Superconformal Algebras
and Lie Superalgebras of the Hodge Theory

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Abstract
We observe a correspondence between the zero modes of superconformal algebras $S'(2,1)$ and $W(4)$ ([8]) and the Lie superalgebras formed by classical operators appearing in the Kähler and hyper-Kähler geometry.

1 Lie superalgebras of the Hodge theory

1.1 Kähler manifolds

Let $M = (M^{2n}, g, I, \omega)$ be a compact Kähler manifold of real dimension $2n$, where $g$ is a Riemannian (Kähler) metric, $I$ is a complex structure on $M$, and $\omega$ is the corresponding closed 2-form defined by $\omega(x, y) = g(x, I(y))$ for any vector fields $x$ and $y$ [10].

A number of classical operators on the Dolbeault algebra $A^*(M)$ is well-known [5]: the exterior differential $d$ and its holomorphic and antiholomorphic parts, $\partial$ and $\bar{\partial}$, and $d_c = i(\partial - \bar{\partial})$, their dual operators, and the associated Laplacians. Recall that $\partial : A^{p,q}(M) \rightarrow A^{p+1,q}(M)$, $\bar{\partial} : A^{p,q}(M) \rightarrow A^{p,q+1}(M)$, $d = \partial + \bar{\partial}$. (1.1)

The Hodge operator $\star : A^{p,q}(M) \rightarrow A^{n-q,n-p}(M)$, satisfies $\star^2 = (-1)^{p+q}$ on $A^{p,q}(M)$. Accordingly, the Hodge inner product is defined on each of $A^{p,q}(M)$: $(\varphi, \psi) = \int_M \varphi \wedge \star \psi$. Recall that $\Delta = dd^* + d^*d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$. In addition, $A^{*,*}(M)$ admits an $\mathfrak{s}(2)$-module structure, where $\mathfrak{s}(2) = \langle E, H, F \rangle$ and the generators satisfy

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F. \quad (1.2)$$

The operator $E : A^{p,q}(M) \rightarrow A^{p+1,q+1}(M)$ is defined by $E(\varphi) = \varphi \wedge \omega$. (Clearly, $\omega$ is a $(1,1)$-form). Let $F = E^* : A^{p,q}(M) \rightarrow A^{p-1,q-1}(M)$ be its dual operator, and $H|_{A^{p,q}(M)} = p + q - n$. According to the Lefschetz theorem, there exists the corresponding action of $\mathfrak{s}(2)$ on $H^*(M)$ [5]. These operators satisfy a series of identities, known as the Hodge identities [5]. Let $\mathcal{K}$ be the Lie superalgebra, whose even part is spanned by $\mathfrak{s}(2) = \langle E, H, F \rangle$ and the Laplace operator $\Delta$, and the odd part is spanned by the
differentials $d$, $d^*$, $d_c$, $d_c^*$. The non-vanishing commutation relations in $\mathcal{K}$ are (1.2) and the following relations (see [5]):

$$[d, d^*] = [d_c, d_c^*] = \Delta, \quad [H, d] = d, \quad [H, d^*] = -d^*, \quad [H, d_c] = d_c, \quad [H, d_c^*] = -d_c^*, \quad [E, d^*] = -d_c, \quad [E, d_c^*] = d_c, \quad [F, d] = d^*, \quad [F, d_c] = -d^*. \quad (1.3)$$

Thus $\mathcal{K} = \mathfrak{sl}(2) \oplus \mathfrak{hei}(0|4)$, where $\mathfrak{hei}(0|4)$ is the Heisenberg Lie superalgebra: $\mathfrak{hei}(0|4)_1 = \langle d, d^* \rangle \oplus \langle d_c, d_c^* \rangle$ is a direct sum of two isotropic subspaces with respect to the non-degenerate symmetric form given by $(d, d^*) = (d_c, d_c^*) = 1$, and $\mathfrak{hei}(0|4)_0 = \langle \Delta \rangle$ is the center. The isotropic subspaces are standard $\mathfrak{sl}(2)$-modules. Since $\mathfrak{sl}(2) \simeq \mathfrak{sp}(2)$, the following is a natural generalization.

