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Fine Structure of the Discrete Transformation for Multicomponent Integrable Systems

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Abstract

It is shown that in the case of multicomponent integrable systems connected with algebras A_n, the discrete transformation T possesses the fine structure and can be represented in the form $T = \prod T_i^{l_i}$, where T_i are n commuting basis discrete transformations and l_i are arbitrary natural numbers. All the calculations are conducted in detail for the case of a 3-wave interacting system.

1 Introduction

In the present paper we consider a 3-wave interacting system in the framework of the discrete transformation theory

$$B_1 = -DE^*, \qquad E_2 = -DB^*, \qquad D_3 = -BE, \tag{1.1}$$

where (D, B, E) are 3 complex-valued unknown functions; and $(x_1 \equiv 1,\ x_2 \equiv 2,\ x_3 \equiv 3)$, three independent arguments of the problem (we assume them to be real). In the case of $(1 + 1)$ space, the operators of differentiation are connected by the additional condition $(\partial_1 + \partial_2 + \partial_3 = 0)$.

The Lax pair for (1.1), in the traditional consideration (by the inverse scattering method), is connected with A_2 algebra [1]. This circumstance distinguishes this case from numerous two-component integrable systems connected with A_1 algebra [2]. Thus, in this example, it is suitable to investigate the structure of the discrete transformation theory for multicomponent integrable systems.

The central idea of the method of discrete substitution consists in that, instead of (1.1), a larger system of equations is considered for 6 unknown functions (D, B, E, Q, P, A)

$$\begin{aligned} &P_1 = -QE, \qquad A_2 = -BQ, \qquad Q_3 = -PA, \\ &B_1 = -AD, \qquad E_2 = -DP, \qquad D_3 = -EB. \end{aligned} \tag{1.2}$$

It is not difficult to see that the additional conditions (of real character)

$$P = B^*, \qquad A = E^*, \qquad Q = D^* \tag{1.3}$$

are self-consistent with (1.3) and are reduced to (1.1).

The goal of the present paper is to show that the discrete transformation of (1.1) can be represented in the form

$$T = T_1^{n_1} T_2^{n_2}, \tag{1.4}$$

where $T_{1,2}$ are two commutative basis transformations, and $n_{1,2}$ are arbitrary natural numbers. For comparison, we mention that the discrete transformation in the case of two-component integrable systems has the form

$$T = T_1^n$$

with the only basis discrete transformation.

To the best of our knowledge, the structure of the discrete transformation for multi-component integrable systems was not yet analyzed.

2 Discrete transformation

The present section presents the following assertion concerning the symmetry properties of the system (1.2).

Assertion 1. *The system (1.2) is invariant with respect to three possible changes of the unknown functions*

T_3

$$\bar{Q} = \frac{1}{D}, \qquad \bar{A} = -\frac{B}{D}, \qquad \bar{P} = \frac{E}{D},$$
$$\bar{B} = D\left(\frac{B}{D}\right)_2, \qquad \bar{E} = -D\left(\frac{E}{D}\right)_1, \qquad \frac{\bar{D}}{D} = DQ - (\ln D)_{1,2};$$

T_1

$$\bar{P} = \frac{1}{B}, \qquad \bar{Q} = \frac{A}{B}, \qquad \bar{E} = -\frac{D}{B},$$
$$\bar{D} = B\left(\frac{D}{B}\right)_2, \qquad \bar{A} = -B\left(\frac{A}{B}\right)_3, \qquad \frac{\bar{B}}{B} = BP - (\ln B)_{2,3};$$

T_2

$$\bar{A} = \frac{1}{E}, \qquad \bar{B} = \frac{D}{E}, \qquad \bar{Q} = -\frac{P}{E},$$
$$\bar{D} = -E\left(\frac{D}{E}\right)_1, \qquad \bar{P} = E\left(\frac{P}{E}\right)_3, \qquad \frac{\bar{E}}{E} = EA - (\ln E)_{1,3}.$$

The validity of this assertion can easily be checked by a direct substitution of the bar quantities into the corresponding equations of motion and by using the equations of motion for nonbar functions.

