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Ehrenpreis Type Representations and Their Riemann–Hilbert Nonlinearisation

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Abstract

We review a new method for studying boundary value problems for evolution PDEs. This method yields explicit results for a large class of evolution equations which include: (a) Linear equations with constant coefficients, (b) certain classes of linear equations with variable coefficients, and (c) integrable nonlinear evolution equations.

1 Introduction

A new method for solving boundary value problems has been recently introduced by the author [1]. The implementation of this method to linear elliptic PDEs in two space variables is discussed in [2–4]. Here we concentrate on evolution equations in one and two space variables, which will be denoted by x and by (x_1, x_2) respectively. We assume that the space variables are on the half line. The case when x is on a finite segment is discussed in [5, 6]. Moving boundary value problems are discussed in [7].

A. Linear PDEs

For linear evolution equations with constant coefficients in *one* space dimension this method involves two novel steps: (a) Find an integral representation for the solution $q(x,t)$ in the complex k-plane in terms of certain transforms of the boundary values of q and its derivatives (see for example (2.1)). (b) Analyze the global relation satisfied by these transforms (see for example (2.10)) in order to express these transforms in terms of the given boundary conditions. This analysis uses two facts: First, there exist certain transformations in the complex k-plane which leave the unknown transforms invariant (for example $k \to \nu(k)$ in § 2). Second, the analyticity properties of the integral representation mentioned in (a) above imply that some of the unknown transforms do *not* contribute to the solution (such as the transforms $c(\nu_j(k), t)$, $j = 1, 2$ in § 2).

For linear evolution equations with constant coefficients in one space dimension the simplest way of finding an integral representation for the solution $q(x,t)$ is to deform the Fourier transform representation from the real line to the complex k-plane [8].

For linear evolution equations *whose coefficients depend on the space variables*, the two steps (a) and (b) mentioned above continue to form the basis of the method. However, in order to obtain an integral representation for $q(x,t)$, one now makes crucial use of the completeness relation of the associated space dependent eigenfunctions (see equation (3.7)). It is remarkable that even for this class of PDEs the global relation can be solved explicitly. Thus such PDEs can be solved with the same level of efficiency as the corresponding PDEs of constant coefficients.

For linear evolution equations with constant coefficients in *two* space variables the method also uses the steps (a) and (b) mentioned earlier. However, one must now work in the complex (k_1,k_2)-planes. Thus for step (a) one constructs an integral representation of $q(x_1,x_2,t)$ in the complex (k_1,k_2)-planes (see (4.3)). Similarly for step (b) one finds appropriate invariant transformations in the complex k_1-plane and the complex k_2-plane, see [8].

B. Integrable Nonlinear PDEs

Before discussing integrable nonlinear evolution equations in one space variable we first make some relevant remarks for linear evolution equations in one space variable. For such equations there exist at least three different ways of obtaining the integral representation for $q(x,t)$. Use: (1) The deformation of the Fourier transform mentioned earlier. (2) A reformulation of Green's theorem [9]. (3) The spectral analysis of an associated 0-differential form (the Lax pair approach) [1]. The starting point of all these approaches is the observation that two-dimensional linear PDEs with constant coefficients can be written as the condition that an appropriate differential 1-form is closed. For example, for the equation

$$q_t + q_x + q_{xxx} = 0, \tag{1.1}$$

such a form is given by

$$W(x,t,k) = e^{-ikx+iw(k)t}\left\{q(x,t)dx - X(x,t,k)dt\right\}, \qquad k \in \mathbb{C}, \tag{1.2}$$

where

$$w(k) = k - k^3, \qquad X = q_{xx} + ikq_x + (1-k^2)q. \tag{1.3}$$

Indeed,

$$dW = e^{-ikx+iw(k)t}(q_t + q_x + q_{xxx})dt \wedge dx,$$

thus $dW = 0$ iff equation (1.1) is valid.

Equation $dW = 0$ is equivalent to

$$\left(e^{-ikx+iw(k)t}q\right)_t + \left(e^{-ikx+iw(k)t}X\right)_x = 0, \tag{1.4}$$

which is the starting point for the construction of the integral representation of $q(x,t)$ via the deformation of the Fourier transform. Also a slight generalization of W, namely

$$\tilde{W}(x,t,k_1,k_2) = e^{-ik_1x-ik_2t}\{q(x,t)dx - X(x,t,k_1)dt\},$$

is the basic object used in the construction involving the reformulation of Green's theorem. Finally equation (1.2) is the starting point of the Lax pair approach: If equation (1.1) is valid in a simply connected domain Ω then W is exact, i.e. there exists a 0-form such that $W = dM$; letting $M = \mu \exp[-ikx + iw(k)t]$ we find

$$d\left[e^{-ikx+iw(k)t}\mu(x,t,k)\right] = W(x,t,k), \qquad k \in \mathbb{C}, \quad (x,t) \in \Omega. \tag{1.5}$$

