Noncentral Extensions as Anomalies in Classical Dynamical Systems

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Abstract

A two cocycle is associated to any action of a Lie group on a symplectic manifold. This allows to enlarge the concept of anomaly in classical dynamical systems considered by F Toppan in [*J. Nonlinear Math. Phys.* **8**, Nr. 3 (2001), 518–533] so as to encompass some extensions of Lie algebras related to noncanonical actions.

1   Introduction

The concept of anomalies in classical dynamical systems was considered by F Toppan in a recent paper [7]. This concept allowed him to establish interesting re-interpretations of some relevant results.

The general situation he discusses is the following: the momentum mappings associated to a canonical action of a Lie group G on a classical system (the Noether charges) yield a nontrivial central extension of $\mathrm{Lie}(G)$, the Lie algebra of G.

The aim of this paper is to generalize this kind of ideas in order to encompass some noncanonical actions of Lie Groups on classical systems that give rise to representations of extensions of $\mathrm{Lie}(G)$. These extensions are in general noncentral.

Our approach is based on the introduction and analysis of a two cocycle on $\mathrm{Lie}(G)$ associated to any action of G on a symplectic manifold (M, ω). This cocycle takes values in $C^\infty(M)$, but, if the action of G is symplectic, its evaluation at any point in M yields a real valued two cocycle which is cohomologous to the cocycle defining the central extension mentioned above (see, for instance, [1] or [8]).

The generalization we propose will allow us to consider in our context an example involving the Mickelsson–Fadeev cocycle by means of classical objects. This cocycle is associated to the quantum anomalous commutator of the constraints of the Gauss-law in a $(3 + 1)$-dimensional Yang–Mills theory interacting with Weyl fermions and it was first

computed by L Fadeev by using path-integral techniques [4] (and also by cohomological ones [3]). Almost simultaneously, J Mickelsson constructed a representation of the extension of the current algebra associated to this cocycle by using topological techniques.

In fact, we shall see that the Mickelsson–Fadeev cocycle is cohomologous, modulo constants, to the cocycle associated to the action of the gauge group on the classical system defined by the effective lagrangian.

2   The canonical two-cocycle and the momentum maps

In this section we consider a symplectic manifold (M, ω) and an action of a Lie group G on M which is not assumed to be symplectic. This action induces a natural action of G on the space $C^\infty(M)$:

$$(g \cdot f)(x) = f\left(g^{-1} \cdot x\right), \quad \forall g \in G.$$

The derivative of this action produces a nontrivial action of $\mathrm{Lie}(G)$ on $C^\infty(M)$ by

$$a \cdot f = -L_{\tilde{X}_a} f,$$

with $\tilde{X}_a$ the infinitesimal generator associated to $a \in \mathrm{Lie}(G)$ and L the Lie derivative.

Under such action $C^\infty(M)$ becomes a $\mathrm{Lie}(G)$-module. In general, $\mathrm{Lie}(G)$ acts on all differential forms on M in the same way: $a \cdot \alpha = -L_{\tilde{X}_a} \alpha$.

In order to define the cohomology of $\mathrm{Lie}(G)$ with coefficients in $C^\infty(M)$, the standard coboundary operator is introduced:

$$(\delta\alpha)(a_1, \ldots, a_{n+1}) = \sum_{i=0}^{n+1} (-1)^{(i+1)} a_i \cdot \alpha(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1})$$
$$+ \sum_{i<j} \alpha([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{n+1})$$

where α is one n cochain on $\mathrm{Lie}(G)$ with values in $C^\infty(M)$ (i.e. an alternate multilineal map), $a_1, a_2, \ldots, a_{n+1} \in \mathrm{Lie}(G)$, the symbol '^' meaning that the variable under it has been deleted and the symbol '·' denoting the action of $\mathrm{Lie}(G)$ on $C^\infty(M)$.

The space $Z^n(\mathrm{Lie}(G), C^\infty(M))$ of n-cocycles consists of n cochain α with $\delta\alpha = 0$ and the space $B^n(\mathrm{Lie}(G), C^\infty(M))$ of n-coboundaries consists of n cochain such that exists some $(n-1)$ cochain β with $\alpha = \delta\beta$.

The cohomology groups are defined as $H^n(\mathrm{Lie}(G), C^\infty) = \dfrac{Z^n(\mathrm{Lie}(G), C^\infty(M)}{B^n(\mathrm{Lie}(G), C^\infty(M))}$.