### 1.2 Hyper-Kähler manifolds

Let $M$ be a compact hyper-Kähler manifold. By definition, $M$ is a Riemannian manifold endowed with three complex structures $I$, $J$, and $K$, such that $I \circ J = -J \circ I = K$ and $M$ is Kähler with respect to each of the complex structures $I$, $J$ and $K$. Let $\omega_I$, $\omega_J$ and $\omega_K$ be the corresponding closed 2-forms on $M$. Having fixed one of the complex structures, for example, $I$, we obtain the Hodge theory as in the Kähler case.

For each of the complex structures $I$, $J$ and $K$, the operators $E_i$, $E_j$ and $E_k$ and their dual operators $F_i$, $F_j$ and $F_k$ with respect to the Hodge inner product are naturally defined. One can also define differentials, $d_c$ and $(d_c)^*$, where $l = i, j, k$ for $I$, $J$ and $K$. Set

$$d_c^l = d_c^* = d_c^* \quad (1.4)$$

Let $Q = \{1, i, j, k\}$ be the set of indices satisfying the quaternionic identities.

In the hyper-Kähler case there is a natural action of the Lie algebra $\mathfrak{so}(5) \simeq \mathfrak{sp}(4)$ on $H^*(M)$ [13]. The $\mathfrak{sp}(4)$ is spanned by the operators $E_i, F_i, K_i$, where $i$ runs through the set $Q \setminus \{1\}$, and $H$. The non-vanishing commutation relations are

$$[E_i, F_j] = H, \quad [H, E_i] = 2E_i, \quad [H, F_i] = -2F_i, \quad [E_i, F_j] = K_{ij}, \quad [K_i, K_j] = -2K_{ij}, \quad K_{ij} = -K_{ji}, \quad [K_i, E_j] = -2E_{ij}, \quad E_{ij} = -E_{ji}, \quad [K_i, F_j] = -2F_{ij}, \quad F_{ij} = -F_{ji}, \quad i \neq j. \quad (1.5)$$

Let $\mathcal{H}$ be a Lie superalgebra, whose even part is spanned by the $\mathfrak{sp}(4)$ and the Laplace operator $\Delta$, and the odd part is spanned by the differentials $d_c^l$ and $(d_l^c)^*$, where $l \in Q$. Thus $\dim \mathcal{H} = (11|8)$. The non-vanishing commutation relations in $\mathcal{H}$ are (1.5) and the following relations (cf. [2, 14, 15]):

$$[d_c^l, (d_c^l)^*] = \Delta, \quad [H, d_c^l] = d_c^l, \quad [H, (d_c^l)^*] = -(d_c^l)^*, \quad [E_i, (d_c^l)^*] = -d_{c^l}^i, \quad d_{c^l}^i = -d_{c^l}^i, \quad [F_i, d_c^l] = (d_{c^l}^i)^*, \quad (d_{c^l}^i)^* = -(d_{c^l}^i)^*, \quad [K_i, d_c^l] = -d_{c^l}^{il}, \quad [K_i, (d_c^l)^*] = -(d_{c^l}^{il})^*. \quad (1.6)$$

Thus $\mathcal{H} = \mathfrak{sp}(4) \oplus \mathfrak{hei}(0|8)$, where $\mathfrak{hei}(0|8)$ is the Heisenberg Lie superalgebra: $\mathfrak{hei}(0|8)_1 = \langle d_c^l, (d_c^l)^* \rangle \ l \in Q$, $\mathfrak{hei}(0|8)_0 = \langle \Delta \rangle$, where $\langle \Delta \rangle$ is the center. $\mathfrak{hei}(0|8)_1 = V_1 \oplus V_2$ is
a direct sum of two isotropic subspaces with respect to the non-degenerate symmetric form: $(d_i^a, (d_i^b)^*) = \delta_{ab}$ for $a, b \in Q$;

