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Toda Equations and σ-Functions of Genera One and Two

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Abstract

We study the Toda equations in the continuous level, discrete level and ultradiscrete level in terms of elliptic and hyperelliptic σ and ψ functions of genera one and two. The ultradiscrete Toda equation appears as a discrete-valuation of recursion relations of ψ functions.

1 Introduction

Recently Kimijima and Tokihiro found solutions of the discrete and ultradiscrete Toda equations in terms of elliptic and hyperelliptic θ functions [17]. In this article we present another type of solution of another type of the discretization of the Toda equation [7] and its ultradiscretization [15] associated with algebraic curves of genera one and two.

Elliptic and hyperelliptic σ functions are related to nonlinear differential equations from the beginning [2, 10, 19]. We study the Toda equations at the continuous level, discrete level and ultradiscrete level in terms of σ functions and ψ functions of genera one and two. We show that these equations have solutions expressed in terms of σ and ψ functions. Here the ψ functions are defined by rational functions of the σ functions.

In [13] it was shown that the ψ functions can be related to discrete nonlinear equations, such as the discrete Painlevé equations. This article can be considered as one of a series in which relations between ψ functions and discrete nonlinear difference equations are unfolded.

Further, as was mentioned in [14], the ultradiscrete sometimes can be regarded as a valuation of a related field. This article shows that, in the case of the Toda equation, it can be also realized as a discrete valuation of a function field over a Jacobi variety.

In Section 2 we concentrate on the genus one case and give concrete solutions. We investigate the genus two version in Section 3. It is shown that all solutions of the Toda equations in this study are connected with the addition formulae of the σ functions.

2 Genus one case

In this Section we deal with an elliptic curve given by

$$C_1: \ \frac{1}{4}\bar{y}^2 = y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$
$$= (x - b_1)(x - b_2)(x - b_3), \tag{2.1}$$

where the b's are complex numbers.

2.1 Continuous Toda equation

Firstly we give a $\wp$ function solution of the continuous Toda equation [16]. We treat the Weierstrass elliptic σ function associated with the curve C_1, which is connected with the Weierstrass $\wp$ function by

$$\wp(u) = -\frac{d^2}{du^2} \log \sigma(u). \tag{2.2}$$

A local parameter u in C_1 is given by

$$u = \int_\infty^{(x,y)} \frac{dx}{2y}, \tag{2.3}$$

and $\wp(u)$ is equal to $x(u)$. Here ∞ is the infinity point of C_1.

The addition formula of the σ functions is given by

$$-[\wp(u) - \wp(v)] = \frac{\sigma(v+u)\sigma(u-v)}{[\sigma(v)\sigma(u)]^2}. \tag{2.4}$$

By differentiating the logarithm of (2.4) by u twice, we have

$$-\frac{d^2}{du^2} \log[\wp(u) - \wp(v)] = \wp(u+v) - 2\wp(u) + \wp(u-v). \tag{2.5}$$

For a constant number u_0, by letting $u = nu_0 + t$, $v = u_0$ and $b := \wp(u_0)$, we have

$$-\frac{d^2}{dt^2} \log[\wp(nu_0 + t) - b] = [\wp((n+1)u_0 + t) - b]$$
$$- 2[\wp(nu_0 + t) - b] + [\wp((n-1)u_0 + t) - b]. \tag{2.6}$$

Further by letting $q_n := \log[\wp((n+1)u_0 + t) - b]$, we have

$$-\frac{d^2}{dt^2}(q_n) = \mathrm{e}^{q_{n+1}} - 2\mathrm{e}^{q_n} + \mathrm{e}^{q_{n-1}}. \tag{2.7}$$

This is identified with the continuous Toda lattice equation. In fact, by letting $q_n = Q_n - Q_{n-1}$, Q_n obeys the nonlinear differential equation of a nonlinear lattice [16]. It is clear that this elliptic solution comes from the addition formula (2.4). In § 3.1 we show that a genus two solution of the Toda equation can be expressed by a similar form.

2.2 Discrete Toda equation and ψ functions

Though there are several models of the discrete Toda equations, we concentrate on a model given in [7]. In this subsection we give elliptic solutions of the discrete Toda equation.