This equation implies that equation (1.1) is equivalent to the compatibility condition of the following Lax pair,

$$\mu_x - ik\mu = q, \quad \mu_t + iw(k)\mu = -X. \tag{1.6}$$

Among the three approaches mentioned above, it is only the third one, i.e. the Lax pair approach, which can be generalized to integrable nonlinear equations [10]. This implies that the two basic steps (a) and (b) of the new method can be "nonlinearized" as follows: (a) The integral representation of $q(x,t)$ involves the eigenfunction $\mu(x,t,k)$ of the associated Lax pair. This function can *not* be written down explicitly but satisfies a matrix Riemann–Hilbert problem. The main simplifying feature of this Riemann–Hilbert problem is that it involves *explicit* x and t dependence of the form $\exp[-ikx + iw(k)t]$ where $w(k)$ is the dispersion relation of the linearized equation. (b) The analysis of the global relation can be used to express the relevant transforms (which are now called spectral functions) in terms of the boundary conditions. But these formulas are not in general explicit but they involve the solution of a nonlinear Volterra integral equation [11]. However, for certain classes of boundary conditions, called linearisable boundary conditions, boundary value problems can be solved with the same level of efficiency as the associated initial value problems.

C. Outline of the paper

In Section § 2 we solve a boundary value problem for equation (1.1) and we also briefly discuss the implications of this new method for the analysis of general (i.e. non integrable) nonlinear evolution equations. In § 3–5 we solve boundary value problems for the following equations: the time-dependent Schrödinger equation with a space-dependent potential

$$iq_t + q_{xx} + \frac{2p^2}{(\cosh px)^2}q = 0, \qquad p > 0, \tag{1.7}$$

the equation

$$q_t = q_{x_1x_1} + q_{x_2x_2} + q_{x_1} + q_{x_2}, \tag{1.8}$$

and the defocusing nonlinear Schrödinger equation

$$iq_t + q_{xx} - 2|q|^2 q = 0. \tag{1.9}$$

For the analysis of equation (1.1) we assume that the initial and the boundary conditions belong to appropriate Sobolev spaces. For the analysis of the other equations we assume that the initial and the boundary conditions are Schwartz functions; however similar results are valid for a less restrictive class of functions.

For linear PDEs the integral representations obtained by the new method are consistent with the Euler–Ehrenpreis–Palamodov representations [12–15]. For nonlinear PDEs the relevant integral representations can be thought as the proper nonlinear analogues of these fundamental representations.

2 Linear equations with constant coefficients in one dimension

Theorem 2.1. *Let $q(x,t)$ satisfy*

$$q_t + q_x + q_{xxx} = 0, \qquad 0 < x < \infty, \quad 0 < t < T,$$
$$q(x,0) = q_0(x) \in H^2(\mathbb{R}^+), \quad q(0,t) = g_0(t) \in H^1([0,T]), \quad q_0(0) = g_0(0),$$

where T is a positive constant. The unique solution of this IBV problem is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-iw(k)t}\hat{q}_0(k)dk + \frac{1}{2\pi}\int_{\partial D_+} e^{ikx-iw(k)t}\hat{g}(k,t)dk, \tag{2.1}$$

where $\omega(k) = k^3 - k$, the curve ∂D_+ is defined by

$$\partial D_+ : \ \operatorname{Im} w(k) = 0, \quad \operatorname{Im} k > 0, \tag{2.2}$$

and the spectral function $\hat{q}(k,t) = \{\hat{q}_0(k), \hat{g}(k,t)\}$, is defined as follows:

$$\hat{q}_0(k) = \int_0^{\infty} e^{-ikx} q_0(x)dx, \qquad \operatorname{Im} k \geq 0, \tag{2.3}$$

$$\hat{g}(k,t) = (1-3k^2)\hat{g}_0(k,t) + \frac{\nu_1 - k}{\nu_2 - \nu_1}\hat{q}_0(\nu_2) + \frac{k - \nu_2}{\nu_2 - \nu_1}\hat{q}_0(\nu_1), \tag{2.4}$$

$$\hat{g}_0(k,t) = \int_0^t e^{iw(k)\tau} g_0(\tau)d\tau, \tag{2.5}$$

$\nu_1(k)$, $\nu_2(k)$ are the two nontrivial roots of $w(k) = w(\nu(k))$.

The rigorous investigation of the above IBV problem involves the following steps, see [16] for details.

Step 1. *Assuming existence: (a) construct the integral representations for $q(x,t)$ and for $\hat{q}(k,t)$; (b) find the global relation.*