\[
V_1 = \langle d_c^1 + \sqrt{-2}d_c^2 - d_c^3, d_c^1 - \sqrt{-2}d_c^2 + d_c^3, (d_c^1)^* + (d_c^2)^* + \sqrt{-2}(d_c^3)^*, \sqrt{-2}(d_c^2)^* + (d_c^3)^* - (d_c^1)^* \rangle,
\]

\[
V_2 = \langle d_c^1 - \sqrt{-2}d_c^2 - d_c^3, d_c^1 - \sqrt{-2}d_c^2 + d_c^3, (d_c^1)^* - (d_c^2)^* - \sqrt{-2}(d_c^3)^*, \sqrt{-2}(d_c^2)^* + (d_c^3)^* + (d_c^1)^* \rangle.
\]

The subspaces $V_1$ and $V_2$ are irreducible $\mathfrak{sp}(4)$-modules.

## 2 Superconformal algebras

A **superconformal algebra** (SCA [8, 9]) is a complex $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_i \mathfrak{g}_i$, such that $\mathfrak{g}$ is simple. $\mathfrak{g}$ contains the centerless Virasoro algebra, i.e. the Witt algebra, $L = \oplus_{n \in \mathbb{Z}} \mathbb{C} L_n$ with the commutation relations $[L_m, L_n] = (m - n)L_{m+n}$ as a subalgebra, and $adL_0$ is diagonalizable with finite-dimensional eigenspaces: $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [L_0, x] = ix\}$, so that $\dim \mathfrak{g}_i < C$, where $C$ is a constant independent of $i$. (Other definitions of superconformal algebras, embracing central extensions, are also popular, see [4]; for an intrinsic definition see [6]).

In general, a SCA is spanned by a number of fields; the Virasoro field is among them. The basic example of a SCA is $W(N)$. Let $\Lambda(N)$ be the Grassmann algebra in $N$ variables $\theta_1, \ldots, \theta_N$. Let $\Lambda(1, N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$ be a supercommutative superalgebra with natural multiplication and with the following parity of generators: $p(t) = 0$, $p(\theta_i) = 1$ for $i = 1, \ldots, N$. By definition, $W(N)$ is the Lie superalgebra of all derivations of $\Lambda(1, N)$. Let $\partial_t$ stand for $\frac{\partial}{\partial t}$ and $\partial_i$ stand for $\frac{\partial}{\partial \theta_i}$.

The superalgebra $W(N)$ contains a one-parameter family of SCAs $S'(N, \alpha)$. By definition,

\[
S(N, \alpha) = \{D \in W(N) \mid \text{Div}(t^\alpha D) = 0\} \quad \text{for } \alpha \in \mathbb{C},
\]

where $\text{Div} \left( f \partial_t + \sum_{i=1}^N f_i \partial_i \right) = \partial_t f + \sum_{i=1}^N (-1)^{p(f_i)} \partial_i f_i$ for $f, f_i \in \Lambda(1, N)$. The derived superalgebra $S'(N, \alpha) = [S(N, \alpha), S(N, \alpha)]$ is simple.

Let $\mathfrak{g} = S'(2, 1)$ or $W(4)$, respectively. Let

\[
L_n = -t^{n+1} \partial_t - \frac{1}{2}(n + 2)t^n \sum_{i=1}^N \theta_i \partial_i,
\]

and let $\mathfrak{g}_0$ be singled out by $L_0$. There exist an isomorphisms $\varphi : K \rightarrow S'(2, 1)_0$, and a monomorphism $\psi : H \rightarrow W(4)_0$ in the case of a compact Kähler or hyper-Kähler manifold, respectively.