In the form presented above, the substitutions T_i can be considered as mappings connecting six initial (nonbar) functions with six final (bar) ones. On the other hand, each substitution can be considered as an infinite- dimensional chain of equations. For instance, the corresponding chain of equations in the case of T_1 substitution has the form

$$\frac{B^{n+1}}{B^n} - \frac{B^n}{B^{n-1}} = -\left(\ln B^n\right)_{2,3}, \qquad D^{n+1} = B^n \left(\frac{D^n}{B^n}\right)_2, \qquad A^{n+1} = -B^n \left(\frac{A^n}{B^n}\right)_3,$$
$$E^{n+1} = -\frac{D^n}{B^n}, \qquad Q^{n+1} = \frac{A^n}{B^n}. \tag{2.1}$$

The first line contains a lattice-like system connecting 3 unknown functions (B, D, A) at each point of the lattice. The first chain for B functions is exactly the well-known two-dimensional Toda lattice.

3 Some properties of the discrete transformations

All the discrete transformations constructed above are invertible. This means that a nonbar unknown function can be represented in terms of the bar ones. For instance, T_3^{-1} looks as

$$D = \frac{1}{\bar{Q}}, \qquad B = -\frac{\bar{A}}{\bar{Q}}, \qquad E = \frac{\bar{P}}{\bar{Q}},$$
$$P = -\bar{Q}\left(\frac{\bar{P}}{\bar{Q}}\right)_2, \qquad A = \bar{Q}\left(\frac{\bar{A}}{\bar{Q}}\right)_1, \qquad \frac{Q}{\bar{Q}} = \bar{D}\bar{Q} - (\ln \bar{Q})_{1,2}.$$

It is not difficult to check by direct computation that discrete transformations T_i are commutative ($T_i T_j = T_j T_i$) on the solutions of the system (1.2).

Below, we present the corresponding calculations to prove that $T_1 T_2 = T_2 T_1 = T_3$. Indeed, the result of the action of T_1 on a certain solution of the system (1.2) is the following

$$P^1 = \frac{1}{B}, \qquad Q^1 = \frac{A}{B}, \qquad E^1 = -\frac{D}{B},$$
$$D^1 = B\left(\frac{D}{B}\right)_2, \qquad A^1 = -B\left(\frac{A}{B}\right)_3, \qquad \frac{B^1}{B} = BP - (\ln B)_{2,3}.$$

The action of the T^2 transformation on this solution leads to

$$A^{21} = \frac{1}{E^1} = -\frac{B}{D}, \qquad B^{21} = \frac{D^1}{E^1} = D\left(\frac{B}{D}\right)_2, \qquad Q^{21} = -\frac{P^1}{E^1} = \frac{1}{D},$$
$$D^{21} = -E^1\left(\frac{D^1}{E^1}\right)_1 = -\frac{D}{B}(B(\ln D)_2 - B_2)_1 = QD^2 - D(\ln D)_{12},$$
$$P^{21} = E^1\left(\frac{P^1}{E^1}\right)_3 = \frac{E}{D}, \qquad E^{21} = \left(E^1\right)^2 A^1 - E^1\left(\ln E^1\right)_{13} = -D\left(\frac{E}{D}\right)_1.$$

The same calculation carried out in the back direction shows that $W^{1,2} = W^{2,1} = W^3$ – the result of application of the T_3 transformation to an initial solution W.

Thus, from each given initial solution $W_0 \equiv (A, P, Q, E, B, D)$ of the system (1.2), it is possible to obtain the chain of solutions labeled by two natural numbers ($(l_1, l_2$, or $(l_3))$ – the number of times of applying the discrete transformations (T_1, T_2, T_3) to it (as it was shown above, $T_1T_2 = T_2T_1 = T_3$).

The generated chain of equations in (D, B, E) functions is exactly two-dimensional Toda lattices. Their general solutions in the case of two fixed ends are well-known [3]. As the reader will see below, this fact allows one to construct many soliton solutions of the 3-wave problem in the most straightforward way.

4 Solution of discrete transformation chains

In the present section, we derive an explicit form of the result of application of the general discrete transformation $T_1^{n_1}T_2^{n_2}$ to a specially chosen initial solution of the system (1.2).

In the first subsection, we present some necessary equalities connecting the determinants of the definite form matrices – the so-called two Jacobi equalities. In the second subsection, these equalities are used for solution of the main problem of the present section.

4.1 Two Jacobi identities

We begin with the following obvious equalities for determinants of n-th order

$$\mathrm{Det}_n(T_n) \equiv D_n \begin{pmatrix} T_{n-1} & a \\ b & \tau \end{pmatrix} = D_{n-1}(T_{n-1}) \left(\tau - bT_{n-1}^{-1}a\right) \equiv D_{n-1}(T_{n-1})\tilde{\tau},$$

where T_{n-1} is $(n-1) \times (n-1)$ matrix, a, b are $(n-1)$-dimensional column (row) vectors, respectively, and τ is a scalar.

