beta \geq \mathbf{0}} \frac{s'^{\beta+1}}{s^\beta} = (\eta(s') - 1) \sum_{\beta \geq \mathbf{0}} \frac{s'^{\beta+1}}{s^\beta}.\end{aligned}$$

Hence the lemma follows. ∎

We now state the main theorem in this section, which shows the existence of the tau function $\tau(t)$ corresponding to a Baker function of the type discussed in Section 3.

Theorem 2. *Let $w(t,z)$ be the Baker function in (3.2) corresponding to an element $\phi \in 1 + P_-$, and let $\widehat{w}(t,z)$ be the associated formal power series given by (3.3). Then there is a function $\tau(t)$ with $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ such that*

$$\widehat{w}(t,z) = G(z)\tau(t)/\tau(t)$$

for $z \in \mathbb{C}^n$ and $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$, where $G(z)$ is the operator given by (4.1).

Proof. By (3.7) we have

$$\mathrm{Res}_z w(t,z) G(s) w^*(t,z) = 0$$

for each $s \in \mathbb{C}^n$. Using this and (4.2), we have

$$\mathrm{Res}_z \widehat{w}(t,z) G(s) \widehat{w}^*(t,z) \left(1 - \sum_{r=1}^{n} \frac{z_r}{s_r}\right)^{-1} = 0.$$

Thus by Lemma 1 we see that

$$s_1 \cdots s_n (\widehat{w}(t,s) G(s) \widehat{w}^*(t,s) - 1) = 0;$$

hence we obtain

$$\widehat{w}(t,s)^{-1} = G(s) \widehat{w}^*(t,s). \tag{4.3}$$

Similarly, we have

$$\mathrm{Res}_z w(t,z) G(s) G(s') w^*(t,z) = 0$$

for all $s, s' \in \mathbb{C}^n$, which implies that

$$\mathrm{Res}_z \widehat{w}(t,z) G(s) G(s') \widehat{w}^*(t,z) \left(1 - \sum_{r=1}^{n} \frac{z_r}{s_r}\right)^{-1} \left(1 - \sum_{r=1}^{n} \frac{z_r}{s'_r}\right)^{-1} = 0.$$

Using this and applying Lemma 2 to the formal power series

$$\phi(z) = \widehat{w}(t,z) G(s) G(s') \widehat{w}^*(t,z),$$

we obtain

$$\widehat{w}(t,s)G(s)G(s')\widehat{w}^*(t,s) = \widehat{w}(t,s')G(s)G(s')\widehat{w}^*(t,s') = 1.$$

By combining this with (4.3) we have

$$\widehat{w}(t,s)(G(s')\widehat{w}(t,s))^{-1} = \widehat{w}(t,s)(G(s)\widehat{w}(t,s'))^{-1}. \tag{4.4}$$

We now set

$$h(t,s) = \ln(\widehat{w}(t,s)).$$

Then by taking the logarithm of both sides of (4.4) we obtain

$$(1 - G(s'))h(t,s) = (1 - G(s))h(t,s').$$

Replacing s and s' by z and ζ, respectively, gives us

$$h(t,z) - G(\zeta)h(t,z) = h(t,\zeta) - G(z)h(t,\zeta). \tag{4.5}$$

For each $k \in \{1,\dots,n\}$ we define the differential operator $\mathcal{D}_k(z)$ by

$$\mathcal{D}_k(z) = \sum_{\alpha\in\mathbb{Z}_+^n} \alpha_k \alpha^{-1} z^{-\alpha-\mathbf{e}_k} \partial_\alpha - \frac{\partial}{\partial z_k},$$

where $\partial_\alpha = \partial_{t_\alpha} = \partial/\partial t_\alpha$ with $\alpha = (\alpha_1,\dots,\alpha_n)$. For any function $\varphi(t)$, we have

$$\begin{aligned}
\mathcal{D}_k(z)G(z)\varphi(t) &= \mathcal{D}_k(z)\varphi\left(\left(t_\alpha - \alpha^{-1}z^{-\alpha}\right)_{\alpha\in\mathbb{Z}_+^n}\right)\\
&= \sum_{\alpha\in\mathbb{Z}_+^n} \alpha_k\alpha^{-1}z^{-\alpha-\mathbf{e}_k}\partial_\alpha\varphi\left(\left(t_\alpha - \alpha^{-1}z^{-\alpha}\right)_{\alpha\in\mathbb{Z}_+^n}\right)\\
&\quad - \sum_{\alpha\in\mathbb{Z}_+^n} \alpha_k\alpha^{-1}z^{-\alpha-\mathbf{e}_k}\partial_\alpha\varphi\left(\left(t_\alpha - \alpha^{-1}z^{-\alpha}\right)_{\alpha\in\mathbb{Z}_+^n}\right) = 0.
\end{aligned}$$

Using this and (4.5), we see that

$$\mathcal{D}_k(z)h(t,z) - G(\zeta)\mathcal{D}_k(z)h(t,z) = \mathcal{D}_k(z)h(t,\zeta) = \sum_{\alpha\in\mathbb{Z}_+^n} \alpha_k\alpha^{-1}z^{-\alpha-\mathbf{e}_k}\partial_\alpha h(t,\zeta).$$

