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The Cauchy Problem for the Nonlinear Schrödinger Equation on a Compact Manifold

Nicolas BURQ, Patrick GÉRARD and Nikolay TZVETKOV

Département de Mathématiques, Université de Paris–Sud, Orsay, France
E-mail: firstname.lastname@math.u-psud.fr

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Abstract

We discuss the wellposedness theory of the Cauchy problem for the nonlinear Schrödinger equation on compact Riemannian manifolds. New dispersive estimates on the linear Schrödinger group are used to get global existence in the energy space on arbitrary surfaces and three-dimensional manifolds, generalizing earlier results by Bourgain on tori. On the other hand, on specific manifolds such as spheres, new instability phenomena are displayed, leading to some kind of illposednesss in higher dimensions.

1 Introduction

The nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = F(u), \tag{1.1}$$

where $F = \frac{\partial V}{\partial \bar{z}}$ and $V : \mathbb{C} \to \mathbb{R}$ satisfies $V(\mathrm{e}^{i\theta} z) = V(z)$ for every $z \in \mathbb{C}$, $\theta \in \mathbb{R}$, plays an important role in many areas of Physics, such as Laser Optics, Plasma Physics, Bose–Einstein condensates, … (see *e.g.* the recent book [23]). Typical nonlinearities of interest are smooth enough and have polynomial growth at infinity. Here we shall assume $V \in S^{2+\alpha}(\mathbb{C})$ for some $\alpha > 0$, namely

$$V \in C^\infty(\mathbb{C}), \qquad |D^k_{z,\bar{z}} V(z)| \le C_k (1 + |z|)^{2+\alpha-k}, \quad k = 0, 1, 2, \ldots .$$

If Δ denotes the usual Laplace operator in $\mathbb{R}^d$, the Cauchy problem for (1.1) has been extensively studied in the last two decades (see [12, 13, 27, 9]). Muchless results are known on bounded domains, with the notable exception of the work of Bourgain (see [3, 4, 11]) on the torus $\mathbb{R}^d/\mathbb{Z}^d$. Our aim in this short paper is to address the same problem posed on an arbitrary Riemannian compact manifold (M, g). Our motivation is of course to evaluate the impact of geometry of M on the wellposedness theory of equation (1.1), having in mind the infinite propagation speed of the Schrödinger equation, which suggests that the global geometry of M may have some influence on the solutions in finite time, contrary to what is known for the wave equation, for instance. Here we discuss slight extensions of the recent papers [6, 7].

2 Statement of the results

From now on (M, g) denotes a Riemannian compact manifold of dimension d, and Δ is the associated Laplace–Beltrami operator. First notice that, with the notation of the introduction, it is easy to check the following classical conservation laws for smooth solutions of (1.1),

$$\int_M |u(t,x)|^2 dx = Q_0, \tag{2.1a}$$

$$\int_M |\nabla u(t,x)|^2 dx + \int_M V(u(t,x))\, dx = E_0. \tag{2.1b}$$

As a consequence of these conservation laws and of the usual Gagliardo–Nirenberg inequalities, if we assume

$$\alpha \le \frac{4}{(d-2)_+} \qquad \text{and} \qquad V(z) \ge -C(1+|z|)^\beta \quad \text{for some} \quad \beta < 2 + \frac{4}{d}\,, \tag{2.2}$$

the finiteness of the norm of $u(t)$ in $H^1(M)$ is equivalent to the finiteness of the conserved quantities Q_0 and E_0.

In what follows we shall emphasize the following two aspects of the wellposedness theory for (1.1).

i) Global existence and uniqueness results for data in the energy space $H^1(M)$ under the assumption (2.2).

ii) Continuous dependence of the solutions with respect to the data.

Of course the question of blow up in finite time (if (2.2) is not satisfied) is also of great interest, but very few results are known on compact manifolds, and we shall content ourselves with mentioning them briefly.

2.1 Positive results in dimensions two and three

If $d = 1$, the control of the H^1 norm of u provided by conservation laws yields an estimate of the L^∞ norm, therefore global wellposedness in H^1 is easy. Thus we start with the first non trivial case $d = 2$.

Theorem 1. *Assume the dimension of M is $d = 2$, and*

$$V(z) \ge -C(1+|z|)^\beta$$

for some $\beta < 4$. Let $s \ge 1$ and $u_0 \in H^s(M)$. Then equation (1.1) has a unique global solution $u \in C(\mathbb{R}, H^s(M))$. In particular, $u \in C^\infty(\mathbb{R} \times M)$ if $u_0 \in C^\infty(M)$. Moreover, u satisfies the conservation laws (2.1) and, for any $T > 0$, for any bounded subset B of $H^s(M)$, the flow map

$$u_0 \in B \mapsto u \in C([-T,T], H^s(M)) \tag{2.3}$$

is Lipschitz continuous.

Of course local wellposedness in $H^s(M)$ for $s > 1 = d/2$ is a classical consequence of energy estimates, therefore the main part of Theorem 1 is global existence of strong solutions in H^1, propagation of their regularity, and the Lipschitz continuity of the flow map (2.3) on bounded subsets of $H^1(M)$. Notice that Theorem 1 was proved in [3] in the particular case $M = \mathbb{R}^2/\mathbb{Z}^2$. Let us also mention that logarithmic estimates in Brezis–Gallouët [5], Vladimirov [26] and Ogawa–Ozawa [18] allow to prove Theorem 1 on any compact surface M in the particular case $\alpha \leq 2$, with the exception of the uniform continuity of the flow map (2.3) if $s = 1$.

We now come to the case of dimension 3. The natural limitation due to Sobolev inequality here reads $\alpha \leq 4$. For $\alpha < 4$, Bourgain was able to prove global existence and regularity on the torus. For a general manifold we only obtain results for nonlinearities with cubic growth.