It is well known that the second cohomology group $H^2(\mathrm{Lie}(G), C^\infty(M))$ is related to the extensions of $\mathrm{Lie}(G)$ by $C^\infty(M)$.

The semidirect sum of $\mathrm{Lie}(G)$ and $C^\infty(M)$ consists of pairs $(a, f) \in \mathrm{Lie}(G) \times C^\infty(M)$ with the Lie commutator

$$[(a, f), (b, g)] = ([a, b],\ a \cdot f - b \cdot g).$$

Let $\alpha \in H^2(\mathrm{Lie}(G), C^\infty(M))$. We can try to define a modified commutator by

$$[(a, f), (b, g)]_\alpha = ([a, b],\ a \cdot f - b \cdot g + \alpha(a, b)].$$

(The Jacobi identity for the modifed commutator is easily seen to be equivalent to the cocycle condition $\delta\alpha = 0$.)

So, each $\alpha \in H^2(\mathrm{Lie}(G), C^\infty(M))$ defines a new Lie algebra.

Let α_1 and $\alpha_2 \in Z^2(\mathrm{Lie}(G), C^\infty(M))$. The Lie algebras formed from these cocycles are isomorphic through a mapping of the type $\phi(a, f) = (a, f + \beta(a))$, where $\beta \in H^1(\mathrm{Lie}(G), C^\infty(M))$.

The condition

$$[\phi(a, f), \phi(b, g)]_{\alpha_1} = \phi([(a, f), (b, g)]_{\alpha_2})$$

is the same as $\alpha_1 - \alpha_2 = \delta\beta$ with $\beta \in H^1(\mathrm{Lie}(G), C^\infty(M))$ (i.e. α_1 and α_2 are cohomologous: $\alpha_1 \simeq \alpha_2$).

Thus, up to an isomorphism of the above type the Lie algebra extensions are parametrized by elements of $H^2(\mathrm{Lie}(G), C^\infty(M))$.

Real valued cocycles and coboundaries can be recovered from the previous construction just by considerig real valued cochains as constant functions.

Now, we introduce the $C^\infty(M)$-valued two-cocycle on $\mathrm{Lie}(G)$ canonically associated to the action of G mentioned above.

Proposition 1. *Let* $\Omega(a, b)(x) = \omega(\tilde{X}_a, \tilde{X}_b)(x) \ \forall \ a, b \in \mathrm{Lie}(G)$ *and* $x \in M$. *Then,* Ω *is a* $C^\infty(M)$*-valued two-cocycle on* $\mathrm{Lie}(G)$.

Proof.

$$\begin{aligned}
(\delta\Omega)(a, b, c) ={}& a \cdot \Omega(b, c) - b \cdot \Omega(a, c) + c \cdot \Omega(a, b) \\
& - \Omega([a, b], c) + \Omega([a, c], b) - \Omega([b, c], a) \\
={}& -L_a\omega(\tilde{X}_b, \tilde{X}_c) + L_{\tilde{X}_b}\omega(\tilde{X}_a, \tilde{X}_c) - L_{\tilde{X}_c}\omega(\tilde{X}_a, \tilde{X}_b) \\
& - \omega(\tilde{X}_c, \tilde{X}_{[a,b]}) + \omega(\tilde{X}_b, \tilde{X}_{[a,c]}) - \omega(\tilde{X}_a, \tilde{X}_{[b,c]}) \\
={}& -\tilde{X}_a\omega(\tilde{X}_b, \tilde{X}_c) + \omega([\tilde{X}_a, \tilde{X}_b], \tilde{X}_c) + \omega(\tilde{X}_b, [\tilde{X}_a, \tilde{X}_c]) \\
& + \tilde{X}_b\omega(\tilde{X}_a, \tilde{X}_c) - \omega([\tilde{X}_b, \tilde{X}_a], \tilde{X}_c) - \omega(\tilde{X}_a, [\tilde{X}_b, \tilde{X}_c]) \\
& - \tilde{X}_c\omega(\tilde{X}_a, \tilde{X}_b) + \omega([\tilde{X}_c, \tilde{X}_a], \tilde{X}_b) + \omega(\tilde{X}_a, [\tilde{X}_c, \tilde{X}_b]) \\
& - \omega([\tilde{X}_a, \tilde{X}_b], \tilde{X}_c) + \omega([\tilde{X}_a, \tilde{X}_c], \tilde{X}_b) - \omega([\tilde{X}_b, \tilde{X}_c], \tilde{X}_a) \\
={}& -\tilde{X}_a\omega(\tilde{X}_b, \tilde{X}_c) + \tilde{X}_b\omega(\tilde{X}_a, \tilde{X}_c) - \tilde{X}_c\omega(\tilde{X}_a, \tilde{X}_b) \\
& + \omega([\tilde{X}_a, \tilde{X}_b], \tilde{X}_c) - \omega([\tilde{X}_a, \tilde{X}_c], \tilde{X}_b) + \omega([\tilde{X}_b, \tilde{X}_c], \tilde{X}_a) \\
={}& -3(d\omega)(\tilde{X}_a, \tilde{X}_b, \tilde{X}_c) = 0,
\end{aligned}$$

