The Scattering Approach for the Camassa–Holm equation

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Received March 8, 2002; Revised April 23, 2002; Accepted April 24, 2002

Abstract

We present an approach proving the integrability of the Camassa–Holm equation for initial data of small amplitude.

1 Introduction

The Camassa–Holm equation

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

in dimensionless space-time variables (x, t) is a model for the unidirectional propagation of two-dimensional waves in shallow water with a flat bottom, ω being a positive constant related to the critical shallow water speed (see [1]), and was first derived as an abstract equation in [8] by using the method of recursion operators. Of physical interest are solutions of (1.1) which decay at infinity cf. [9]. The Camassa–Holm equation models wave breaking [5] and admits wave solutions that exist indefinitely in time [3]. An aspect of considerable interest is the fact that the solitary waves of (1.1) are solitons (see [2] for numerical evidence and [10] for the complete description).

In terms of the momentum $m = u - u_{xx}$ the Camassa–Holm equation can be expressed cf. [2] as the condition of compatibility between

$$\psi_{xx} = \frac{1}{4}\psi + \lambda(m + \omega)\psi \tag{1.2}$$

and

$$\psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi. \tag{1.3}$$

Equation (1.2) is the isospectral problem associated to (1.1) so that the Camassa–Holm equation is formally integrable. In the absence of bound states for (1.2), the direct and inverse scattering problem was discussed in [4]. Our purpose is to indicate how the scattering approach can be pursued in the more general case when finitely many bound states are present. In particular, this allows us to solve the Camassa–Holm initial-value problem via inverse scattering for initial data of small amplitude. The importance of the scattering problem in the case of finitely many bound states is emphasized by the fact that in the case of one soliton there is precisely one bound state for (1.2) cf. [6] and the 2-soliton solution corresponds to an isospectral problem (1.2) with two bound states (see [10]).

2 Direct and inverse scattering

If the momentum $m \in H^3(\mathbb{R})$ is such that $m + \omega > 0$ and

$$\int_{\mathbb{R}} (1 + |x|)|m(x)|dx < \infty,$$

then the continuous spectrum of (1.2) is $(-\infty, -\frac{1}{4\omega}]$ and there are at most finitely many eigenvalues in the interval $(-\frac{1}{4\omega}, 0)$ cf. [4]. In absence of bound states the scattering data consists of the transmission and reflection coefficients associated to the elements in the continuous spectrum. More precisely, if $\psi(x, t)$ is an eigenfunction corresponding to some λ in the continuous spectrum of the isospectral problem (1.2), then

$$\psi(t, x) \approx \begin{cases} e^{-ikx} + \mathfrak{R}(t, k)\, e^{ikx} & \text{as} \quad x \to \infty, \\ \mathfrak{T}(t, k)\, e^{-ikx} & \text{as} \quad x \to -\infty, \end{cases} \tag{2.1}$$

for some complex transmission coefficient $\mathfrak{T}$ and a reflection coefficient $\mathfrak{R}$, where $k \geq 0$ satisfies $k^2 = -\frac{1}{4} - \lambda\omega \geq 0$. The evolution of $\mathfrak{T}(t, k)$ and $\mathfrak{R}(t, k)$ under the Camassa–Holm flow is given by (see [4])

$$\mathfrak{T}(t, k) = \mathfrak{T}(0, k), \qquad \mathfrak{R}(t, k) = \mathfrak{R}(0, k) \exp\left(\frac{ik}{\lambda}\, t\right), \qquad t \geq 0. \tag{2.2}$$

Since (1.2) is the isospectral problem, the bound states are constants of motion for the Camassa–Holm equation [1]. In order to solve the scattering problem in presence of finitely many bound states it is necessary to find the proper normalization constants for the eigenfunctions associated with the discrete spectrum. This question was left open in [4], due to the fact that the choice suggested by analogy with the classical Schrödinger equation [7] is not appropriate (the time evolution cannot be determined). A proper family of normalization constants can be defined as follows. The Liouville transformation

$$\phi(y) = (m(x) + \omega)^{1/4}\, \psi(x),$$

where

$$y = \int_0^x \sqrt{m(\xi) + \omega}\, d\xi - \int_0^\infty \frac{m(\xi)}{\sqrt{\omega} + \sqrt{m(\xi) + \omega}}\, d\xi$$

converts (1.2) into

$$-\frac{d^2\phi}{dy^2} + Q\phi = \mu\phi. \tag{2.3}$$

Here

$$Q(y) = \frac{1}{4q(y)} + \frac{q_{yy}(y)}{4\, q(y)} - \frac{3q_y^2(y)}{16q^2(y)} - \frac{1}{4\omega} \qquad \text{with} \quad q(y) = m(x) + \omega,$$

and the spectral parameter is $\mu = -\frac{1}{4\omega} - \lambda$. If $\psi_n(x, t)$ is an eigenfunction for (1.2) corresponding to the eigenvalue $\lambda_n \in (-\frac{1}{4\omega}, 0)$, then