The elliptic ψ function is given by

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma(u)^{n^2}} \tag{2.8}$$

and, due to the addition formula (2.4), it satisfies the recursion relation [18],

$$\psi_{n+m}\psi_{m-n} = \begin{vmatrix} \psi_{m-1}\psi_n & \psi_m\psi_{n+1} \\ \psi_m\psi_{n-1} & \psi_{m+1}\psi_n \end{vmatrix}. \tag{2.9}$$

Further ψ_n can also be computed using the Brioches–Kiepert relation [3, 9],

$$\psi_n(u) = \frac{(-1)^{n-1}}{[1!2!\cdots(n-1)!]^2} \begin{vmatrix} \wp'(u) & \wp''(u) & \cdots & \wp^{(n-1)}(u) \\ \wp''(u) & \wp'''(u) & \cdots & \wp^{(n)}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \wp^{(n-1)}(u) & \wp^{(n)}(u) & \cdots & \wp^{(2n-3)}(u) \end{vmatrix}. \tag{2.10}$$

Here derivatives of u are denoted by $'$ and $^{(n)}$. By noting $\frac{d}{du} = 2y\frac{d}{dx}$, the ψ function is a polynomial of x and y over the complex number $\mathbf{C}$. In fact the ψ function can be explicitly obtained as,

$$\begin{aligned} \psi_1(u) &= 1, \\ \psi_2(u) &= -2y, \\ \psi_3(u) &= 3x^4 + 4\lambda_2x^3 + 6\lambda_1x^2 + 12\lambda_0x - {\lambda_1}^2 + 4\lambda_2\lambda_0, \\ \psi_4(u) &= -4y\left[x^6 + 2\lambda_2x^5 + 5\lambda_1x^4 + 20\lambda_0x^3 + \left(20\lambda_2\lambda_0 - 5{\lambda_1}^2\right)x^2\right. \\ &\quad \left. + \left(8{\lambda_2}^2\lambda_0 - 2\lambda_2{\lambda_1}^2 - 4\lambda_1\lambda_0\right)x + 4\lambda_2\lambda_1\lambda_0 - {\lambda_1}^3 - 8{\lambda_0}^2\right]. \end{aligned} \tag{2.11}$$

For ψ_n, $n > 4$, we have the recursion relations

$$\begin{aligned} \psi_{2n+1} &= \psi_{n+2}\psi_n^3 - \psi_{n+1}^3\psi_{n-1}, \\ \psi_{2n} &= \psi_n(\psi_{n+2}\psi_{n-1}^2 - \psi_{n+1}^2\psi_{n-2})/\psi_2. \end{aligned} \tag{2.12}$$

Thus we know that

$$\begin{aligned} &\psi_n(u) \in \mathbf{C}[x, \lambda_0, \lambda_1, \lambda_2] \quad \text{for odd } n, \\ &\psi_n(u) \in \mathbf{C}[x, \lambda_0, \lambda_1, \lambda_2]y \quad \text{for even } n. \end{aligned} \tag{2.13}$$

When n is odd, ψ_n is a polynomial of x whose order is $(n^2-1)/2$ and, for an even n, the order of x for ψ_n/y is $(n^2-4)/2$. For specific curves, we give explicit forms of ψ_n in the Appendix.

We comment on the properties of ψ functions. We note that $\sigma(u)$ is characterized by the property that it has no singularity with respect to $u \in \mathbf{C}$ and its zeros are identified with a lattice points generated by the periodicity 2ω of $\wp(u)$, $\wp(u + 2\omega) = \wp(u)$. In other words the zero of σ is congruent to the origin of the local parameter u modulo the lattice. Accordingly, as ψ function is a function over the curve C, we conclude that a point satisfying

$$\psi_n(u) = 0 \tag{2.14}$$

is a point for which nu is equal to the lattice point again. In other words we have n-cyclic points as zeros of ψ_n. Conversely it can be shown that the polynomial of x and y whose zeros multiplied by n are lattice points must be ψ_n modulo constant factors.