where d denotes the usual exterior differential operator.   ∎

Definition. The cocycle

$$\Omega(a, b)(x) = \omega(\tilde{X}_a, \tilde{X}_b)(x)$$

will be called the **canonical cocycle** associated to the action of G on (M, ω).

Thus, each action of G on M yields an extension of $\mathrm{Lie}(G)$ by $C^\infty(M)$.

When the action of G is symplectic, $\Omega(a,b)(x)$ and $\Omega(a,b)(x_0)$ are cohomologous as real valued two cocycles on $\mathrm{Lie}(G)$ for every $x, x_0 \in M$. Then, in this case, the element $[\Omega(a,b)(x)]$ in $H^2(\mathrm{Lie}(G), \mathbb{R})$ is independent of x. We shall denote it by $\tilde{\Omega}(a,b)$. In this way, a central extension of $\mathrm{Lie}(G)$ can be defined.

On the other hand, assuming, as henceforth we shall do for the symplectic actions to be considered, that the action of G admits momentum mappings J_a (that is, for each $a \in \mathrm{Lie}(G)$ there exists $J_a \in C^\infty(M)$ such that $dJ_a = i_{\tilde{X}_a}\omega$), $\forall\, a, b \in \mathrm{Lie}(G)$ the function

$$\Sigma(a,b)(x) := \{J_a, J_b\}(x) - J_{[a,b]}(x),$$

with $\{\ ,\ \}$ the Poisson bracket associated to ω, turns out to be constant on M and, as a function of (a,b), it is a real valued two cocycle on $\mathrm{Lie}(G)$ [1, 8]. This cocycle will be denoted by $\tilde{\Sigma}(a,b)$.

Since for any $x_0 \in M$, $J_{[a,b]}(x_0)$ is trivial in $H^2(\mathrm{Lie}(G), \mathbb{R})$, then $\tilde{\Sigma}(a,b)$ is equivalent to $\{J_a, J_b\}(x_0) = \Omega(a,b)(x_0)$ in this cohomology. Hence, as real two cocycles on $\mathrm{Lie}(G)$,

$$\{J_a, J_b\}(x) - J_{[a,b]}(x) \simeq \tilde{\Omega}(a,b).$$

Thus, for symplectic actions, the momenta give rise to a representation of the central extension of $\mathrm{Lie}(G)$ determined by $\tilde{\Omega}(a,b)$.

According to F Toppan [7], an anomaly appears in a classical dynamical system when the central extension associated to a symplectic action of G on (M, ω) is nontrivial. Let us recall that a necessary condition for it to occur is that symplectic potentials are not preserved by the action [1].

Now, for a not necessarily symplectic action of G, we shall show that, under some additional hypotheses, the extension of $\mathrm{Lie}(G)$ defined by Ω can still be represented by the Poisson brackets of some functions associated to the action of G.