Thus for each $\beta \in \mathbb{Z}_+^n$ we obtain

$$\begin{aligned}
&\mathrm{Res}_z z^\beta \mathcal{D}_k(z)h(t,z) - G(\zeta)\,\mathrm{Res}_z z^\beta\mathcal{D}_k(z)h(t,z)\\
&\quad = \mathrm{Res}_z \sum_{\alpha\in\mathbb{Z}_+^n} \alpha_k\alpha^{-1}z^{-\alpha+\beta-\mathbf{e}_k}\partial_\alpha h(t,\zeta) = \beta_k(\beta + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\beta+\mathbf{1}-\mathbf{e}_k}h(t,\zeta),
\end{aligned}$$

where we used the fact that the k-component of $\beta + \mathbf{1} - \mathbf{e}_k$ is β_k. Thus, if we set $a_{\alpha,k} = \mathrm{Res}_z z^\alpha\mathcal{D}_k(z)h(t,z)$ for each $\alpha \in \mathbb{Z}_+^n$, we have

$$(1 - G(\zeta))a_{\alpha,k} = \alpha_k(\alpha + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\alpha+\mathbf{1}-\mathbf{e}_k}h(t,\zeta) \tag{4.6}$$

for all $\alpha \in \mathbb{Z}_+^n$ and $k \in \{1, \ldots, n\}$. Hence we obtain

$$\begin{aligned}
&\alpha_k(\alpha + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\alpha+\mathbf{1}-\mathbf{e}_k} a_{\beta,k} - \beta_k(\beta + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\beta+\mathbf{1}-\mathbf{e}_k} a_{\alpha,k} \\
&\qquad = G(\zeta)\left(\alpha_k(\alpha + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\alpha+\mathbf{1}-\mathbf{e}_k} a_{\beta,k} - \beta_k(\beta + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\beta+\mathbf{1}-\mathbf{e}_k} a_{\alpha,k}\right),
\end{aligned}$$

which implies that

$$\alpha_k(\alpha + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\alpha+\mathbf{1}-\mathbf{e}_k} a_{\beta,k} = \beta_k(\beta + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\beta+\mathbf{1}-\mathbf{e}_k} a_{\alpha,k}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$. Therefore there is a function $\tau(t)$ such that

$$a_{\alpha,k} = -\alpha_k(\alpha + \mathbf{1} - \mathbf{e}_k)^{-1}\partial_{\alpha+\mathbf{1}-\mathbf{e}_k} \ln \tau(t).$$

By combining this with (4.6) we obtain

$$\partial_{\alpha+\mathbf{1}-\mathbf{e}_k} h(t,\zeta) = \alpha_k^{-1}(\alpha + \mathbf{1} - \mathbf{e}_k)(1 - G(\zeta))a_{\alpha,k} = -(1 - G(\zeta))\partial_{\alpha+\mathbf{1}-\mathbf{e}_k} \ln \tau(t)$$

for all $\alpha \in \mathbb{Z}_+^n$ and $k \in \{1, \ldots, n\}$; hence we see that

$$h(t,\zeta) = -(1 - G(\zeta)) \ln \tau(t).$$

Thus it follows that

$$\ln(\widehat{w}(t,\zeta)) = h(t,\zeta) = -\ln \tau(t) + G(\zeta) \ln \tau(t) = \ln(G(\zeta)\tau(t)/\tau(t)).$$

Thus we obtain

$$\widehat{w}(t,\zeta) = G(\zeta)\tau(t)/\tau(t),$$

and therefore the proof of the theorem is complete. ■

5 Concluding remarks

As is mentioned in the introduction, Baker functions associated to single-variable pseudodifferential operators provide formal solutions of soliton equations. Baker functions for pseudodifferential operators of several variables also determine solutions of soliton equations, and by Theorem 2 we see that the Baker function in (3.2) can be written in the form

$$w(t,z) = \widehat{w}(t,z)e^{\xi(t,z)} = (G(z)\tau(t)/\tau(t))e^{\xi(t,z)},$$

where $\xi(t,z) = \sum_{\alpha \in \mathbb{Z}_+^n} t_\alpha z^\alpha$. The function $\tau(t)$ with $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ is a tau function for pseudodifferential operators of several variables. Thus we have extended the expression of a Baker function in term of the corresponding tau function to the case of pseudodifferential operators of several variables.

References

[1] Arbarello E, De Concini C, Kac V G and Procesi C, Moduli Spaces of Curves and Representation Theory, *Comm. Math. Phys.* **117** (1988), 1–36.

[2] Belokolos E, Bobenko A, Enol'skii V, Its A and Matveev V, Algebro-Geometric Approach to Nonlinear Integrable Equations, Springer-Verlag, Heidelberg, 1994.

[3] Carroll R, Topics in Soliton Theory, North-Holland, Amsterdam, 1991.

[4] Cherednik I, Basic Methods of Soliton Theory, World Scientific, Singapore, 1996.

[5] Date E, Kashiwara M, Jimbo M and Miwa T, Transformation Groups for Soliton Equations, in Nonlinear Integrable Systems — Classcal Theory and Quantum Theory (Kyoto, 1981), 39–119, World Scientific, Singapore, 1983.

[6] Dickey L, Soliton Equations and Hamiltonian Systems, World Scientific, Singapore, 1991.

[7] Krichever A, Methods of Algebraic Geometry in the Theory of Non-Linear Equations, *Russian Math. Surveys* **32** (1977), 185–213.

[8] Kupershmidt B, KP or mKP: Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems, Amer. Math. Soc., Providence, 2000.

[9] Lee M H, Pseudodifferential Operators of Several Variables and Baker Functions, *Lett. Math. Phys.*, **60** (2002), 1–8.

[10] Osipov D, Krichever Correpondence for Algebraic Varieties, *Izvestiya RAN: Ser. Math.* **65** (2001), 91–128.

[11] Parshin A, The Krichever Correspondence for Algebraic Surfaces, *Funct. Anal. Appl.* **35** (2001), 74–76.

[12] Parshin A, On a Ring of Formal Pseudo-Differential Operators, *Proc. Steklov Inst. Math.* **224** (1999), 266–280.

[13] Segal G and Wilson G, Loop Groups and Equations of KdV, *Publ. Math. I.H.E.S.* **61** (1985), 5–65.