For the same reason, the following formula takes place

$$D_n \begin{pmatrix} T_{n-2} & a^1 & a^2 \\ b^1 & \tau_{11} & \tau_{12} \\ b^2 & \tau_{21} & \tau_{22} \end{pmatrix} = D_{n-2}(T_{n-2}) D_2 \begin{pmatrix} \tau_{11} - b^1T_{n-2}^{-1}a^1 & \tau_{12} - b^1T_{n-2}^{-1}a^2 \\ \tau_{21} - b^2T_{n-2}^{-1}a^1 & \tau_{11} - b^2T_{n-2}^{-1}a^2 \end{pmatrix},$$

where a^i, b^i are $(n-2)$-dimensional column (row) vectors; and $\tau_{i,j}$ components of a 2-dimensional matrix. It is evident how relations of these types can be continued.

Now using the above results, we transform the following expression

$$\begin{aligned} & D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^1 & \tau_{11} \end{pmatrix} D_n \begin{pmatrix} T_{n-1} & a^2 \\ b^2 & \tau_{22} \end{pmatrix} - D_n \begin{pmatrix} T_{n-1} & a^2 \\ b^1 & \tau_{12} \end{pmatrix} D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^2 & \tau_{21} \end{pmatrix} \\ & \quad = D_{n-1}^2(T_{n-1}) D_2 \begin{pmatrix} \tau_{11} - b^1T_{n-1}^{-1}a^1 & \tau_{12} - b^1T_{n-1}^{-1}a^2 \\ \tau_{21} - b^2T_{n-1}^{-1}a^1 & \tau_{11} - b^2T_{n-1}^{-1}a^2 \end{pmatrix} \\ & \quad = D_{n-1}D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ b^1 & \tau_{11} & \tau_{12} \\ b^2 & \tau_{21} & \tau_{22} \end{pmatrix}. \end{aligned}$$

We will treat the last equality as the first Jacobi identity. By the same technique, it is not difficult to show that the following equality holds valid:

$$D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^1 & \tau \end{pmatrix} D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ d^1 & \nu & \mu \\ b^2 & \rho & \tau \end{pmatrix}$$
$$- D_n \begin{pmatrix} T_{n-1} & a^1 \\ b^2 & \rho \end{pmatrix} D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ d^1 & \nu & \mu \\ b^1 & \tau & \sigma \end{pmatrix}$$
$$= D_n \begin{pmatrix} T_{n-1} & a^1 \\ d^1 & \nu \end{pmatrix} D_{n+1} \begin{pmatrix} T_{n-1} & a^1 & a^2 \\ b^2 & \rho & \tau \\ b^1 & \tau & \sigma \end{pmatrix}$$

that is named the second Jacobi identity. These identities can be generalized to the case of an arbitrary semisimple group. The reader can find the corresponding results in [4].

4.2 Direct calculations of the discrete transformation chains

We take an initial solution in the form

$$Q = A = P = 0, \qquad B \equiv B(2), \qquad E \equiv E(1), \qquad D_3 = -BE. \tag{4.1}$$

Application of each of the inverse transformations T_i^{-1} to this solution is meaningless because of zeroes in denominators (see Section 2). The chain of equations under this boundary condition will be called as the chain with a fixed end from the left (from one side).

The result of application of l_3 times T_3 transformation to such an initial solution looks as (to check this fact, only two Jacobi identities of the previous subsection are necessary):

$$\begin{aligned} &Q^{(l_3} = (-1)^{l_3-1}\frac{\Delta_{l_3-1}}{\Delta_{l_3}}, \qquad D^{(l_3} = (-1)^{l_3}\frac{\Delta_{l_3+1}}{\Delta_{l_3}}, \qquad \Delta_0 = 1, \\ &A^{(l_3} = (-1)^{l_3}\frac{\Delta^B_{l_3}}{\Delta_{l_3}}, \qquad P^{(l_3} = \frac{\Delta^E_{l_3}}{\Delta_{l_3}}, \qquad \Delta^B_0 = \Delta^E_0 = 0, \\ &B^{(l_3} = \frac{\Delta^B_{l_3+1}}{\Delta_{l_3}}, \qquad E^{(l_3} = (-1)^{l_3}\frac{\Delta^E_{l_3+1}}{\Delta_{l_3}}, \qquad \Delta_{-1} = 0. \end{aligned} \tag{4.2}$$

where Δ_n are minors of the nth order of an infinite-dimensional matrix

$$\Delta = \begin{pmatrix} D & D_2 & D_{22} & \cdots \\ D_1 & D_{12} & D_{122} & \cdots \\ D_{11} & D_{112} & D_{1122} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{4.3}$$

and $\Delta^E_{l_3}$, $\Delta^B_{l_3}$ are the minors of l_3 order in the matrices of which the last column (or row) is changed to the derivatives of the corresponding order with respect to argument 1 of the E function (with respect to argument 2 of the B function in the second case).