More precisely, we shall see that if ω can be written as

$$\omega = \omega_i + \Delta\omega$$

with ω_i G-invariant and $\Delta\omega$ a closed form (not necessarily non-degenerate) on M such that $L_{\tilde{X}_a}\omega = L_{\tilde{X}_a}\Delta\omega$ and if J_a are momentum maps corresponding to the symplectic action of G on (M, ω_i), we have

Proposition 2. *Let ΔX_a denote the vector field $\tilde{X}_a - X_{J_a}$. Under the additional assumption*

$$\omega(\Delta X_a, \Delta X_b) = 0 \quad \forall\, a, b \in \mathrm{Lie}(G)$$

it holds

$$\{J_a, J_b\} - J_{[a,b]} \simeq \Omega(b,a) \quad as \quad C^\infty(M)\text{-valued cocycle on} \quad \mathrm{Lie}(G).$$

Proof.

$$\begin{aligned}
\{J_a, J_b\} = \omega(X_{J_a}, X_{J_b}) &= \omega(\tilde{X}_a, \tilde{X}_b) - \omega(\tilde{X}_a, \Delta X_b) - \omega(\Delta X_a, \tilde{X}_b)\\
&= \omega(\tilde{X}_a, \tilde{X}_b) + \omega(\Delta X_b, \tilde{X}_a) - \omega(\Delta X_a, \tilde{X}_b)\\
&= \omega(\tilde{X}_a, \tilde{X}_b) + \Delta\omega(\tilde{X}_b, \tilde{X}_a) - \Delta\omega(\tilde{X}_a, \tilde{X}_b)\\
&= \omega(\tilde{X}_a, \tilde{X}_b) - 2\Delta\omega(\tilde{X}_a, \tilde{X}_b).
\end{aligned}$$

On the other hand,

$$(\delta J)(a,b) = a \cdot J_b - b \cdot J_a - J_{[a,b]} = -L_{\tilde{X}_a} J_b + L_{\tilde{X}_b} J_a - J_{[a,b]}$$
$$= -dJ_b(\tilde{X}_a) + dJ_a(\tilde{X}_b) - J_{[a,b]}$$
$$= -\omega_i(\tilde{X}_b, \tilde{X}_a) + \omega_i(\tilde{X}_a, \tilde{X}_b) - J_{[a,b]} = 2\omega_i(\tilde{X}_a, \tilde{X}_b) - J_{[a,b]}.$$

So, $J_{[a,b]} = 2\omega_i(\tilde{X}_a, \tilde{X}_b) - (\delta J)(a,b)$. Then, we conclude that:

$$\{J_a, J_b\} - J_{[a,b]} = \omega(\tilde{X}_a, \tilde{X}_b) - 2\Delta\omega(\tilde{X}_a, \tilde{X}_b) - 2\omega_i(\tilde{X}_a, \tilde{X}_b) + (\delta J)(a,b)$$
$$= \Omega(a,b) + (\delta J)(a,b)$$

as we wanted. ∎

Thus, under some additional hypothesis, noncentral extensions associated to nonsymplectic actions can also be represented by physically relevant functions and then considered as anomalies of classical dynamical systems.

Remark 1. The additional hypothesis in the previous proposition is obviously fulfilled by any symplectic action since, in this case, $\Delta X_a = 0 \ \forall \ a \in \mathrm{Lie}(G)$. Then, as a $C^\infty(M)$-valued two cocycle on $\mathrm{Lie}\,(G)$,

$$\Sigma(a,b)(x) := \{J_a, J_b\}(x) - J_{[a,b]}(x) \simeq \Omega(b,a)(x).$$

On the other hand, as mentioned above, $\Sigma(a,b)(x)$ is constant on M and, as real valued cocycle,

$$\tilde{\Sigma}(a,b) = [\Sigma(a,b)(x)] \simeq \Omega(b,a)(x),$$
$$-\tilde{\Sigma}(a,b) = -[\Sigma(a,b)(x)].$$

Nevertheless, the opposite signs do not yield a contradiction. In fact, from the proof of Proposition 2 we have

$$\tilde{\Sigma}(a,b) = \{J_a, J_b\}(x_0) - J_{[a,b]}(x_0) = \Omega(a,b)(x_0) + (\delta J)(a,b)(x_0)$$
$$= \Omega(a,b)(x_0) + 2\omega_i(\tilde{X}_a, \tilde{X}_b)(x_0) - J_{[a,b]}(x_0) = \Omega(a,b)(x_0) - J_{[a,b]}(x_0).$$

To wit,

$$\tilde{\Sigma}(a,b) - \omega_i(\tilde{X}_a, \tilde{X}_b)(x_0) = -J_{[a,b]}(x_0).$$

So, in this case, it follows from the Proposition 2 that, as expected,

$$\tilde{\Sigma}(a,b) \simeq \tilde{\Omega}(a,b)$$

as real valued cocycles.