(a) It is shown in [8] that $q(x,t)$ is given by equation (2.1) where $\hat{q}_0(k)$ is defined by equation (2.3), while $\hat{g}(k,t)$ is defined by

$$\hat{g}(k,t) = (1-k^2)\hat{g}_0(k,t) + ik\hat{g}_1(k,t) + \hat{g}_2(k,t), \tag{2.6}$$

$$\hat{g}_j(k,t) = \int_0^t e^{iw(k)\tau} g_j(\tau)d\tau, \qquad j = 0,1,2, \quad k \in \mathbb{C}, \tag{2.7}$$

and

$$g_j(\tau) = \partial_x^j q(0,\tau), \qquad j = 0,1,2. \tag{2.8}$$

(b) The equation

$$\oint_{\partial\Omega} W(x,t,k) = 0,$$

where W is defined by equation (1.2) and $\partial\Omega$ is the boundary of the domain $\{0 < x < \infty, 0 < \tau < t\}$ yields, see Figure 2.1,

$$\hat{q}_0(k) + \hat{g}(k,t) = e^{iw(k)t}c(k,t), \tag{2.9}$$

where

$$c(k,t) = \int_0^\infty e^{-ikx}q(x,t)dx, \qquad \operatorname{Im} k \le 0.$$

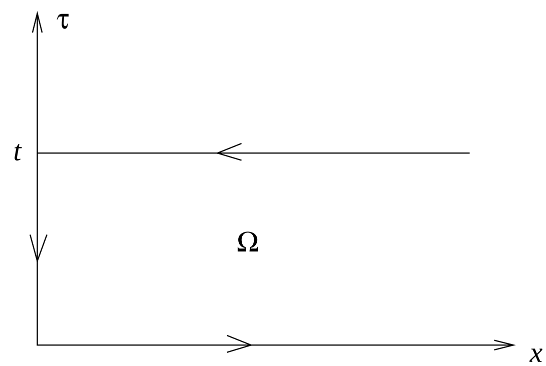

Figure 2.1

Step 2. *Assuming the validity of the global relation, prove existence: Namely assume that there exist functions* $q_0(x)$, $\{g_j(t)\}_0^2$, *such that the functions* $\hat{q}_0(k)$ *and* $\hat{g}(k,t)$ *defined by equations (2.3), (2.6), (2.7), satisfy equation (2.9), where* $c(k)$ *is some function holomorphic for* $\operatorname{Im} k < 0$ *and of* $O(1/k)$ *as* $k \to \infty$. *Then prove that if* $q(x,t)$ *is defined by equation (2.1), (a)* $q(x,t)$ *solves equation (1.1); (b)* $q(x,0) = q_0(x)$; *(c)* $\partial_x^j q(0,t) = g_j(t)$, $j = 0,1,2$.

The proof of (a) is a direct consequence of the exponential dependence of (x,t). The proof of (b) follows from the fact that $\exp(-iw(k)t)\hat{g}(k,t)$ is analytic and bounded in D_+,

$$D_+ = \{k \in \mathbb{C},\ \operatorname{Im} w(k) > 0,\ \operatorname{Im} k > 0\}.$$

The proof of (c) is based on the global relation and on appropriate contour deformations.

Step 3. *Given boundary conditions, analyze the global relation.*

Using the definition of $\hat{g}(k,t)$, the global relation (2.9) becomes

$$\hat{q}_0(k) + (1-k^2)\hat{g}_0(k,t) + ik\hat{g}_1(k,t) + \hat{g}_2(k,t) = e^{iw(k)t}c(k,t), \quad \operatorname{Im} k \le 0. \tag{2.10}$$

The crucial observation is that $\hat{g}_j(k,t)$, $j = 0,1,2$, depend on k only through $w(k)$. Thus these functions are invariant if $k \to \nu(k)$, where $\nu(k)$ is defined by

$$w(k) = w(\nu(k)).$$

This equation has two nontrivial roots: If $\nu_1(k) \in D_1$ then $k \in D_+$, and if $\nu_2(k) \in D_2$, then $k \in D_+$. Thus evaluating equation (2.10) at $\nu_1(k)$ and $\nu_2(k)$ we find

$$\begin{aligned}\hat{q}_0(\nu_j(k)) &+ (1-\nu_j^2(k))\hat{g}_0(k,t) + i\nu_j(k)\hat{g}_1(k,t) + \hat{g}_2(k,t)\\ &= e^{iw(k)t}c(\nu_j(k),t), \quad j = 1,2, \quad k \in D_+.\end{aligned}$$