In what follows, we will use the notation: W^{l_3,l_1} and (W^{l_3,l_2}) are the result of application of the discrete transformation $T^{l_3}T^{l_1}$ $(T^{l_3}T^{l_2})$ to the corresponding component of the 3-wave field; Δ^{l_3,l_1} (Δ^{l_3,l_2}) is the determinant of l_3+l_1 (l_3+l_2) orders, with the following structure of its determinant matrix: The first l_3 rows (columns) of it coincide with the matrix of (4.3) and the last l_1, (l_2) rows (columns) constructed from the derivatives of B, (E) functions with respect to arguments 2 (1).

The result of further application of l_1 times T_1 transformation to the solution (4.2) looks as

$$
\begin{aligned}
&P^{(l_3,l_1} = \frac{\Delta_{l_3,l_1-1}}{\Delta_{l_3,l_1}}, \qquad B^{(l_3,l_1} = \frac{\Delta_{l_3,l_1+1}}{\Delta_{l_3,l_1}}, \qquad \Delta_0 = 1, \qquad \Delta^{l_3,-1} \equiv \Delta^E_{l_3}, \\
&Q^{(l_3,l_1} = (-1)^{l_3+l_1-1}\frac{\Delta_{l_3-1,l_1}}{\Delta_{l_3,l_1}}, \qquad D^{(l_3,l_1} = (-1)^{l_3+l_1}\frac{\Delta_{l_3+1,l_1}}{\Delta_{l_3,l_1}}, \\
&E^{(l_3,l_1} = (-1)^{l_3+l_1}\frac{\Delta_{l_3+1,l_1-1}}{\Delta_{l_3,l_1}}, \qquad A^{l_3,l_1} = (-1)^{l_3+l_1}\frac{\Delta_{l_3-1,l_1+1}}{\Delta_{l_3,l_1}},
\end{aligned}
\tag{4.4}
$$

We do not present the explicit form for components $W^{(l_3,l_2}$ that can be obtained without any difficulties from (4.4) by the corresponding change of the arguments and initial functions B and E.

5 Multisoliton solution of the scalar 3-wave problem

As is mentioned in the Introduction, the system (1.2) allows the following reduction (under a further assumption that all operators of differentiation are the real ones $\partial_\alpha = \partial^*_\alpha$), given above by (1.3) $(P = B^*, \quad A = E^*, \quad Q = D^*)$. And in this case, the system (1.2) is reduced to the three equations; $B_1 = -DE^*$, $E_2 = -DB^*$, $D_3 = -BE$, given above by (1.1), for three complex-valued unknown functions (E, B, D).

Now we would like to demonstrate how the multisoliton solutions of the system (1.1) can be obtained with the help of the technique of discrete transformation in the most straightforward way.

With this aim, we consider the action of the direct and inverse T_i, T_i^{-1} transformations on the reduced solution of the system (1.1). The trick consists in that the discrete transformation does not preserve the condition of realness (1.3), and starting with the solution of the reduced system, we come back to the solution of the nonreduced one, and only in some special cases, visa versa. We denote the three-dimensional vector (Q, P, A) by the single symbol $\vec{Q}$ and the three-dimensional vector (D, B, E) by the symbol $\vec{D}$. Then the result of action of the direct and inverse transformations on the solution satisfying the condition of realness $\vec{Q} = \vec{D}^*$ is the following:

$$
T_i^n(\vec{D}, \vec{D}^*) = (t_i)^n(\vec{q}, \vec{d}), \qquad T_i^{-n}(\vec{D}, \vec{D}^*) = (\vec{d}^*, \vec{q}^*)
$$

where t_i are point-like symmetries of the system (1.2)

$$
\begin{aligned}
&t_3(Q, P, A, D, B, E) = (Q, -P, -A, D, -B, -E), \\
&t_2(Q, P, A, D, B, E) = (-Q, -P, A, -D, -B, E), \\
&t_3(Q, P, A, D, B, E) = (-Q, P, -A, -D, B, -E).
\end{aligned}
$$

It is obvious that $t_i^2 = 1$. Thus, if we apply the discrete transformation $2n$ times to the initial bad (nonreduced) solution $(0, \vec{D})$ and, as a result, obtain $(t^n \vec{D}^*, 0)$, then in the middle of the chain, we will have a solution satisfying the condition of realness that coincides with an n-soliton solution of the reduced system (1.1).