Remark 2. It is shown in [2] that, under the same hypothesis as in the previous proposition, for any C^∞-valued two cochain $h(a, b)$ on $\mathrm{Lie}(G)$ such that $dh(a, b) = (\delta\alpha)(a, b)$, $\alpha(a)$ being a local symplectic potential of $-L_{\tilde{X}_a}\omega$, the following equality holds:

$$d\{J_a, J_b\} - dJ_{[a,b]} = dh(a, b),$$

and then,

$$\{J_a, J_b\} - J_{[a,b]} = h(a, b) + c(a, b),$$

with $c(a, b)$ a real valued two cocycle on $\mathrm{Lie}(G)$.

Proposition 2 tells us that, by taking $h(a, b) = \Omega(a, b) + (\delta J)(a, b)$, we get rid of $c(a, b)$. It is worth to notice that no explicit expression for $h(a, b)$ nor for $c(a, b)$ is given in [2].

A simple example. Let $G = \mathbb{R}^2$ acting by translations on $Q = \mathbb{R}^2$ and lift the action to $TQ \approx \mathbb{R}^4$.

Let the lagrangian $L : T\mathbb{R}^2 \simeq \mathbb{R}^4 \to \mathbb{R}$ be defined as

$$L(q^1, q^2, \dot{q}^1, \dot{q}^2) = \frac{1}{2}\left((\dot{q}^1)^2 + (\dot{q}^2)^2\right) + (q^1)^2\dot{q}^2.$$

It is clear that L can be written as $L = L^i + \Delta L$ with $L^i = \frac{1}{2}\left((\dot{q}^1)^2 + (\dot{q}^2)^2\right)$ invariant under the action of $\mathbb{R}^2$ and $\Delta L = (q^1)^2\dot{q}^2$.

The Legendre transform is given by

$$FL(q^1, q^2, \dot{q}^1, \dot{q}^2) = \left(q^1, q^2, \dot{q}^1, \dot{q}^1 + (q^1)^2\right).$$

The lagrangian two form $\omega_L = FL^*\left(dq^1 \wedge dp_1 + dq^2 \wedge dp_2\right)$ turns out to be

$$\omega_L = \frac{\partial^2 L}{\partial\dot{q}^1\partial q^2} dq^1 \wedge dq^2 + \frac{\partial^2 L}{\partial\dot{q}^1\partial\dot{q}^2} dq^1 \wedge d\dot{q}^2 + \frac{\partial^2 L}{\partial\dot{q}^2\partial q^1} d\dot{q}^1 \wedge dq^1$$

$$+ \frac{\partial^2 L}{\partial\dot{q}^2\partial\dot{q}^1} dq^2 \wedge d\dot{q}^1 + \frac{\partial^2 L}{\partial\dot{q}^1\partial\dot{q}^1} dq^1 \wedge d\dot{q}^1 + \frac{\partial^2 L}{\partial\dot{q}^2\partial\dot{q}^2} dq^2 \wedge d\dot{q}^2.$$

So,

$$\omega_L = dq^1 \wedge d\dot{q}^1 + dq^2 \wedge d\dot{q}^2 - 2q^1 dq^1 \wedge dq^2.$$

For $a = (a_1, a_2) \in \mathbb{R}^2$, the infinitesimal generator is

$$\tilde{X}_a = (a_1, a_2, 0, 0) = a_1\frac{\partial}{\partial q^1} + a_2\frac{\partial}{\partial q^2}.$$

The momentum mapping $J_a : \mathbb{R}^2 \to \mathbb{R}$ must satisfy

$$dJ_a = i_{\tilde{X}_a}\omega_L^i = i_{\tilde{X}_a}\omega_L^i = a_1 d\dot{q}^1 + a_2 d\dot{q}^2$$

and

$$dJ_a = \frac{\partial J_a}{\partial q^1} dq^1 + \frac{\partial J_a}{\partial q^2} dq^2 + \frac{\partial J_a}{\partial\dot{q}^1} d\dot{q}^1 + \frac{\partial J_a}{\partial\dot{q}^2} d\dot{q}^2.$$