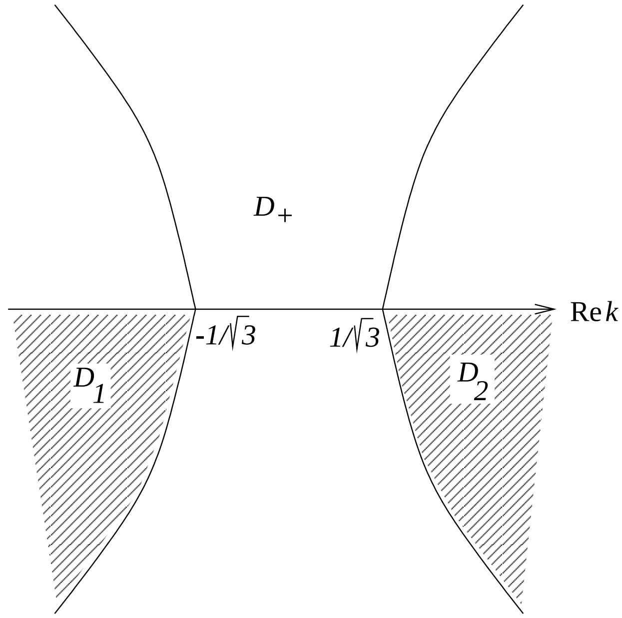

Figure 2.2

Solving these two equations for $\hat{g}_1(k,t)$, $\hat{g}_2(k,t)$ and substituting the resulting expressions in equation (2.6) we find that $\hat{g}(k,t)$ is given by equation (2.4) plus an additional term involving $e^{iw(k)t}$ multiplied by a certain combination of $c(\nu_1(k),t)$ and $c(\nu_2(k),t)$. However, this additional term does *not* contribute to $q(x,t)$. Indeed, $\exp[ikx]$, as well as $c(\nu_j(k),t)$, are bounded and analytic for $k \in D_+$, thus Cauchy's theorem implies that this additional term vanishes.

Remark 2.1. Let $\hat{g}(k)$ be defined by equation (2.4) where $\hat{g}_0(k,t)$ is replaced by

$$\hat{g}_0(k) = \int_0^T e^{iw(k)\tau} g_0(\tau) d\tau.$$

It is straightforward to show that $q(x,t)$ is also given by an expression similar to the rhs of equation (2.1) where $\hat{g}(k,t)$ is replaced by $\hat{g}(k)$.

This alternative representation is very convenient for computing the asymptotic properties of $q(x,t)$. These include the long time asymptotics [17] as well as the small dispersion limit.

Remark 2.2. Suppose that $q(x,t)$ satisfies the forced version of equation (1.1), i.e.

$$\begin{aligned} &q_t + q_x + q_{xxx} = f(x,t), \quad 0 < x < \infty, \quad 0 < t < T, \\ &q(x,0) = q_0(x), \quad 0 < x < \infty; \quad q(0,t) = g_0(t), \quad 0 < t < T, \\ &q_0(0) = g_0(0), \end{aligned}$$

where $f(x,t)$ is a given function with appropriate smoothness and decay. Then

$$q(x,t) = \tilde{q}(x,t) + F(x,t),$$

where

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx - iw(k)t}\left(\int_0^t d\tau \int_0^\infty d\xi e^{-ik\xi + iw(k)\tau} f(\xi,\tau)\right)$$

and $\tilde{q}(x,t)$ solves equation (1.1) with

$$\tilde{q}(x,0) = q_0(x), \quad 0 < x < \infty; \quad \tilde{q}(0,t) = g_0(t) - F(0,t).$$

Nonlinear PDEs can be considered as forced linear PDEs. By using the explicit formulas for forced linear PDEs derived by the new method, it should be possible to study the well-posedness of a large class of nonlinear PDEs. At least this should yield existence for small time, or for boundary conditions which have small norms in an appropriate functional space.

3 Linear equations with variable coefficients

Theorem 3.1. *Let $q(x,t)$ satisfy*

$$iq_t + q_{xx} + \frac{2p^2}{(\cosh px)^2} q = 0, \qquad 0 < x < \infty, \quad 0 < t < T, \tag{3.1}$$

$$q(x,0) = q_0(x) \in S(\mathbb{R}^+), \qquad q(0,t) = g_0(t) \in C^1[0,T], \tag{3.2}$$

where p is a positive constant and S denotes the space of Schwartz functions. The unique solution of this IBV problem is given by

$$\begin{aligned} q(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ik^2 t}\psi(x,k)\rho_0(k) + \frac{1}{2\pi} e^{ip^2 t}\varphi(x) C_0 \\ & + \frac{1}{2\pi}\int_{\partial D_+} dk \psi(x,k)\Bigg\{ 2(k+ip)e^{-ik^2 t}\hat{g}_0(k,t) \\ & - \frac{1}{k-ip}\left[(k+ip)e^{-ik^2 t}\rho_0(-k) - i\sqrt{\frac{p}{\pi}} e^{ip^2 t} C_0\right]\Bigg\}, \end{aligned} \tag{3.3}$$

where ∂D_+ is the oriented boundary of the first quadrant of the complex k-plane, see Figure 3.1, and

$$\psi(x,k) = \frac{k + ip\tanh px}{k+ip} e^{ikx}, \quad \varphi(x) = \frac{\sqrt{\pi p}}{\cosh px}, \quad k \in \mathbb{C}; \tag{3.4}$$

$$\hat{q}_0(k) = \int_0^\infty dx e^{-ikx} q_0(x), \quad \operatorname{Im} k \le 0; \quad \hat{g}_0(k,t) = \int_0^t d\tau e^{ik^2\tau} g_0(\tau), \tag{3.5}$$

$$\rho_0(k) = \int_0^\infty dx q_0(x)\overline{\psi(x,\bar{k})}, \quad \operatorname{Im} k \le 0; \quad C_0 = \int_0^\infty dx q_0(x)\overline{\varphi(x)}. \tag{3.6}$$

The derivation of this result makes crucial use of the completeness relation [18]

$$2\pi\delta(x - x') = \int_{-\infty}^{\infty} dk \psi(x,k)\overline{\psi(x,k)} + \varphi(x)\overline{\varphi(x)}. \tag{3.7}$$