Theorem 2. *Assume the dimension of M is $d = 3$, and*

$$\alpha \leq 2, \qquad V(z) \geq -C(1 + |z|)^{\beta}$$

for some $\beta < 10/3$. Let $s \geq 1$ and $u_0 \in H^s(M)$. Then equation (1.1) has a unique global solution $u \in C(\mathbb{R}, H^s(M))$. In particular, $u \in C^\infty(\mathbb{R} \times M)$ if $u_0 \in C^\infty(M)$. Moreover, u satisfies the conservation laws (2.1) and, for any $T > 0$, for any $s > 1$, for any bounded subset B of $H^s(M)$, the flow map

$$u_0 \in B \mapsto u \in C([-T, T], H^s(M)) \tag{2.4}$$

is Lipschitz continuous.

Notice that here the difference between the energy regularity $s = 1$ and the Sobolev threshold $s > d/2$ is $1/2$ derivative, therefore even the local in time part of Theorem 2 is not trivial if $s \in [1, 3/2]$. We do not know whether the flow map (2.4) is still uniformly continuous if $s = 1$. This is related to the specific method of proof of Theorem 2 if $s = 1$, which is a combination of compactness arguments and uniqueness of weak solutions. In contrast, Theorem 1 and the part $s > 1$ of 2 can be obtained through some contraction argument in suitable spaces.

In dimensions $d \geq 4$, global wellposedness for equation (1.1) on general manifolds is completely open. To our knowledge, the only positive result concerns quadratic nonlinearities on the four-dimensional torus and is due to Bourgain [4]. This limitation may not be just technical, as suggested by the result of the next subsection.

2.2 A negative result in dimension six

We consider the following particular case of equation (1.1) on $M = S^6$

$$i\partial_t u + \Delta u = (1 + |u|^2)^{\alpha/2} u \tag{2.5}$$

with $\alpha \leq 1$ $(= 4/(d-2))$.

Theorem 3. *Let $\alpha \in]0, 1]$ and $s \in [1, 5/4[$. There exist sequences (u_0^n), $(\tilde{u}_0^n)$ of functions in $C^\infty(S^6)$ such that*

$$\sup_n \|u_0^n\|_{H^s} + \|\tilde{u}_0^n\|_{H^s} < +\infty, \qquad \|u_0^n - \tilde{u}_0^n\|_{H^s} \to 0,$$

and if there exist C^∞ solutions u^n, $\tilde{u}^n$ of (2.5) with $u^n(0) = u_0^n$, $\tilde{u}^n(0) = \tilde{u}_0^n$, on $[0,T] \times S^6$ with $T > 0$ independent of n, then

$$\liminf_n \sup_{0 \le t \le T} \|u^n(t) - \tilde{u}^n(t)\|_{H^s} > 0.$$

Theorem 3 is in strong contrast with uniform continuity of the flow map stated in dimensions 2 and 3 in the previous subsection, as well as with results for equation (2.5) on $\mathbb{R}^6$, for which one can prove global existence, regularity and Lipschitz continuity of the flow map on H^1 (see Appendix A). The instability property which is displayed in Theorem 3 is strongly related to the existence of families of eigenfunctions which concentrates on a closed geodesic curve. It also appears in lower dimensions, but only for regularities $s < 1$. Unfortunately it is not clear to generalize this phenomenon to other manifolds having the same geometric property since our proof also uses a global information about the distribution of eigenvalues. Let us mention that variants of this instability property on spheres for other dimensions and other regularities s can be found in [7], as well as the case of a boundary value problem on the two–dimensional disc in [8].

2.3 Blow up results

As already mentioned, the blow up in finite time of solutions of (1.1) on a compact manifold is a widely open problem. To our knowledge, the only examples of such phenomena are given by the following result, due to Ogawa–Tsutsumi [19] if $d = 1$ and generalized to the case $d = 2$ in [8].

Theorem 4. *Let (M, g) be a compact Riemannian manifold of dimension $d = 1$ or $d = 2$. Assume there exists $x^0 \in M$ and a system of coordinates near x^0 in which*

$$g = \sum_{j=1}^{d} dx_j^2.$$

Then there exist smooth solutions $u \in C^\infty([0, T[\times M)$ of

$$i\partial_t u + \Delta u = -|u|^{4/d} u \tag{2.6}$$

such that, as $t \to T$,

$$|u(t,x)|^2 \rightharpoonup \|Q\|^2_{L^2(\mathbb{R}^d)} \delta(x - x^0),$$

where Q is the ground state solution on $\mathbb{R}^d$ of

$$\Delta Q + Q^{1+4/d} = Q. \tag{2.7}$$

Notice that the assumption of Theorem 4 is always fulfilled if $d = 1$, and holds at every point of the two dimensional torus if g is the standard metric. The proof of Theorem 4 consists in modifying properly explicit blow up solutions of equation (2.6) on $\mathbb{R}^d$, obtained from (2.7) and some pseudoconformal invariance of (2.6). We refer to [19] and [8] for details.