Hence, if n is a factor of m, *i.e.*, $n|m$, it is clear that ψ_m is divided by ψ_n,

$$\psi_n|\psi_m. \tag{2.15}$$

For mutually coprime numbers p, q and an integer n_0, we introduce

$$\phi_i^{\ j} := \psi_{n_0+pi+qj}. \tag{2.16}$$

By letting $n \equiv n_0 + pi + qj$ we have

$$\begin{aligned}\psi_{n+p}\psi_{n-p} &= \psi_n^2\psi_{p+1}\psi_{p-1} - \psi_p^2\psi_{n+1}\psi_{n-1},\\ \psi_{n+q}\psi_{n-q} &= \psi_n^2\psi_{q+1}\psi_{q-1} - \psi_q^2\psi_{n+1}\psi_{n-1}.\end{aligned} \tag{2.17}$$

The components $\psi_{n+1}\psi_{n-1}$ in both formulae give a relation, viz

$$\left(\psi_{n+p}\psi_{n-p} - \psi_n^2\psi_{p+1}\psi_{p-1}\right)\psi_q^2 = \left(\psi_{n+q}\psi_{n-q} - \psi_n^2\psi_{q+1}\psi_{q-1}\right)\psi_p^2. \tag{2.18}$$

Noting $n \equiv n_0 + pi + qj$ (2.18) can be regarded as an evolution equation for i and j when we consider ψ_q, ψ_p and $\psi_{q\pm1}$ as initial values. Let us assume that ψ_q, ψ_p and $\psi_{p\pm q}$ are not equal to zero by choosing the parameter u. By letting $\delta := \psi_q/\psi_p$ and $c(1-\delta) := -\psi_{p+q}\psi_{p-q}/(\psi_q)^2$ we have

$$\phi_i^{\ j+1}\phi_i^{\ j-1} - c\left(1-\delta^2\right)\phi_i^{\ j}\phi_i^{\ j} - \delta^2\phi_{i+1}^{\ \ j}\phi_{i-1}^{\ \ j} = 0. \tag{2.19}$$

For later convenience we do not fix c and define

$$V_i^{\ j} := \left(\frac{\phi_{i+1}^{\ \ j}\phi_{i-1}^{\ \ j}}{\phi_i^{\ j}\phi_i^{\ j}}\right) - c. \tag{2.20}$$

Then we obtain

$$\log\left(\frac{\left(c+V_i^{\ j}\right)^2}{\left(c+V_i^{\ j+1}\right)\left(c+V_i^{\ j-1}\right)}\right) = \log\left(\frac{\left(c+\delta^2V_i^{\ j}\right)^2}{\left(c+\delta^2V_{i+1}^{\ \ j}\right)\left(c+\delta^2V_{i-1}^{\ \ j}\right)}\right). \tag{2.21}$$

When $c = 1$, this equation is one of discrete versions of the Toda equation, which appeared in [7].

The condition $c = 1$ means that

$$\psi_{p+q}\psi_{p-q} + \psi_p\psi_p - \psi_q\psi_q = 0, \tag{2.22}$$

which is an equation of (p^2+q^2-2)-order with respect to x. In other words, for a point u satisfying (2.22), we have a solution of the discrete Toda equation in [7] in terms of ψ functions.

Further we introduce $U_i{}^j := \left(V_i{}^j + c\right)$ which satisfies

$$\left(\frac{\left(U_i{}^j\right)^2}{\left(U_i{}^{j+1}\right)\left(U_i{}^{j-1}\right)}\right) = \left(\frac{\left(c\left(1-\delta^2\right)+\delta^2 U_i{}^j\right)^2}{\left(c\left(1-\delta^2\right)+\delta^2 U_{i+1}^{\ j}\right)\left(c\left(1-\delta^2\right)+\delta^2 U_{i-1}^{\ j}\right)}\right). \tag{2.23}$$

We investigate this equation with general c and go on to call it the discrete Toda equation in this article. We note that this solution is due to the recursion relation (2.9) which comes from the addition formula (2.4).

2.3 Periodic solutions of discrete Toda equation

For general c in (2.20) we consider a periodic solution of (2.23). It is obvious that, when $\psi_n = 0$, $\psi_{nr} = 0$. Thus there may exist a point, u_1, such that

$$\psi_i(u_1) = \psi_{n+i}(u_1). \tag{2.24}$$

In fact we have solutions of (2.23) for a curve $y^2 = x^2(x+1/4)$ and its point $x = -1$ (see the Appendix). The ψ function has values as in Table 1.

Table 1. ψ_n at $x = -1$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
ψ_n	0	1	$-\sqrt{-3}$	2	$-\sqrt{-3}$	1	0	-1	$\sqrt{-3}$	-2	$\sqrt{-3}$	-1	0

For $(p,q) = (3,2)$ and $n_0 = 0$, *i.e.*, $\delta^2 = -3/4$ and $c(1-\delta^2) = 1/4$, we have a periodic solution of (2.23).