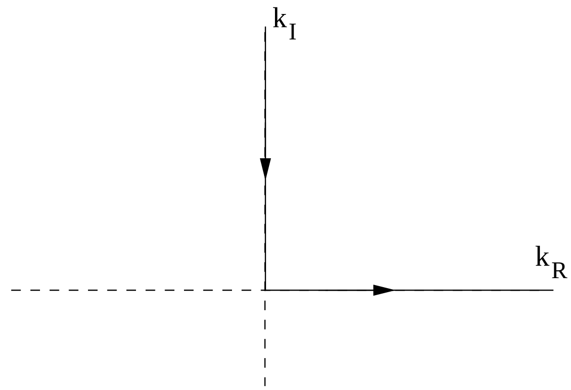

Figure 3.1. The contour ∂D_+ for equation (3.1)

4 Linear equations with constant coefficients in two space dimensions

Theorem 4.1. *Let $q(x_1, x_2, t)$ satisfy*

$$q_t = q_{x_1x_1} + q_{x_2x_2} + q_{x_1} + q_{x_2}, \quad 0 < x_j < \infty, \quad j = 1, 2, \quad 0 < t < T, \tag{4.1}$$

$$\begin{aligned} &q(x_1, x_2, 0) = q_0(x_1, x_2), \quad 0 < x_j < \infty, \quad j = 1, 2, \\ &q(0, x_2, t) = g_0^{(1)}(x_2, t), \quad 0 < x_2 < \infty, \quad 0 < t < T, \\ &q(x_1, 0, t) = g_0^{(2)}(x_1, t), \quad 0 < x_1 < \infty, \quad 0 < t < T, \end{aligned} \tag{4.2}$$

where q_0, $g_0^{(1)}$, $g_0^{(2)}$ are given Schwartz functions, which are compatible at $x_1 = t = 0$ and at $x_2 = t = 0$, i.e. $q_0(0, x_2) = g_0^{(1)}(x_2, 0)$ and $q_0(x_1, 0) = g_0^{(2)}(x_1, 0)$. The solution of this IBV problem is given by

$$\begin{aligned} q(x,t) = {}& \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 e^{ikx - w(k)t} \hat{q}_0(k) \\ &+ \frac{1}{(2\pi)^2} \int_{\partial D_+^{(1)}} dk_1 \int_{\partial D_+^{(2)}} dk_2 e^{ikx - w(k)t} \hat{g}(k) \\ &+ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_2 \int_{\partial D_+^{(1)}} dk_1 e^{ikx - w(k)t} \hat{g}^{(1)}(k, t) \\ &+ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_1 \int_{\partial D_+^{(2)}} dk_2 e^{ikx - w(k)t} \hat{g}^{(2)}(k, t), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} &x = (x_1, x_2), \quad k = (k_1, k_2), \quad xk = x_1k_1 + x_2k_2, \\ &w(k) = w_1(k_1) + w_2(k_2), \quad w_j(k) = k^2 - ik, \quad j = 1, 2, \end{aligned}$$

$\partial D_+^{(j)}$ is the oriented curve, see Figure 4.1, defined by

$$\operatorname{Re} w_j(k) = 0, \qquad \operatorname{Im} k > 0,$$

and the functions $\hat{g}^{(1)}$, $\hat{g}^{(2)}$, $\hat{g}$ are defined as follows:

$$\hat{g}^{(1)}(k,t) = -\hat{q}_0(i-k_1,k_2) - (2ik_1+1)\hat{g}_0^{(1)}(k,t),$$
$$\hat{g}^{(2)}(k,t) = -\hat{q}_0(k_1,i-k_2) - (2ik_2+1)\hat{g}_0^{(2)}(k,t),$$
$$\hat{g}(k) = \hat{q}_0(i-k_1,i-k_2), \tag{4.4}$$

where

$$\hat{q}_0(k_1,k_2) = \int_0^\infty dx_1 \int_0^\infty dx_2 e^{-ikx} q_0(x_1,x_2),$$
$$\hat{g}_0^{(1)}(k,t) = \int_0^t d\tau \int_0^\infty dx_2 e^{-ik_2x_2+w(k)\tau} g_0^{(1)}(x_2,\tau),$$
$$\hat{g}_0^{(2)}(k,t) = \int_0^t d\tau \int_0^\infty dx_1 e^{-ik_1x_1+w(k)\tau} g_0^{(2)}(x_1,\tau). \tag{4.5}$$

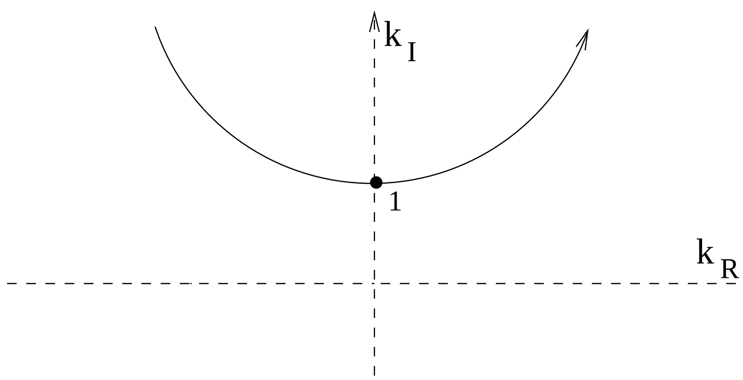

Figure 4.1. The contour ∂D_+ for equation (4.1)