3 Outline of the proof of Theorems 1 and 2

3.1 Dispersive estimates and generalized Strichartz inequalities

It is now classical that the proof of wellposedness results for equation (1.1) on $\mathbb{R}^d$ in the class $H^s(M)$ for $s < \frac{d}{2}$ is based on Strichartz inequalities (see [12, 13, 27, 9, 14]). In order to state these inequalities, let us recall the usual definition of admissible pair. A pair (p, q) of exponents in $[2, \infty]$ is said to be d-admissible if

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad p \geq 2, \qquad (p, q) \neq (2, \infty). \tag{3.1}$$

Then the Strichartz inequalities for the linear Schrödinger group on $\mathbb{R}^d$ read as

$$\|v\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^d))} \leq C\|v_0\|_{L^2(\mathbb{R}^d)}, \qquad \text{if} \quad v(t) = e^{it\Delta}v_0, \tag{3.2a}$$

$$\|w\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^d))} \leq C\|f\|_{L^{\overline{p'}}(\mathbb{R}, L^{\overline{q'}}(\mathbb{R}^d))}, \qquad \text{if} \quad w(t) = \int_{-\infty}^{t} e^{i(t-\tau)\Delta} f(\tau)d\tau \tag{3.2b}$$

for any admissible pairs (p, q), (p', q') (where $\bar{r}$ denotes the conjugate exponent of r). The usual way for deriving such inequalities is to rely on the following dispersive estimate

$$\|e^{it\Delta}f\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{|t|^{d/2}}\|f\|_{L^1(\mathbb{R}^d)}, \tag{3.3}$$

which is a trivial consequence of the explicit formula for the kernel of $e^{it\Delta}$. Then (3.2) follow from the combination of (3.3) with the following functional analytic device, which we borrow from the paper by Keel–Tao [14], where it is proved in its widest generality.

Lemma 1 ($TT^\star$ lemma, Keel–Tao). *Let $(X, \mathcal{S}, \mu)$ be a σ-finite measured space, and U be a weakly measurable map from $\mathbb{R}$ to the space of bounded operators on $L^2(X, \mathcal{S}, \mu)$. Assume that U satisfies the following two estimates for some $A > 0$, $B > 0$, $\sigma > 0$,*

$$(i) \quad \|U(t)\|_{L^2 \to L^2} \leq A, \qquad t \in \mathbb{R},$$

$$(ii) \quad \|U(t_1)U(t_2)^\star\|_{L^1 \to L^\infty} \leq \frac{B}{|t_1 - t_2|^\sigma}, \qquad t_1, t_2 \in \mathbb{R}.$$

Then for every 2σ-admissible pairs (p, q), (p', q'), one has

$$\left(\int_{\mathbb{R}} \|U(t)f\|_{L^q}^p dt\right)^{\frac{1}{p}} \leq C\|f\|_{L^2},$$

$$\|w\|_{L^p(\mathbb{R}; L^q)} \leq C\|f\|_{L^{\overline{p'}}(\mathbb{R}; L^{\overline{q'}})}, \qquad \textit{if} \quad w(t) = \int_{\tau < t} U(t)U(\tau)^\star f(\tau)d\tau.$$

The problem is that the dispersive estimate (3.3) fails on every compact manifold, not only globally in time (which is trivial by testing on $f = 1$) but also locally in time, due to the existence of point spectrum. Starting from this fact, Bourgain introduced a direct approach to Strichartz estimates to (1.1) on the torus $\mathbb{R}^d/\mathbb{Z}^d$, based on the Fourier series representation

$$e^{it\Delta}v_0(x) = \sum_{k \in (2\pi\mathbb{Z})^d} e^{-it|k|^2} e^{ik \cdot x} \widehat{v_0}(k). \tag{3.4}$$

In particular, using ingredients from analytic number theory, he was able to prove L^p estimates for (3.4) in terms of the $\|v_0\|_{H^s(\mathbb{T}^d)}$ for suitable $s \geq 0$ as well as bounds on the distributional function of the restriction of the exponential sum (3.4) on $|k| \leq N$, $N \gg 1$. It is worth noticing that the L^4 estimates of Bourgain for the linear Schrödinger group on $\mathbb{T}^d$ coincide (with exception of the end point) with the L^4 estimates obtained on $\mathbb{R}^d$ by combining (3.2a) with $p = 4$ and Sobolev inequalities. Unfortunately, the generalization of this approach to an arbitrary compact manifold M seems unrealistic, because of the lack of information about the distribution of eigenvalues of the Laplace–Beltrami operator.

Our strategy is to use the following weakened version of dispersive estimate.

Proposition 1. *Let $\varphi \in C_0^\infty(\mathbb{R})$. There exists $\alpha > 0$ such that, for any $h \in]0, 1]$, for any $t \in [-\alpha h, \alpha h]$,*

$$\|\varphi(h^2\Delta)e^{it\Delta} f\|_{L^\infty(M)} \leq \frac{C}{|t|^{d/2}} \|f\|_{L^1(M)}. \tag{3.5}$$

The proof of Proposition 1 is based on the following elementary observation. By rescaling the time variable to $t = hs$, the function $w = \varphi(h^2\Delta)e^{ihs\Delta} f$ satisfies the semiclassical problem

$$ih\partial_s w + h^2\Delta w = 0, \qquad w(0) = \varphi(h^2\Delta)f$$

and therefore can be described by a Fourier integral operator, using the WKB method. The dispersive estimate (3.5) is then a consequence of the stationary phase formula. Let us mention that the length of the time interval on which (3.5) holds can not be $\gg h$, due to the Weyl asymptotics. Also notice that (3.5) was already known in the particular case $M = \mathbb{T}^d$ (see [25, 15]).

Combining (3.5) with the $TT^\star$ lemma, we obtain the same Strichartz estimates as on $\mathbb{R}^d$, but for the truncated evolution $\varphi(h^2\Delta)e^{ihs\Delta}$, and only for very small time intervals. Namely, for all d-admissible pairs (p, q), (p', q'),

$$\|\varphi(h^2\Delta)v\|_{L^p(J,L^q(M))} \leq C\|v_0\|_{L^2(M)}, \qquad |J| \lesssim h, \tag{3.6a}$$

if $v(t) = e^{it\Delta}v_0$ and

$$\|w\|_{L^p(J,L^q(M)} \leq C\|f\|_{L^{\overline{p'}}(J,L^{\overline{q'}}(M))}, \qquad |J| \lesssim h, \qquad \operatorname{supp}(f) \subset J \times M, \tag{3.6b}$$

if $w(t) = \int_{-\infty}^t e^{i(t-\tau)\Delta} f(\tau)d\tau$.