### 2.1 Kähler manifolds

Let $N = 2$. $S'(2, 1)$ is spanned by 4 bosonic fields $L_n, H_n, E_n, F_n$, where $E_n, F_n$ and $H_n$ form the loop algebra of $\mathfrak{sl}(2)$, and 4 fermionic fields $X_n^i, Y_n^i, i = 1, 2$: $H_n = t^n(\theta_1 \partial_1 - \theta_2 \partial_2)$, $E_n = t^n \theta_1 \partial_2$, $F_n = t^n \theta_2 \partial_1$. 

\[ X_n^1 = t^n \theta_1 \partial_1 + (n + 1)t^{n-1} \theta_1 \theta_2 \partial_2, \quad X_n^2 = -t^{n+1} \partial_2, \]
\[ Y_n^1 = -t^{n+1} \partial_1, \quad Y_n^2 = t^n \theta_2 \partial_t + (n + 1)t^{n-1} \theta_1 \partial_1. \] (2.3)

The commutation relations between \( L_n \) and the fields, defined by (2.3), are
\[
[L_n, H_m] = -m H_{n+m}, \quad [L_n, E_m] = -m E_{n+m}, \quad [L_n, F_m] = -m F_{n+m},
\] (2.4)
\[
[L_n, X_m^i] = \left( \frac{n}{2} - m \right) X_{n+m}^i, \quad [L_n, Y_m^i] = \left( \frac{n}{2} - m \right) Y_{n+m}^i, \quad i = 1, 2. \] (2.5)

Clearly, the fields \( X_n^i \) and \( Y_n^i \), where \( i = 1, 2 \), generate \( S'(2, 1) \).

The Lie superalgebra \( S'(2, 1)_0 \) is isomorphic to the Lie superalgebra \( \mathcal{K} \) of classical operators in Kähler geometry. The isomorphism \( \varphi \) is as follows:
\[
\varphi(\Delta) = L_0, \quad \varphi(H) = H_0, \quad \varphi(E) = E_0, \quad \varphi(F) = F_0,
\]
\[
\varphi(d) = X_0^1, \quad \varphi(d^*) = Y_0^1, \quad \varphi(d_c) = X_0^2, \quad \varphi(d_c^*) = Y_0^2. \] (2.6)

### 2.2 Hyper-Kähler manifolds

Let \( N = 4 \). The following 10 bosonic fields span a subalgebra of \( W(4) \) isomorphic to the loop algebra of \( \mathfrak{sp}(4) \):
\[
H_n = t^n (\theta_1 \partial_1 + \theta_2 \partial_2 - \theta_3 \partial_3 - \theta_4 \partial_4), \quad E_n^i = t^n (\theta_1 \partial_4 + \theta_2 \partial_3),
\]
\[
F_n^i = t^n (\theta_3 \partial_2 + \theta_4 \partial_1), \quad E_n^j = it^n (\theta_1 \partial_3 + \theta_2 \partial_4), \quad F_n^j = -it^n (\theta_3 \partial_1 + \theta_4 \partial_2),
\]
\[
E_n^k = t^n (\theta_1 \partial_3 - \theta_2 \partial_4), \quad F_n^k = t^n (\theta_3 \partial_1 - \theta_4 \partial_2),
\]
\[
K_n^i = it^n (\theta_1 \partial_1 - \theta_2 \partial_3), \quad K_n^j = t^n (\theta_1 \partial_2 - \theta_2 \partial_1 + \theta_3 \partial_4 - \theta_4 \partial_3),
\]
\[
K_n^k = -it^n (\theta_1 \partial_2 + \theta_2 \partial_1 - \theta_3 \partial_4 + \theta_4 \partial_3). \] (2.7)

Define 8 fermionic fields \( X_n^l, Y_n^l \), where \( l \in Q \). Let
\[
A_n^m = t^n \theta_m \partial_t + (n + 1)t^{n-1} \theta_m \sum_{i=1}^4 \theta_i \partial_i, \quad m = 1, \ldots, 4. \] (2.8)

Let \( \mathbf{X} = (x_{lm}) \) and \( \mathbf{Y} = (y_{lm}) \) be the following complex \( 4 \times 4 \) matrices, where \( l = 1, i, j, k \) and \( m = 1, \ldots, 4 \):
\[
\mathbf{X} = \begin{pmatrix} 1 & -1 & i & i \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & -1 & -1 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} -1 & 1 & i & i \\ -i & -i & 1 & 1 \\ -1 & -1 & -i & i \\ -i & i & 1 & 1 \end{pmatrix}. \] (2.9)

Set
\[
X_n^l = \frac{1}{2} \sum_{m=1}^2 x_{lm} A_n^m + \frac{1}{2} \sum_{m=3}^4 x_{lm} t^{n+1} \partial_m, \]
\[
Y_n^l = \frac{1}{2} \sum_{m=1}^2 y_{lm} t^{n+1} \partial_m + \frac{1}{2} \sum_{m=3}^4 y_{lm} A_n^m. \] (2.10)
The commutation relations between \( L_n \) and the fields, defined by (2.7) and (2.10), are analogs of the relations (2.4) and (2.5), respectively.