$$\phi_n(y, t) = (m(x, t) + \omega)^{1/4}\, \psi_n(x, t)$$

is an eigenfunction for (2.3) corresponding to the eigenvalue $\mu_n = -\frac{1}{4\omega} - \lambda_n < 0$. Requiring

$$\int_{\mathbb{R}} \phi_n^2(y,t)\,dy = 1,$$

the normalization constants $c_n(t)$ are determined by

$$\phi_n(y,t) \approx c_n(t) e^{-k_n y} \qquad \text{for} \quad y \to \infty,$$

with $k_n = \sqrt{-\mu_n}$. It turns out that as $m(x,t)$ evolves according to the Camassa–Holm equation,

$$c_n(t) = c_n(0) \exp\left(-\frac{k_n \sqrt{\omega}}{2\lambda_n}\, t \right), \qquad t \geq 0,$$

as a rather intricate analysis shows. Therefore the evolution of the scattering data (the reflection and transmission coefficients together with the previously defined normalization constants) under the Camassa–Holm flow has been explicitly determined. At this point the method presented in [4] can be implemented to solve the inverse scattering problem for the Camassa–Holm equation.

The presented approach is best exemplified by the fact that it shows that the solitary waves of (1.1) are solitons: the associated spectral problem is reflectionless and has a single eigenvalue. This important feature of the Camassa–Holm equation was explained in [6] by means of trace formulas and eigenvalue estimates for Schrödinger operators. An application of our technique provides a simpler and more direct proof.

It is known (see [11] and [6]) that solitary wave solutions $u(x,t) = \varphi(x - ct)$ propagating at the speed $c > 0$ exist only for $c > 2\omega$. Moreover, the function φ, determined uniquely up to translations (henceforth we choose φ with the crest positioned at $x = 0$), is smooth and positive with a profile decreasing symmetrically from its crest of height $(c - 2\omega)$. No mathematical expression in closed form is available for φ so that our analysis depends on the equations

$$-c\varphi + c\varphi_{xx} + \frac{3}{2}\varphi^2 + 2\omega\varphi = \varphi\varphi_{xx} + \frac{1}{2}\varphi_x^2, \tag{2.4}$$

and

$$\varphi_x^2 (c - \varphi) = \varphi^2 (c - 2\omega - \varphi), \tag{2.5}$$

which are both obtained from (1.1). For $u(x,0) = \varphi(x)$ we have

$$m(x,0) + \omega = \frac{\omega c^2}{[c - \varphi(x)]^2}$$

so that the Liouville transformation can be performed. In combination with (2.4)–(2.5), it yields

$$Q(y) = -\frac{\varphi(x)}{c[c - \varphi(x)]}.$$

Straightforward (but long) calculations relying repeatedly on (2.4)–(2.5) show that $Q(y)$ satisfies the differential equation

$$Q_y^2 = Q^2\left(2Q + \frac{c-2\omega}{c\omega}\right).$$

Hence (see [7])

$$Q(y) = -\frac{c-2\omega}{2c\omega}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{c-2\omega}{c\omega}}\,y\right).$$

With the above potential it is well-known that the problem (2.3) is reflectionless and has $\mu = \frac{2\omega-c}{4c\omega}$ as the only eigenvalue cf. [7]. While performing the Liouville transformation, let us note that

$$\begin{cases} y(x) - \sqrt{\omega}\,x \to 0 & \text{as} \quad x \to \infty, \\ y(x) - \sqrt{\omega}\,x \to \displaystyle\int_{-\infty}^{\infty}\left(\sqrt{\omega} - \sqrt{m(\xi)+\omega}\right)d\xi & \text{as} \quad x \to -\infty. \end{cases} \tag{2.6}$$

From (1.1) we infer that $C = \int_{-\infty}^{\infty}\left(\sqrt{\omega} - \sqrt{m(\xi)+\omega}\right)d\xi$ is preserved under the Camassa–Holm flow. Therefore (2.6) can be used to deduce that

$$\omega^{-1/4}\phi(t,y) \approx \begin{cases} e^{-i\sqrt{\mu}y} + \mathfrak{R}(t,k)\,e^{i\sqrt{\mu}y} & \text{as} \quad y \to \infty, \\ \mathfrak{T}(t,k)\,e^{-i\sqrt{\mu}(y-C)} & \text{as} \quad y \to -\infty. \end{cases} \tag{2.7}$$

As an outcome of (2.2) and (2.7), the solution of (1.1) with initial data $u(x,0) = \varphi(x)$ is reflectionless at any time $t > 0$. Hence the solitary waves of the Camassa–Holm equation are solitons.

Acknowledgements

The author is grateful to A Constantin for bringing this problem to his attention and for stimulating conversations. The author thanks the referee for valuable remarks on an earlier draft of this note.

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