By slicing finite time intervals into intervals of length $\lesssim h$, we infer from (3.6a)

$$\|\varphi(h^2\Delta)v\|_{L^p(I,L^q(M))} \leq C(I)h^{-\frac{1}{p}}\|\varphi(h^2\Delta)v_0\|_{L^2(M)}, \qquad |I| < \infty.$$

Then, using the Littlewood–Paley inequality, we obtain finally the

Theorem 5. *If $d \geq 2$, (p, q) is d-admissible, and I a finite interval,*

$$\|v\|_{L^p(I,L^q(M))} \leq C(I)\|v_0\|_{H^{\frac{1}{p}}(M)}, \qquad \text{if} \quad v(t) = e^{it\Delta}v_0. \tag{3.7}$$

Notice that the idea of slicing time intervals in order to derive Strichartz inequalities with loss of derivatives was already used by Bahouri–Chemin [1] and Tataru [24] in the context of the wave equation with non smooth coefficients. It should also be observed that similar estimates to (3.7) were derived by Staffilani–Tataru [22] by a different method. In our context it is interesting to observe that at least one of the estimates in Theorem 3 is optimal from the view point of the loss of derivatives $\frac{1}{p}$. Indeed, if $M = S^d$ and v_0 is a zonal spherical harmonic associated to eigenvalue $\lambda = n(n+d-1)$ it is classical (see e.g. Sogge [21]) that for $\lambda \gg 1$,

$$\|v_0\|_{L^q(M)} \approx \sqrt{\lambda}^{s(q)}, \qquad s(q) = \frac{d-1}{2} - \frac{d}{q} \qquad \text{if} \quad q \geq \frac{2(d+1)}{d-1}.$$

If we set $d \geq 3$, $p = 2$, $q = 2^\star := \frac{2d}{d-2}$ in (3.7), observing that $v(t) = e^{-it\lambda} v_0$, we notice that

$$\frac{1}{p} = \frac{1}{2} = \frac{d-1}{2} - \frac{d}{2^\star} = s(2^\star),$$

thus this slicing method leads to an optimal result in this case. However some intermediate Strichartz inequalities on spheres ($2 < p < +\infty$) can be improved (see [6]).

3.2 Applications to the nonlinear equation

We now come to the proofs of Theorem 1 and Theorem 2. First observe that estimate (3.7) implies its non-homogeneous version

$$\|w\|_{L^p([0,T],L^q(M))} \leq C(T)\|f\|_{L^1([0,T],H^{\frac{1}{p}}(M))}, \quad \text{if} \quad w(t) = \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau. \qquad (3.8)$$

Using (3.7) and (3.8) if $d = 2$, one proves easily local wellposedness for (1.1) in $H^1(M)$ by a contraction argument in

$$X^1_T = C([-T,T], H^1(M)) \cap L^p([-T,T], W^{\sigma,q}(M)),$$

where $p \in]\alpha, +\infty[$, $\sigma = 1 - \frac{1}{p} > \frac{2}{q} = 1 - \frac{2}{p}$, so that $W^{\sigma,q}(M) \subset L^\infty(M)$. The global existence is then derived classically from conservation laws, while propagation of regularity is a consequence of the key information $u \in L^p_{\text{loc}}(\mathbb{R}, L^\infty(M))$ for all $p < +\infty$.

A similar approach leads to local wellposedness in $H^s(M)$ for $s > 1$ and $d = 3$, but we have to be more careful with the growth of nonlinearities. Since the case $s > \frac{3}{2}$ is trivial, we shall assume $s \in]1, \frac{3}{2}]$. Our starting point is the following general estimate which is valid in any dimension (see Appendix B).

Lemma 2. *Let $G \in S^2(\mathbb{C})$, $G(0) = 0$, $s \in [1,2]$ and $u \in L^\infty \cap H^s$, then $G(u) \in H^s$ and*

$$\|G(u)\|_{H^s} \leq C(1 + \|u\|_{L^\infty})\|u\|_{H^s}.$$

As a consequence of this lemma, we get, if $F \in S^3(\mathbb{C})$,

$$\|F(u) - F(v)\|_{H^s} \leq C(M_\infty^2 \|u - v\|_{H^s} + M_\infty N_s \|u - v\|_{L^\infty})$$

if $M_\infty = 1 + \|u\|_{L^\infty} + \|v\|_{L^\infty}$, $N_s = 1 + \|u\|_{H^s} + \|v\|_{H^s}$. Let (p, q) be a 3-admissible pair with $p > 2$ so close to 2 such that

$$\sigma = s - \frac{1}{p} > \frac{3}{q} = \frac{3}{2} - \frac{2}{p}.$$

Then the above inequality combined with (3.8) and (3.7) allows to solve (1.1) by a contraction argument in

$$X_T^s = C([-T, T], H^s(M)) \cap L^p([-T, T], W^{\sigma,q}(M)).$$

Unfortunately, this argument breaks down if $s = 1$, which is our key regularity for global existence in view of conservation laws (2.1). Therefore we appeal to another kind of argument if $s = 1$. Notice that global existence of weak solutions in the class H^1 is an easy consequence of (2.1) and of a compactness argument. We shall show that these weak solutions are in fact strong and regular if their Cauchy data are regular. Unfortunately because of the critical character of regularity $s = 1$, estimates (3.7) and (3.8) are not sufficient and we have to come back to truncated evolutions. The following lemma is crucial.