The zero modes of the Virasoro field \( L_n \) and of the fields, defined by (2.7) and (2.10), span a Lie superalgebra, which is isomorphic to the Lie superalgebra \( \mathcal{H} \) of classical operators in hyper-Kähler geometry.

The monomorphism \( \psi \) is given by the following formulas:

\[
\begin{align*}
\psi(\Delta) &= L_0, & \psi(H) &= H_0, & \psi(E_i) &= E_i^0, & \psi(F_i) &= F_i^0, \\
\psi(K_i) &= K_i^0, & \psi(d_i^l) &= X^i_l, & \psi(d_i^r) &= Y^i_l,
\end{align*}
\]

where \( i \) runs through the set \( Q \setminus \{1\} \), and \( l \in Q \).

**Statement.** The fields \( X^i_l \) and \( Y^i_l \), for all \( l \in Q \), generate \( W(4) \).

### 3 Conclusion

It is natural to expect that “affinization” of the classical operators in the case of an infinite-dimensional manifold gives a SCA, which should act on a relevant cohomology complex. Recall that the Weil complex is used for definition of the equivariant differential forms. The infinite-dimensional generalization of the classical Weil complex is the semi-infinite Weil complex of a graded Lie algebra \([1]\).

In particular, let \( \tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \) be the loop algebra of a complex finite-dimensional Lie algebra \( \mathfrak{g} \). Naturally, \( \tilde{\mathfrak{g}} = \oplus_{n \in \mathbb{Z}} \tilde{\mathfrak{g}}_n \). Let \( \tilde{\mathfrak{g}}' = \oplus_{n \in \mathbb{Z}} \tilde{\mathfrak{g}}'_n \) be the restricted dual of \( \tilde{\mathfrak{g}} \). The linear space \( U = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}' \) can be naturally endowed with non-degenerate skew-symmetric and symmetric bilinear forms: \( \langle \cdot, \cdot \rangle \) and \( \{\cdot, \cdot\} \). The Weyl and Clifford algebras, \( W(\tilde{\mathfrak{g}}) \) and \( C(\tilde{\mathfrak{g}}) \), are the quotients of the tensor algebra \( T^*(U) \) modulo the two-sided ideals generated by the elements of the form \( xy - yx - \{x, y\} \) and \( xy + yx - \{x, y\} \) for any \( x, y \in U \), respectively; here \( xy := x \otimes y \). Let \( u \) run through a fixed basis of \( \tilde{\mathfrak{g}} \) and \( u' \) run through the dual basis. Let \( \beta(u_m), \gamma(u'_m) \) and \( \tau(u_m), \varepsilon(u'_m) \), where \( m \in \mathbb{Z} \), be generators of \( W(\tilde{\mathfrak{g}}) \) and \( C(\tilde{\mathfrak{g}}) \), respectively. We can realize \( S'(2, 1) \) in terms of the following quadratic expansions:

\[
\begin{align*}
L_n &= \sum_{u,m} m : \tau(u_{m-n}) \varepsilon(u'_m) : + m : \beta(u_{m-n}) \gamma(u'_m) : - \frac{n}{2} : \beta(u_m) \gamma(u'_{m+n}) : , \\
H_n &= - \sum_{u,m} : \beta(u_m) \gamma(u'_m) + : , & E_n &= - \frac{i}{2} \sum_{u,m} \gamma(u'_m) \gamma(u'_{n-m}) , , \\
F_n &= - \frac{i}{2} \sum_{u,m} \beta(u_m) \beta(u_{m-n}) , , & X^1_n &= \sum_{u,m} \gamma(u'_m) \tau(u_m) , , \\
X^2_n &= i \sum_{u,m} m \gamma(u'_{m-n}) \varepsilon(u'_m) , , & Y^1_n &= \sum_{u,m} m \beta(u_{m-n}) \varepsilon(u'_m) , , \\
Y^2_n &= i \sum_{u,m} \beta(u_m) \tau(u_{m-n}) , ,
\end{align*}
\]