The solution of the chain with the boundary conditions $\vec{Q} = 0$ at the left end of the chain and $\vec{D} = 0$ at the right end will be the chain with the both fixed ends. Really, the condition $\vec{D} = 0$ is the system of equations, from which the initial functions D, B, E (see (1.1) can be determined as the solutions of ordinary differential equations.

6 Matrix three-wave problem in the space of three dimensions and its discrete transformation

In this section, we will consider an integrable system unknown until now in the three-dimensional space for 6 unknown functions taking values in some semisimple algebra. The $L - A$ formalism is not applicable to a system of that sort, but the formalism of discrete transformations works well. We emphasize that this system is the local one in all three independent arguments.

In all the above calculations, we have not used (except for the detailed resolving of discrete transformation chains) the condition that operators of differentiation are connected by the expression

$$\partial_1 + \partial_2 + \partial_3 = 0.$$

(Provided we do not want to derive explicit multisoliton solutions in the $(1 + 1)$ space of the real physical 3-wave problem.)

So, we can consider the system (1.2) where all three operators are independent of each other and correspond to differentiation with respect to one of coordinates of the three-dimensional space. The second generalization consists in the possibility to treat the unknown function in (1.2) as the operator-valued one. Of course, in this case, the order of the multipliers is essential and should exactly coincide with the one fixed in the system (1.2).

In this case, the following assertion takes place:

Assertion 2. *The system (1.2) with operator-valued unknown functions* (Q, A, P, D, E, B) *is invariant under the following three transformations of the unknown functions*

T_3

$$\bar{Q} = D^{-1}, \qquad \bar{A} = -BD^{-1}, \qquad \bar{P} = D^{-1}E,$$
$$\bar{B} = \left(BD^{-1}\right)_2 D, \qquad \bar{E} = -D\left(D^{-1}E\right)_1, \qquad D^{-1}\bar{D} = QD - \left(D^{-1}D_2\right)_1;$$

T_1

$$\bar{P} = B^{-1}, \qquad \bar{Q} = B^{-1}A, \qquad \bar{E} = -DB^{-1},$$
$$\bar{D} = \left(DB^{-1}\right)_2 B, \qquad \bar{A} = -B\left(B^{-1}A\right)_3, \qquad \bar{B}B^{-1} = BP - \left(B_3B^{-1}\right)_2;$$

T_2

$$\bar{A} = E^{-1}, \qquad \bar{B} = E^{-1}D, \qquad \bar{Q} = -PE^{-1},$$
$$\bar{D} = -E\left(E^{-1}D\right)_1, \qquad \bar{P} = \left(PE^{-1}\right)_3 E, \qquad E^{-1}\bar{E} = AE - \left(E^{-1}E_3\right)_1.$$

The validity of this assertion can be verified by a direct rather simple calculation, like in Section 2.

As in the scalar case, these discrete transformations of the functions under consideration are commutative. The arising chains of equations for (E, B, D) operator-valued functions (the matrices of finite dimensions, for instance) coincide with the matrix Toda chains investigated above. Explicit solutions to these chains of equations with the fixed ends can be found in [5]. Combining these results, it is possible to construct multisoliton solutions of the matrix 3-wave problem in three dimensions in a way similar to that proposed in [6] for construction of multisoliton solutions to the matrix Devay–Stewartson equation.

7 Outlook

The concrete results of the present paper consist in deriving explicit formulae of discrete transformations for the 3-wave problem (Section 2) and their generalization to the matrix case (Section 6).

However, it is not less important to understand how the method of the discrete transformations can be generalized to the case of multicomponent systems connected with the semisimple algebras of higher ranks r. From results of the present paper it is clear that in the case of an arbitrary semisimple algebra, there are r independent basis commuting discrete transformations. How these commutative objects are connected with the main ingredients of the representation theory of groups is a very interesting and intriguing question for further investigation.

And the last comment: The chain with two fixed ends cannot be considered as the basis for some finite-dimensional representation of the group of the discrete transformations, if at all, it is possible to use the term "group" in this case. On the basis function at the end point of the chain on the right, it is impossible to act by a direct discrete transformation and by an inverse transformation at the left end. What is the discrete transformation from the group-theoretical point of view in this case? At present, we have no answer to this question.

Acknowledgments

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