5 Integrable nonlinear equations

In what follows we discuss the three steps (analogues to the three steps presented in § 2) needed for the analysis of the defocusing NLS equation on the half line:

$$iq_t + q_{xx} - 2|q|^2 q = 0, \quad 0 < x < \infty, \quad 0 < t < T, \tag{5.1}$$
$$q(x,0) = q_0(x) \in S(\mathbb{R}^+), \quad q(0,t) = g_0(t) \in C^1(0,T), \quad q_0(0) = g_0(0), \tag{5.2}$$

where T is a given positive constant.

It is more convenient to work with the analogue of $\hat{g}(k)$ instead of $\hat{g}(k,t)$, see Remark 2.1.

Step 1. *Assuming existence: (a) Construct the integral representations of $q(x,t)$ and of the spectral function $\hat{q}(k)$; the former involves the formulation of a RH problem and the latter involves the solution of certain linear Volterra integral equations. (b) Derive the global relation satisfied by $\hat{q}(k)$.*

If A is a 2×2 matrix, define $\hat{\sigma}_3 A$ by $[\sigma_3, A]$, $\sigma_3 = \mathrm{diag}(1,-1)$; then it follows that

$$e^{\hat{\sigma}_3} A = e^{\sigma_3} A e^{-\sigma_3}.$$

Step 1 is based on the fact that the defocusing NLS equation (5.1) is equivalent to

$$d\left[e^{(ikx+2ik^2t)\hat{\sigma}_3}\mu(x,t,k)\right] = W(x,t,k), \qquad k \in \mathbb{C}, \tag{5.3}$$

where μ is a 2×2 matrix, and the differential 1-form W is defined by

$$W = e^{(ikx+2ik^2t)\hat{\sigma}_3}\left(Q(x,t)\mu(x,t,k)dx + \tilde{Q}(x,t,k)\mu(x,t,k)dt\right), \tag{5.4}$$

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \bar{q}(x,t) & 0 \end{pmatrix}, \tag{5.5a}$$

$$\tilde{Q}(x,t,k) = 2kQ - iQ_x\sigma_3 - i|q|^2\sigma_3. \tag{5.5b}$$

The derivation of (a) involves the spectral analysis of equation (5.3).