We can take

$$J_a = a_1 \dot{q^1} + a_2 \dot{q^2}.$$

The hamiltonian vector associated to the function J_a by ω_L is

$$X_{J_a} = \tilde{X}_a - 2q^1 a_2 \frac{\partial}{\partial \dot{q^1}} - 2q^1 a_1 \frac{\partial}{\partial \dot{q^2}}.$$

Then,

$$\Delta X_a = -2q^1 \left(a_2 \frac{\partial}{\partial \dot{q^1}} + a_1 \frac{\partial}{\partial \dot{q^2}} \right).$$

Now, it is easy to see that, for this example, it holds

$$\omega_L(\Delta X_a, \Delta X_b) = 0 \quad \forall\, a, b \in \mathrm{Lie}\,(\mathbb{R}^2).$$

For $a = (a_1, a_2)$ and $b = (b_1, b_2) \in \mathbb{R}^2$, the canonical cocycle turns out to be

$$\Omega(b, a) = \omega_L(\tilde{X}_b, \tilde{X}_a) = -2q_1 a_1 b_2 + 2q^1 b_1 a_2.$$

Thus,

$$\Omega((1,0),(0,1)) = -2q^1.$$

On the other hand, a direct computation yields

$$(\delta J)(a, b) = -L_{\tilde{X}_a} \tilde{X}_b + L_{\tilde{X}_b} \tilde{X}_a = -dJ_a(\tilde{X}_b) + dJ_b(\tilde{X}_a) = 0.$$

So,

$$\{J_a, J_b\}_{\omega_L} = \omega_L(X_{J_a}, X_{J_b}) = -2q_1 a_1 b_2 + 2q^1 b_1 a_2.$$

3 An extension involving the Mickelsson–Faddev cocycle

The Mickelsson–Fadeev extension of $\mathrm{Map}\big(S^3, SU(3)\big)$ is a non central one of interest in Quantum Field Theory. It is related to the anomalous commutator of the constraints of the Gauss-law appearing in a $(3+1)$-dimensional Yang–Mills theory interacting with Weyl fermions.

Fifteen years ago, Faddeev [3] computed this anomalous commutator by means of functional integration techniques in the following way:

The lagrangian of the theory is given by

$$\mathcal{L} = tr F^2 + i\bar{\psi}\mathcal{D}_A\psi,$$

where ψ is a Weyl fermion, $A \in \mathcal{A}$, the space of connections on a trivial bundle over the manifold S^4 with structure group $SU(3)$, $F = dA$ is it associated curvture and $\mathcal{D}_A$ is the covariant operator Dirac coupled to A.

The quantization of the theory is carried out in stages. First, the fermionic path integral is computed and then, once the effective lagrangian is obtained, the bosonic integration is performed.

The first step yields a result which is not gauge invariant:

$$W[A] = \int D\psi D\bar{\psi} \exp(i\bar{\psi}\mathcal{D}_A\psi) = \det \mathcal{D}_A.$$

In fact, since $\mathcal{D}_A$ is an unbounded operator, its determinant is divergent, a regularizing method must be applied, and this procedure gives rise to a not gauge invariant result.

Let us notice that the variation of $W[A]$ coincides with the variation of the Wess–Zumino–Witten lagrangian and is the integrated anomaly of the theory:

$$W[A^g] = \exp(i\alpha_1(A;g))W[A],$$

where $g \in \mathcal{G}$, the gauge group, and $\alpha_1(A;g)$ is a 1-cocycle on $\mathcal{G}$ with values in $C^\infty(\mathcal{A})$.

Now, for the second step, the lagrangian that must be considered is the effective one:

$$\mathcal{L}_E(A) = \frac{1}{2} \operatorname{tr} F_{\mu\nu}F^{\mu\nu} + W[A].$$

Now, the constraints of the Gauss-law are just the momenta (i.e. the Noether charges) associated to the action of the gauge group with respect to the original lagrangian.

By using the Johnson–Bjrken–Low method [5], Fadeev obtained the noncentral extension of $\operatorname{Lie}(G)$ represented by the equal-time commutators of the operators $\tilde{\boldsymbol{G}}_a$, the quantum representants of the momenta $\boldsymbol{G}_a$:

$$[\tilde{\boldsymbol{G}}_a(x), \tilde{\boldsymbol{G}}_b(y)][A] = C_{ab}^c \tilde{\boldsymbol{G}}_c(y)\delta(x-y) + S_{(a,b)}[A](x,y),$$

where $S_{(a,b)}[A](x,y) = \delta(x-y) \cdot \operatorname{MF}_{(a,b)}[A]$.