Lemma 3. *Let $u \in C_w(\mathbb{R}, H^1(M))$ be a weak solution of (1.1) with $\alpha \le 2$ and $d = 3$, and $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, $0 < h < 1$, $T > 0$. Then*

$$\|\varphi(h^2\Delta)u\|_{L^2([0,T],L^6(M))} \le C h^{\frac{1}{2}} \|\varphi(h^2\Delta)u\|_{L^2([0,T],H^1(M))} + N_1 h^{\frac{1}{2}+\delta} \tag{3.9}$$

for some $\delta > 0$, where N_1 only depends on a bound of $\|u\|_{L^\infty([0,T],H^1(M))}$.

The idea for proving Lemma 3 is once again to slice interval $[0, T]$ into small pieces of length $\lesssim h$ and to apply (3.6a) and (3.6b) with $p = p' = 2$ on each piece. If we apply this strategy to the solution u itself, we obtain contributions of the value of $\varphi(h^2\Delta)u$ at the bottom of each piece, which is not easy to compare with $\|\varphi(h^2\Delta)u\|_{L^2([0,T],H^1)}$. Therefore the trick is to apply the nonhomogeneous estimate (3.6b) with $p = p' = 2$ to $\chi_h(t)\varphi(h^2\Delta)u$, where χ_h is a cutoff function which localizes in the piece. Summing up these contributions leads to (3.9).

By combining lemma 3 with Littlewood–Paley theory, one infers, for every weak solution of (1.1) with $\alpha \le 2$ and every $q < +\infty$,

$$\|u\|_{L^2([0,T],L^q(M))} \le N_1(\sqrt{qT} + 1). \tag{3.10}$$

Inequality (3.10) allows a strategy which goes back to Yudovitch in the framework of fluid mechanics. Indeed, if u, $\tilde{u}$ are two weak solutions of (1.1) with $u(0) = \tilde{u}(0)$, estimate (3.10) allows to control the evolution of $g(t) = \|u(t) - \tilde{u}(t)\|^2_{L^2(M)}$, specifically,

$$g'(t) \le C(\|u(t)\|^2_{L^q(M)} + \|\tilde{u}(t)\|^2_{L^q(M)})g(t)^{1-\frac{3}{q}}$$

thus

$$g(t) \le \left(N_1\left(t + \frac{1}{q}\right)\right)^{\frac{q}{3}}$$

which goes to zero as q tends to infinity if $t>0$ is small enough. This yields the uniqueness for the Cauchy problem. Similarly, regularity in $H^s(M)$, $s>1$, is obtained by deriving the following other consequence of Lemma 3,

$$\|u\|_{L^2([0,T],L^\infty(M))} \le N_1([T\log(2+\|u\|_{L^2([0,T],H^s(M))})]^{\frac{1}{2}}+1) \tag{3.11}$$

which, plugged into a Gronwall process, leads to the result. ■

4 Outline of the proof of Theorem 3

In order to justify the choice $d=6$, we work a priori on

$$M = S^d = \{x\in\mathbb{R}^{d+1},\ x_1^2+\cdots+x_{d+1}^2=1\}, \qquad d\ge 2.$$

For $n\ge 1$, the polynomial $(x_1+ix_2)^n$ is harmonic and homogeneous of degree n, hence its restriction ψ_n to S^d is an eigenfunction of $-\Delta$ associated to the eigenvalue $\lambda_n = n(n+d-1)$. Moreover ψ_n concentrates on the circle $x_1^2+x_2^2=1$, so that, for any $p>0$,

$$\int_M |\psi_n|^p\,dx = \beta_p n^{-\frac{d-1}{2}}\left(1+\mathcal{O}\left(n^{-1}\right)\right), \qquad \beta_p>0. \tag{4.1}$$

Let $s<\frac{d-1}{4}$ and set

$$\phi_n = n^{\frac{d-1}{4}-s}\psi_n. \tag{4.2}$$

Then $\|\phi_n\|_{H^s}\approx 1$ and, for any $\alpha>0$, for some $\gamma>0$,

$$\frac{\|\phi_n\|_{L^{\alpha+2}}^{2+\alpha}}{\|\phi_n\|_{L^2}^2} = \gamma\, n^{\alpha(\frac{d-1}{4}-s)}\left(1+\mathcal{O}\left(n^{-1}\right)\right) \longrightarrow +\infty \quad \text{as } n \text{ tends to } +\infty. \tag{4.3}$$

Let $\kappa\in[\frac{1}{2},1]$, consider the equation

$$i\partial_t u_n + \Delta u_n = \langle u_n\rangle^\alpha u_n, \qquad u_n(0)=\kappa\phi_n, \tag{4.4}$$

with $\alpha\in]0,1]$. Assume u_n exists as a C^∞ function on $[0,T]\times M$, where $T>0$ is independent of n, and denote by $c_n(t)$ the component of $u_n(t)$ along ϕ_n. Let us look for an ansatz for $c_n(t)$ an $n\to+\infty$. If we start with a rough analysis which ignores the dynamics orthogonally to ϕ_n we are led to $|c_n(t)|\approx 1$ (by conservation of the L^2 norm) and $c_n(t)$ is approximated by the solution $c_n^\#(t)$ of the ordinary differential equation

$$i\frac{d}{dt}c_n^\# - \lambda_n c_n^\# = \omega_n(\kappa)c_n^\#, \qquad c_n^\#(0)=\kappa,$$

where $\omega_n(\kappa)$ is the component of $\langle\kappa\phi_n\rangle^\alpha\phi_n$ along ϕ_n, namely

$$\begin{aligned}\omega_n(\kappa) &= \frac{1}{\|\phi_n\|_{L^2}^2}\int_M \langle\kappa\phi_n(x)\rangle^\alpha|\phi_n(x)|^2dx \\ &= \kappa^\alpha\gamma n^{\alpha(\frac{d-1}{4}-s)}(1+\epsilon_n(\kappa)),\end{aligned} \tag{4.5}$$

where $\epsilon_n(\kappa)\to 0$ uniformly with respect to $\kappa\in[\frac{1}{2},1]$. This leads to

$$c_n^\#(t) = \kappa e^{-it(\lambda_n+\omega_n(\kappa))}.$$

It turns out that, for suitable values of α, s, t, this ansatz is correct.