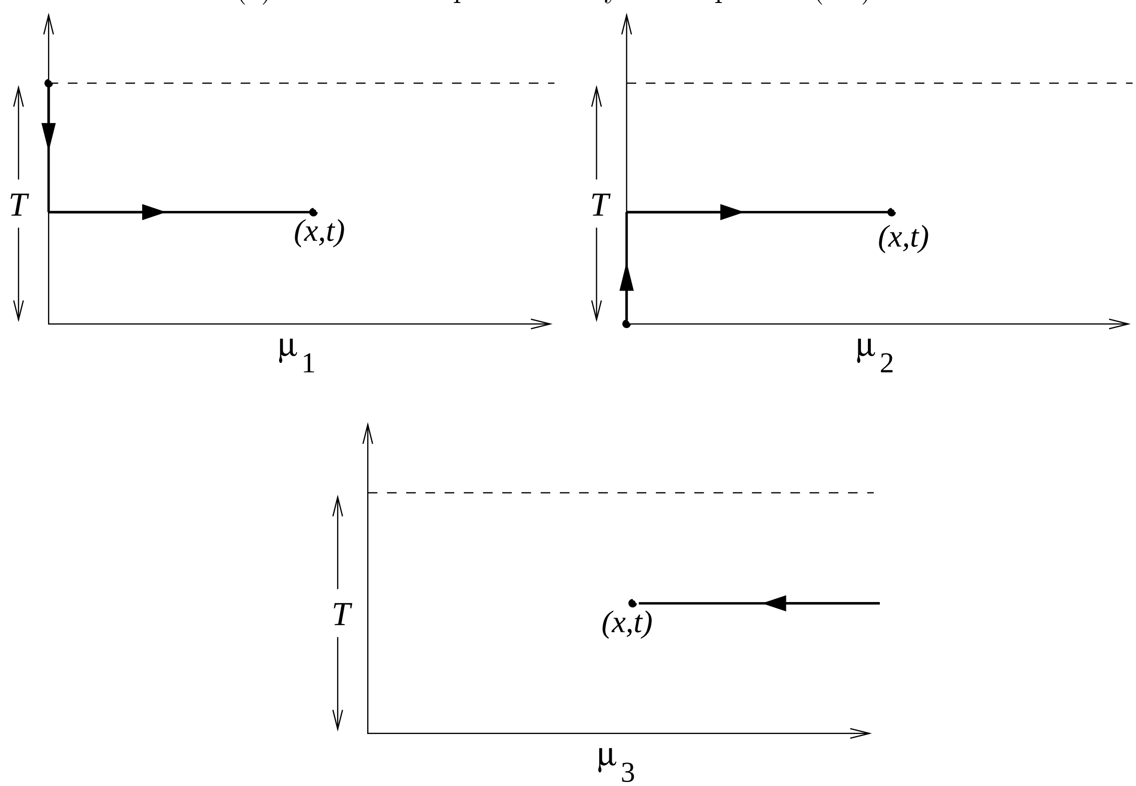

Figure 5.1

A solution of equation (5.3) is

$$\mu_j(x,t,k) = I + e^{-(ikx+2ik^2t)\hat{\sigma}_3}\int_{(x_j,t_j)}^{(x,t)} W(\xi,\tau,k), \tag{5.6}$$

where I is the 2×2 identity matrix and (x_j, t_j) are the three corners depicted in Figures 5.1. These matrices are simply related by the matrix analogues of $\hat{q}_0(k) = \mu_3(0,0,k)$ and of

$\hat{g}(k) = (e^{2ik^2T\hat{\sigma}_3}\mu_2(0,T,k))^{-1}$. Due to certain symmetries these matrices have the form

$$\hat{q}_0(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \overline{b(\bar{k})} & a(k) \end{pmatrix}, \qquad \hat{g}(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \overline{B(\bar{k})} & A(k) \end{pmatrix}. \tag{5.7}$$

The matrices $\mu_3(x,0,k)$ and $\mu_2(0,t,k)$ satisfy linear integral equations, thus $\{a(k), b(k), A(k), B(k)\}$ *cannot* be written in closed form. The matrices μ_j have certain analyticity properties which can be used to define a RH problem. This RH problem, in contrast to the case of linear PDEs, is *not* a scalar RH problem, thus it *cannot* be solved in closed form.

Using $\int_{\partial\Omega} W = 0$, with $\mu = \mu_3$ in the definition of W, it is straightforward to derive the global relation satisfied by the spectral function.

Step 2. *Existence under the assumption that the spectral functions satisfy the global relation.*

Given $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$, the space of Schwartz functions on the positive real axis, define $\{a(k), b(k)\}$. Assume that there exist smooth functions $g_0(t)$ and $g_1(t)$ such that if $\{A(k), B(k)\}$ are defined in terms of them, then $\{a(k), b(k), A(k), B(k)\}$ satisfy the global relation. Define $q(x,t)$ through the solution of the RH problem formulated in Step 1. Then prove that: (a) $q(x,t)$ is defined for all $0 < x < \infty$, $0 < t < T$; (b) $q(x,t)$ solves the NLS; (c) $q(x,0) = q_0(x)$, $0 < x < \infty$ and $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $0 < t < T$.

We give the definitions of $\{a(k), b(k), A(k), B(k)\}$ and the main theorem.

Definition of $a(k)$, $b(k)$. Let $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$. The map

$$\mathbf{S} : \{q_0(k)\} \to \{a(k), b(k)\} \tag{5.8}$$

is defined as follows:

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \varphi(0,k), \tag{5.9}$$

where the vector-valued function $\varphi(x,k)$ is defined in terms of $q_0(x)$ by

$$\begin{aligned} &\partial_x\varphi(x,k) + 2ik \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(x,k) \\ &\quad = \begin{pmatrix} 0 & q_0(x) \\ \bar{q}_0(x) & 0 \end{pmatrix} \varphi(x,k), \qquad 0 < x < \infty, \quad \operatorname{Im} k \geq 0, \\ &\lim_{x\to\infty} \varphi(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{5.10}$$

Definition of $A(k)$, $B(k)$. Let $\{g_0(t), g_1(t)\}$ be smooth functions for $0 < t < T$. The map

$$\tilde{\mathbf{S}} : \{g_0(t), g_1(t)\} \to \{A(k), B(k)\} \tag{5.11}$$

is defined as follows

$$\begin{pmatrix} -e^{-4ik^2T}B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \Phi(T,k), \tag{5.12}$$

where the vector-valued function $\Phi(t,k)$ is defined by

$$\partial_t\Phi(t,k) + 4ik^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi(t,k) = \tilde{Q}(t,k)\Phi(t,k), \qquad 0 < t < T, \quad k \in \mathbb{C},$$

$$\Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.13}$$

and $\tilde{Q}(t,k)$ is given by:

$$\tilde{Q}(t,k) = 2k \begin{pmatrix} 0 & g_0(t) \\ \bar{g}_0(t) & 0 \end{pmatrix} - i \begin{pmatrix} 0 & g_1(t) \\ \bar{g}_1(t) & 0 \end{pmatrix} \sigma_3 - i|g_0(t)|^2\sigma_3.$$

Theorem 5.1. *Given $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ define $\{a(k), b(k)\}$ according to the definition (5.9). Suppose that there exist smooth functions $\{g_0(t), g_1(t)\}$ satisfying $g_0(0) = q_0(0)$, $g_1(0) = \partial_x q(0)$, such that the functions $\{A(k), B(k)\}$ which are defined from $\{g_l(t)\}_0^1$ according to definition (5.12) satisfy the global relation*

$$a(k)B(k) - b(k)A(k) = e^{4ik^2T}c(k), \quad \operatorname{Im} k \geq 0, \tag{5.14}$$

where $c(k)$ is analytic and bounded for Im $k > 0$ and is of $O(1/k)$, $k \to \infty$.