where the double colons \( : \) denote a normal ordering operation:

\[
: \tau(u_j) \varepsilon(v'_i) := \begin{cases} 
\tau(u_j) \varepsilon(v'_i) & \text{if } i \leq 0, \\
- \varepsilon(v'_i) \tau(u_j) & \text{if } i > 0
\end{cases}
\]

with the similar formula for \( \beta \) and \( \gamma \), but without the minus sign.
The semi-infinite Weil complex of \( \tilde{\mathfrak{g}} \) is

\[
\left\{ S_{\infty}^+ (\tilde{\mathfrak{g}}) \otimes \Lambda_{\infty}^{+*} (\tilde{\mathfrak{g}}), \ d + h \right\},
\]

where \( S_{\infty}^+ (\tilde{\mathfrak{g}}) \) and \( \Lambda_{\infty}^{+*} (\tilde{\mathfrak{g}}) \) are semi-infinite analogs of the modules of symmetric and exterior powers (see [1]), \( d \) is the analog of the differential for Lie algebra (co)homology and \( h \) is the analog of the Koszul differential:

\[
d = \sum_{u,v,i,j} \frac{1}{2} : \tau([u_i, v_j]) \varepsilon(v'_j) \varepsilon(u'_i) : + : \beta([u_i, v_j]) \gamma(v'_j) \varepsilon(u'_i) :, \\
h = \sum_{u,i} \gamma(u'_i) \tau(u_i),
\]

The quadratic operators (3.1) define a projective action of \( S'(2,1) \) on the semi-infinite Weil complex of \( \tilde{\mathfrak{g}} \). The cocycle is (see [8])

\[
c(L_n, L_k) = \frac{n^3}{12} \delta_{n,-k}, \quad c(E_n, F_k) = \frac{n-1}{6} \delta_{n,-k}, \quad c(L_n, H_k) = -\frac{n}{6} \delta_{n,-k}, \\
c(H_n, H_k) = \frac{n}{3} \delta_{n,-k}, \quad c(X^i_n, Y^i_k) = \frac{n(n-1)}{6} \delta_{n,-k}, \quad i = 1, 2.
\]

If \( \mathfrak{g} \) has a non-degenerate invariant symmetric bilinear form, then this action commutes with \( d \), and the action on the (relative) semi-infinite cohomology is well-defined (see [12]).

The restriction of this action to the zero modes defines a representation of \( \mathcal{K} \). Note that in this way \( d \) and \( d^* \) act as the semi-infinite Koszul differential \( h \) and the semi-infinite homotopy operator, respectively; this is a generalization of Howe’s construction [7]. Observe that our superalgebras \( \mathcal{K} \) and \( \mathcal{H} \) differ from the ones usually considered in examples of Howe duality on (hyper)Kähler manifolds but are contractions of \( \mathfrak{osp}(2|2) \) and \( \mathfrak{osp}(2|4) \), cf. [11].

It was shown in [3] that the relative semi-infinite complex of a \( \mathbb{Z} \)-graded complex Lie algebra with coefficients in a graded Hermitian module has a structure similar to that of the de Rham complex in Kähler geometry. In [12] we described operators on the relative semi-infinite Weil complex of the loop algebra of a complex Lie algebra, which are analogs of the classical ones in Kähler geometry and span a Lie superalgebra isomorphic to \( \mathcal{K} \). Note that in this realization (for which no “affinization” seems to be possible) \( d \) acts as the differential \( d \).

An interesting problem is to define operators acting on a (relative) semi-infinite Weil complex, which are analogous to the classical ones in hyper-Kähler geometry, and obtain the corresponding field expansions.

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References