Proposition 2. *Assume*

$$\frac{d-1}{4} > s > \frac{d-1}{4} - \frac{1}{2-\alpha}. \tag{4.6}$$

Then there exists T_n such that

$$T_n n^{\alpha(\frac{d-1}{4}-s)} \longrightarrow +\infty$$

and, uniformly with respect to $t \in [0, T_n]$, $\kappa \in [\frac{1}{2}, 1]$,

$$\|\Pi_n(u_n(t)) - \kappa e^{-it(\lambda_n+\omega_n(\kappa))}\phi_n\|_{H^s} \longrightarrow 0 \quad \textit{as } n \textit{ tends to } +\infty,$$

where Π_n is the orthogonal projector onto the line directed by ϕ_n.

Let us show how Proposition 2 implies Theorem 3. Let (u_n) be the above solution with $\kappa = 1$, and let $(\tilde{u}_n)$ be the solution corresponding to $\kappa = \kappa_n$, where $\kappa_n \in [\frac{1}{2}, 1]$ is to be fixed such that $\kappa_n \to \kappa$. Assume $u_n, \tilde{u}_n \in C^\infty([0,T] \times M)$ for a fixed $T > 0$. Then, for $t \in [0, T_n]$,

$$\begin{aligned} \|u_n(t) - \tilde{u}_n(t)\|_{H^s} &\geq \|\Pi_n(u_n(t) - \tilde{u}_n(t))\|_{H^s} \\ &\gtrsim \left| e^{-it\omega_n(1)} - \kappa_n e^{-it\omega_n(\kappa_n)} \right| - o(1) \\ &\gtrsim \left| \sin \frac{t}{2}[\omega_n(1) - \omega_n(\kappa_n)] \right| - o(1). \end{aligned}$$

In view of (4.5), we have

$$\omega_n(1) - \omega_n(\kappa_n) = \gamma n^{\alpha(\frac{d-1}{4}-s)}(1 - \kappa_n^\alpha + \epsilon_n),$$

where $\epsilon_n \to 0$ independently of the choice of (κ_n). As a consequence, we may choose (κ_n) such that

$$1 - \kappa_n^\alpha \ll 1, \qquad T_n n^{\alpha(\frac{d-1}{4}-s)}(1 - \kappa_n^\alpha) \longrightarrow +\infty \quad \text{as } n \text{ tends to } +\infty,$$

so that

$$T_n[\omega_n(1) - \omega_n(\kappa_n)] \longrightarrow +\infty \quad \text{as } n \text{ tends to } +\infty,$$

and therefore

$$\liminf_n \sup_{0\leq t\leq T_n} \|u^n(t) - \tilde{u}^n(t)\|_{H^s} > 0.$$

Finally, notice that, in order that the interval $]\frac{d-1}{4} - \frac{1}{2-\alpha}, \frac{d-1}{4}[$ contains 1 for every $\alpha \in]0,1]$ it is necessary and sufficient that $d = 6$ or $d = 7$. This leads to Theorem 3 if $d = 6$.

Let us now come to the proof of Proposition 2. We set

$$u_n(t) = \kappa e^{-it(\lambda_n+\omega_n(\kappa))}[\phi_n + z_n(t)\phi_n + q_n(t)]$$

with $q_n(t) \perp \phi_n$. Since ϕ_n is unbounded in L^∞, the linearized equation around ϕ_n is not easily tractable. Fortunately, we shall derive valuable information from the conservation laws (2.1), which read as

$$|1+z_n(t)|^2\|\phi_n\|_{L^2}^2 + \|q_n(t)\|_{L^2}^2 = \|\phi_n\|_{L^2}^2, \tag{4.7}$$

$$\begin{aligned}|1+z_n(t)|^2\|\nabla\phi_n\|_{L^2}^2 + \|\nabla q_n(t)\|_{L^2}^2 + \frac{2}{(2+\alpha)\kappa^2}\int_M \langle u_n(t,x)\rangle^{\alpha+2}dx \\ = \|\nabla\phi_n\|_{L^2}^2 + \frac{2}{(2+\alpha)\kappa^2}\int_M \langle \kappa\phi_n(x)\rangle^{\alpha+2}dx.\end{aligned} \tag{4.8}$$

If we substract λ_n times (4.7) to (4.8), we finally obtain

$$\begin{aligned}\|\nabla q_n(t)\|_{L^2}^2 - \lambda_n\|q_n(t)\|_{L^2}^2 &\le \frac{2}{(2+\alpha)\kappa^2}\int_M \langle\kappa\phi_n(x)\rangle^{\alpha+2}dx \\ &\le C\, n^{\alpha(\frac{d-1}{4}-s)-2s}.\end{aligned} \tag{4.9}$$

The miracle is that the left hand side of (4.9) controls the L^2 norm of $q_n(t)$. Indeed, if we denote by R_θ the rotation of angle θ in the plane (x_1,x_2), we observe that

$$\phi_n(R_\theta x) = e^{in\theta}\phi_n(x).$$

Because of the invariance of (4.4) by rotations and by $u \mapsto e^{in\theta}u$, and uniqueness of the Cauchy problem in C^∞, we conclude

$$u_n(t, R_\theta x) = e^{in\theta}u_n(t,x).$$

But elementary considerations on homogeneous polynomials show that, if a spherical harmonic h satisfies $h(R_\theta x) = e^{in\theta}h(x)$, then either its degree is $\ge n+1$, or h is parallel to ϕ_n. As a consequence, the decomposition of $q_n(t)$ into spherical harmonics only contains spherical harmonics of degree $\ge n+1$, and thus