Define $M(x,t,k)$ as the solution of the following 2×2 matrix RH problem:

(a) M is holomorphic for k in $\mathbb{C}\backslash\mathcal{L}$, where $\mathcal{L}$ is the union of the real and of the imaginary axes of the complex k-plane.

(b)

$$M(x,t,k) = I + O(1/k), \quad k \to \infty,$$

(c)

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathcal{L},$$

where J is defined in terms of a, b, A, B by (see Figure 5.2)

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\theta} & 1 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 1 & -\overline{\Gamma(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ \bar{\gamma}(k)e^{2i\theta} & 1 - |\gamma(k)|^2 \end{pmatrix},$$

where

$$\gamma(k) = \frac{b(k)}{a(k)}, \quad k \in \mathbb{R}; \qquad \Gamma(k) = \frac{1}{a(k)\left[a(k)\frac{\overline{A(\bar{k})}}{\overline{B(\bar{k})}} - b(k)\right]};$$

$$\theta(x,t,k) = kx + 2k^2 t.$$

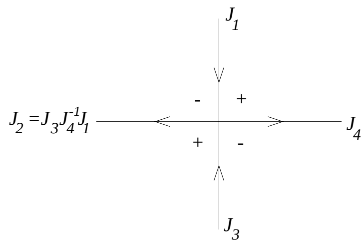

Figure 5.2

Then $M(x,t,k)$ exists and is unique.
Define $q(x,t)$ by

$$q(x,t) = 2i \lim_{k\to\infty} (kM(x,t,k))_{12}.$$

Then $q(x,t)$ solves the NLS equation with

$$q(x,0) = q_0(x), \qquad q(0,t) = g_0(t), \qquad q_x(0,t) = g_1(t).$$

Step 3. *Analyze the Global Relation.*

The global relation together with the definition of $\{A(k), B(k)\}$ yield a *nonlinear Volterra integral equation* for $g_1(t)$ in terms of $g_0(t)$ and $q_0(t)$. It is shown in [11] that this nonlinear equation has a global solution.

We recall that the analogous step for linear evolution equations was solved by algebraic manipulations. This was based on the invariance of the global relation under $k \to \nu(k)$. Unfortunately, the global relation now involves $\Phi(t,k)$ which in general breaks this invariance. However, for a *particular* class of boundary conditions this invariance survives. This is precisely the class of "linearizable problems", namely a class of problems for which $\{A(k), B(k)\}$ can be explicitly written in terms of $\{a(k), b(k)\}$.

Some linearizable cases are given below: Recall that the basic RH problem has a jump matrix which is uniquely defined in terms of the scalar functions $a(k)$, $b(k)$, and $\Gamma(k)$, where $\Gamma(k)$ involves $a(k)$, $b(k)$, and $B(k)/A(k)$. The basic RH problems for the KdV with dominant surface tension and for the sine Gordon have a similar form [10].

In [10] the following concrete linearizable cases are solved.

NLS:

$$q_x(0,t) - \chi q(0,t) = 0, \qquad \chi = \text{const}, \quad \chi \geq 0.$$

sG:

$$q(0,t) = \chi, \qquad \chi = \text{const}.$$

KdV:

$$q(0,t) = \chi, \quad q_{xx}(0,t) = \chi + 3\chi^2, \qquad \chi = \text{const.}$$

For each of these cases, B/A, and hence $\Gamma(k)$, can be explicitly given in terms of $a(k), b(k)$:

$$NLS: \quad \frac{B(k)}{A(k)} = -\frac{2k + i\chi}{2k - i\chi}\frac{b(-k)}{a(-k)},$$

$$\text{KdV, sG}: \quad \frac{B(k)}{A(k)} = \frac{f(k)b(\nu(k)) - a(\nu(k))}{f(k)a(\nu(k)) - b(\nu(k))},$$

where for the sG,

$$\nu(k) = \frac{1}{k}, \qquad f(k) = i\frac{k^2+1}{k^2-1}\,\frac{\sin\chi}{\cos\chi - 1},$$

while for the KdV,

$$\nu^2 + k\nu + k^2 + \frac{1}{4} = 0, \qquad f(k) = \frac{\nu + k}{\nu - k}\left(1 - \frac{4\nu k}{\chi}\right).$$

We emphasize that since $\{a(k), b(k)\}$ are determined in terms of the initial conditions and since $B(k)/A(k)$ and therefore $\Gamma(k)$ is explicitly written in terms of $\{a(k), b(k)\}$, it follows that linearizable initial boundary value problems on the half line are solved as effectively as initial value problems on the line.

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