The two cocycle $\operatorname{MF}_{(a,b)}[A] = \dfrac{1}{24\pi} \displaystyle\int_{S^3} d^3x \operatorname{tr}(A[da,db])$ is **the Mickelsson–Faddeev cocycle**.

In order to relate the Mickelsson–Fadeev cocycle to a canonical one, we consider the symplectic manifold $(T\mathcal{A}, \omega_{\mathcal{L}_E})$, where $\omega_{\mathcal{L}_E}$ is the lagrangian form associated to the effective lagrangian $\mathcal{L}_E$.

Notice that $\omega_{\mathcal{L}_E}$ can be written as $\omega_{\mathcal{L}_E} = \omega_{\mathrm{can}} + \Delta\omega_E$, where ω_{can} is the symplectic and gauge-invariant structure associated to $\mathcal{L}_{\mathrm{YM}} = \frac{1}{2} \operatorname{tr} F_{\mu\nu}F^{\mu\nu}$.

Let us consider the momentum map associated to the lift of the action of the gauge group $\mathcal{G}$ on $\mathcal{A}$

$$g \cdot A = A^g = g^{-1}Ag + g^{-1}dg$$

to $(T\mathcal{A}, \omega_{\mathrm{can}})$.

These maps $\boldsymbol{G}_a : T\mathcal{A} \to \mathbb{R}$ are given by $d\boldsymbol{G}_a = i_{\tilde{X}_a}\omega_{\mathrm{can}} \ \forall \ a \in \operatorname{Lie}(\mathcal{G})$ where $\tilde{X}_a$ is the infinitesimal generator associated to $a \in \operatorname{Lie}(\mathcal{G})$.

As mentioned above, these maps are the constraints of the Gauss-law [4].

Let us consider the cocycle $\Omega_{\omega_{\mathcal{L}_E}}$ defined on $\operatorname{Lie}(\mathcal{G})$ with values in the $\operatorname{Lie}(\mathcal{G})$-module $C^\infty(T\mathcal{A})$ canonically associated to the action de $\mathcal{G}$ on $\mathcal{A}$.

The following lemma shows that the additional hypothesis is fulfilled in this case.

Lemma. *If $\Delta X_a = \tilde{X}_a - X_{\boldsymbol{G}_a}$, where $X_{\boldsymbol{G}_a}$ is the Hamiltonian vector field of the momentum map $\boldsymbol{G}_a$ corresponding to $\omega_{\mathcal{L}_E}$,*

$$\omega_{\mathcal{L}_E}(\Delta X_a, \Delta X_b) = 0 \quad \forall\, a, b \in \mathrm{Lie}(\mathcal{G}).$$

Proof. For any form ω as in Proposition 2 we have

$$i_{\Delta X_a}\omega(\cdot) = i_{\tilde{X}_a - X_{\boldsymbol{G}_a}}\omega(\cdot) = i_{\tilde{X}_a}\omega(\cdot) - i_{X_{\boldsymbol{G}_a}}\omega(\cdot) = i_{\tilde{X}_a}\omega(\cdot) - i_{\tilde{X}_a}\omega_{\mathrm{can}}(\cdot) = i_{\tilde{X}_a}\Delta\omega(\cdot).$$

Since in the example we are considering , $i_{\Delta X_a}\Delta\omega_E = 0 \ \forall\, a \in \mathrm{Lie}(\mathcal{G})$ (formula 4.37 in [6]), then

$$\omega_{\mathcal{L}_E}(\Delta X_a, \Delta X_b) = \Delta\omega_E(\tilde{X}_a, \Delta X_b) = 0 \quad \forall\, a, b \in \mathrm{Lie}(\mathcal{G}). \qquad \blacksquare$$

Now, it follows from Proposition 2 that

$$\{J_a, J_b\} - J_{[a,b]} \simeq \Omega(b, a).$$

On the other hand, as shown in [6],

$$\{J_a, J_b\} - J_{[a,b]} \simeq \mathrm{MF}_{(a,b)} + c(a, b).$$

Then, modulo constants, $\mathrm{MF}_{(a,b)}$ turns out to be cohomologous to the canonical cocycle $\Omega(b, a)$.

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