$$\|\nabla q_n(t)\|_{L^2}^2 - \lambda_n\|q_n(t)\|_{L^2}^2 \ge n\,\|q_n(t)\|_{L^2}^2$$

which, by (4.9), implies

$$\|q_n(t)\|_{L^2}^2 \le C\,n^{\alpha(\frac{d-1}{4}-s)-2s-1} \ll \|\phi_n\|_{L^2}^2$$

since $\alpha(\frac{d-1}{4}-s) < 1$ as a consequence of hypothesis (4.6). It is now possible to project the equation onto ϕ_n, in order to control the evolution of z_n. In view of the above estimate on q_n and (4.7), we obtain finally

$$\begin{aligned}&i\dot z_n = \mathcal{O}\left(n^{\frac{d-1}{4}-s}|z_n|^2 + n^{(\alpha+1)(\frac{d-1}{4}-s)-1} + n^{\frac{3\alpha}{2}(\frac{d-1}{4}-s)-\frac12}\right),\\ &z_n(0) = 0.\end{aligned}$$

In view of assumption (4.6), this implies the existence of some T_n such that $T_n n^{\alpha(\frac{d-1}{4}-s)} \to +\infty$ and $\sup_{0\le t\le T_n}|z_n(t)| \to 0$ as n tends to infinity. ■

Finally, let us mention that, in some cases, it is possible to have an ansatz for $u_n(t)$ itself (see [7]) and even to evaluate the remainder terms $z_n(t)$ and $q_n(t)$ sharply (see [2]).

A A Lipschitz bound for the nonlinear Schrödinger flow on $H^1(\mathbb{R}^6)$

Let $\alpha \in]0,1[$. In Kato [13], it is proved in particular that, given $u_0 \in H^1(\mathbb{R}^6)$, there exists a unique $u \in C(\mathbb{R}, H^1(\mathbb{R}^6)) \cap L^p_{\mathrm{loc}}(\mathbb{R}, W^{1,q}(\mathbb{R}^6))$, ($(p,q)$ 6-admissible) such that

$$i\partial_t u + \Delta u = \langle u \rangle^\alpha u, \qquad u(0) = u_0, \tag{A.1}$$

where $\langle u \rangle := (1+|u|^2)^{1/2}$. In this appendix, we prove the following additional property of the flow map of (A.1).

Proposition 3. *For any $T > 0$, for any bounded subset $B \subset H^1(\mathbb{R}^6)$, the map*

$$\Phi \, : \, u_0 \in B \longmapsto u \in C([-T,T], H^1(\mathbb{R}^6))$$

is Lipschitz continuous.

Proof. First of all we show the following Strichartz bounds,

$$\sup_{u \in \Phi(B)} \|\nabla u\|_{L^p([-T,T], L^q(\mathbb{R}^6))} < +\infty \tag{A.2}$$

for every 6-admissible pair (p,q). Notice that the case $p = \infty$, $q = 2$, $T = \infty$, follows from the conservation of energy. Next we observe that

$$|\nabla(\langle u \rangle^\alpha u)| \lesssim |\nabla u| + |u|^\alpha |\nabla u|.$$

Applying the nonhomogeneous Strichartz inequality to the equation satisfied by ∇u, we infer

$$\|\nabla u\|_{L^p([-T,T],L^q)} \leq C \left(1 + T + \| |u|^\alpha |\nabla u| \|_{L^{\overline{p'}}([-T,T],L^{\overline{q'}})} \right), \tag{A.3}$$

where (p',q') is any other 6-admissible pair. We first choose $q, q' \in]2,3]$ such that

$$\frac{1}{\overline{q'}} = \frac{\alpha}{q^\star} + \frac{1}{q}, \qquad \frac{1}{q^\star} = \frac{1}{q} - \frac{1}{6},$$

which is possible since $\frac{\alpha}{q^\star} + \frac{1}{q}$ ranges in $[\frac{\alpha}{6} + \frac{1}{3}, \frac{\alpha}{3} + \frac{1}{2}[$, while $\frac{1}{\overline{q'}}$ ranges in $]\frac{1}{2}, \frac{2}{3}]$ and $0 < \alpha < 1$. Then using the Sobolev embedding $W^{1,q}(\mathbb{R}^6) \subset L^{q^\star}(\mathbb{R}^6)$ and suitable Hölder inequalities, we observe that

$$\| |u|^\alpha |\nabla u| \|_{L^{\overline{p'}}([-T,T],L^{\overline{q'}})} \leq C \|\nabla u\|^\alpha_{L^r([-T,T],L^q)} \|\nabla u\|_{L^p([-T,T],L^q)}, \tag{A.4}$$

where

$$\frac{1}{\overline{p'}} = \frac{\alpha}{r} + \frac{1}{p},$$

thus

$$\begin{aligned} \frac{\alpha}{r} - \frac{\alpha}{p} &= 1 - \left(\frac{1}{p'} + \frac{\alpha+1}{p} \right) = 1 - \frac{3}{2}(\alpha+2) + 3\left(\frac{1}{q'} + \frac{\alpha+1}{q} \right) \\ &= 1 - \frac{3}{2}(\alpha+2) + 3\left(1 + \frac{\alpha}{6}\right) = 1 - \alpha > 0. \end{aligned}$$

Therefore $r < p$, and (A.3), (A.4) together with Hölder inequality in time yield

$$\|\nabla u\|_{L^p([-T,T],L^q)} \leq C\left(1 + T + T^{1-\alpha}\|\nabla u\|^{1+\alpha}_{L^p([-T,T],L^q)}\right).$$

We therefore obtain (A.2) for this choice of (p,q) and T small enough. Coming back to (A.3) with any other choice of (p,q) we obtain (A.2) for T small enough. Then the generalization to any T follows from the energy conservation.

We now turn to the Lipschitz bound. Of course we may assume that B is convex, so that we are led to show that the differential of our flow map is bounded. If $u_0 \in B$, $\Phi(u_0) = u$, we have $\Phi'_{u_0}(v_0) = v$, with

$$\begin{aligned} &i\partial_t v + \Delta v = f(u,v), \qquad v(0) = v_0, \\ &f(u,v) = \left(\langle u\rangle^\alpha + \frac{\alpha}{2}\langle u\rangle^{\alpha-2}|u|^2\right)v + \frac{\alpha}{2}\langle u\rangle^{\alpha-2}u^2\bar{v}. \end{aligned}$$

Notice that

$$|\nabla f(u,v)| \lesssim |v||\nabla u| + (1+|u|)|\nabla v|. \tag{A.5}$$

Therefore by Strichartz, Sobolev and Hölder inequalities

$$\begin{aligned} \|\nabla v\|_{L^\infty([0,T],L^2)} &\lesssim \|\nabla v_0\|_{L^2} + \|v\|_{L^\infty([0,T],L^3)}\|\nabla u\|_{L^2([0,T],L^3)} \\ &\quad + T\|\nabla v\|_{L^\infty([0,T],L^2)} + \|u\|_{L^2([0,T],L^6)}\|\nabla v\|_{L^\infty([0,T],L^2)} \\ &\lesssim \|\nabla v_0\|_{L^2} + \left(T + \|\nabla u\|_{L^2([0,T],L^3)}\right)\|\nabla v\|_{L^\infty([0,T],L^2)}. \end{aligned}$$

If $T + \|\nabla u\|_{L^2([0,T],L^3)}$ is smaller than some fixed constant δ, we infer the desired estimate

$$\|\nabla v\|_{L^\infty([0,T],L^2)} \leq C\|\nabla v_0\|_{L^2}. \tag{A.6}$$

In order to generalize estimate (A.6) to an arbitrary finite time interval I, we observe that it is possible to slice I into a finite number N of intervals $I_1, \ldots, I_N$ such that

$$|I_k| + \|\nabla u\|_{L^2(I_k,L^3)} \leq \delta$$

and say

$$N \lesssim |I| + \|\nabla u\|_{L^2(I,L^3)}.$$

Iterating the estimate $(A.6)$, we obtain

$$\|\nabla v\|_{L^\infty([0,T],L^2)} \leq C^N\|\nabla v_0\|_{L^2},$$

which completes the proof. ∎

Remark 1. Notice that, though the nonlinearity is H^1 subcritical, the estimate (A.5) is of critical type. For this reason, we had to use the above critical slicing argument (see also Keraani [16], where the same argument was used for a critical equation). For the same reason, our proof does not extend to dimension $d \geq 7$, even if $\alpha < \frac{4}{d-2}$. Therefore the Lipschitz continuity of the flow for these high dimensions seems to be an open problem.

B A tame estimate

In this last section, we give a simple proof of the following estimate, which yields immediately Lemma 2.

Proposition 4. *If $G \in C^2(\mathbb{C})$, $G(0) = 0$, and G'' is bounded, then, for any $s \in]1, 2[$, for any $u \in L^\infty \cap H^s$, $G(u) \in H^s$ and*

$$\|G(u)\|_{H^s} \le C(1 + \|u\|_{L^\infty})\|u\|_{H^s}. \tag{B.1}$$

Remark 2. Notice that the Bony–Meyer paradifferential decomposition (see e.g. [17]) does not provide the correct power of $\|u\|_{L^\infty}$. However a discussion around the proof of Proposition 4 and of more general estimates (with the sharp power of $\|u\|_{L^\infty}$) can be found in the book of Runst and Sickel [20], Chapter 5. Our only goal here is to propose a short proof of (B.1).

Proof of Proposition 4. It is enough to work in local coordinates, therefore we may assume that u is defined on $\mathbb{R}^d$. Notice that estimate (B.1) holds if G is a quadratic polynomial, because of the bilinear estimate

$$\|uv\|_{H^s} \lesssim \|u\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{L^\infty}, \qquad s \ge 0$$

(see e.g. [10]). Next we use the equivalent norm for $s \in]1, 2[$,

$$\|u\|_{H^s} \approx \|u\|_{H^1} + \|\nabla u\|_{\dot{H}^{s-1}}, \qquad \|\nabla u\|_{\dot{H}^{s-1}} = \left(\int_{\mathbb{R}^{2d}} \frac{|\nabla u(x) - \nabla u(y)|^2}{|x-y|^{d+2(s-1)}} dxdy\right)^{\frac{1}{2}}.$$

Since $\|G(u)\|_{H^1} \le C(1 + \|u\|_{L^\infty})\|u\|_{H^1}$ trivially, we just need to prove that

$$\|\nabla G(u)\|_{\dot{H}^{s-1}} \lesssim (1 + \|u\|_{L^\infty})\|u\|_{H^{s-1}}.$$

But

$$\begin{aligned} \nabla(G(u))(x) - \nabla(G(u))(y) = {} & G'(u(x))(\nabla u(x) - \nabla u(y)) \\ & + (G'(u(x)) - G'(u(y)))\nabla u(y). \end{aligned} \tag{B.2}$$

Since $|G'(u)| \le C(1 + |u|)$, the contribution to $\|\nabla G(u)\|_{\dot{H}^{s-1}}$ of the first term in the right hand side of (B.2) is of course bounded by $C(1 + \|u\|_{L^\infty})\|\nabla u\|_{H^{s-1}}$. As for the second term, it is bounded by

$$C|u(x) - u(y)||\nabla u(y)|,$$

since G'' is bounded. Hence its contribution to $\|\nabla G(u)\|_{\dot{H}^{s-1}}$ is precisely the one we would obtain in the particular case $G(u) = u^2$. Since estimate (B.1) is true in this case, it is true for every G. ∎

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