Table 2. U_i^j ($p = 3$, $q = 2$ case)

$j\backslash i$	0	1	2	3
0	∞	0	∞	0
1	1/3	3	1/3	3
2	1/3	3	1/3	3
3	∞	0	∞	0

For $(p,q) = (2,3)$ and $n_0 = 0$, *i.e.*, $\delta^2 = -4/3$ and $c(1-\delta^2) = 1/3$, another periodic solution of (2.23) is given in Table 3.

Table 3. U_i^j ($p = 2$, $q = 3$ case)

$j\backslash i$	0	1	2	3
0	∞	0	0	∞
1	1/4	-2	-2	1/4
2	∞	0	0	∞
3	1/4	-2	-2	1/4

2.4 Ultradiscrete Toda equations

In this subsection we investigate the ultradiscrete version of the Toda equation using ψ functions.

For the elliptic curve C_1 a local parameter t should be characterized by

$$\begin{array}{lll} \text{for a generic point } x_0 \text{ in } C_1, & : & t = x - x_0, \\ \text{for a finite branch point } b_i \text{ in } C_1, & : & t = \sqrt{x - b_i}, \\ \text{for the infinity point } \infty \text{ in } C_1, & : & t = 1/\sqrt{x}. \end{array} \tag{2.25}$$

Let a localization of the commutative ring $R = \mathbf{C}[x, y]/(y^2 - f(x))$ at u_0 be denoted by R_{u_0}. Let K_{u_0} be a field of Laurent transformations at u_0 of rational functions. The valuation of the field K_{u_0} is given that for $f \in K_{u_0}$, let $\mathrm{val}(f) = \infty$ if $f = 0$, and if f is given by

$$f(u) = a(u - u_0)^m + \mathcal{O}\left((u - u_0)^{m+1}\right) \tag{2.26}$$

for $a \neq 0$, let $\mathrm{val}(f) = m$ [6]. Denoting set of integers by $\mathbf{Z}$, the discrete valuation is known as a map

$$\mathrm{val} : K_{u_0} \to \mathbf{Z} + \infty, \tag{2.27}$$

which satisfies

$$\begin{aligned} &\mathrm{val}(fg) = \mathrm{val}(f) + \mathrm{val}(g), \\ &\mathrm{val}(f + g) \geq \min(\mathrm{val}(f), \mathrm{val}(g)). \end{aligned} \tag{2.28}$$

For example the inequality in (2.28) appears due to a case, $k = m$ and $a = -b$ for $f = a(u - u_0)^m + \cdots$ and $g = b(u - u_0)^k + \cdots$ with $(a, b \neq 0)$. Inversely, as long as we avoid such a case, we can regard the second relation in (2.28) as an equality.

R_{u_0} can be expressed as

$$R_{u_0} = \{\, f \in K_{u_0} \mid \mathrm{val}(f) \geq 0 \,\}. \tag{2.29}$$

$R_{u_0}^{\times} := \{\, f \in K_{u_0} \mid \mathrm{val}(f) = 0 \,\}$ is a multiplication group in R_{u_0}. An element in $R_{u_0}^{\times}$ is called unit. The subset $\mathbf{m} := \{\, f \in K_{u_0} \mid \mathrm{val}(f) > 0 \,\}$ of R_{u_0} is a unique maximal ideal in R_{u_0} and thus we have a filter structure,

$$\mathbf{m}^k \supset \mathbf{m}^{k+1}. \tag{2.30}$$

Here the multiplication among ideals is given as a set of sum of multiplications of elements in the ideals. Due to the filter structure there naturally appears a nonarchimedean distance given by

$$|f - g|_{\mathrm{val}} := \exp(-\mathrm{val}(f - g)). \tag{2.31}$$

Thus an element f in $\mathbf{m}$ is a smaller element than unity, *i.e.*, $|f|_{\mathrm{val}} < 1$. When δ behaves like a small parameter [7], we regard it as an element of $\mathbf{m}$, *i.e.*,

$$\delta \equiv \frac{\psi_q}{\psi_p}(u) \in \mathbf{m}. \tag{2.32}$$

We now consider the point satisfying

$$c\left(1-\delta^2\right)=\frac{\psi_{p+q}(u)\psi_{p-q}(u)}{\psi_p(u)^2}\in R_u^\times. \tag{2.33}$$

Define

$$f_i^{\,j}:=-\mathrm{val}(U_i^j),\qquad d:=-\mathrm{val}\left(\delta^2\right). \tag{2.34}$$

When we expand them as $U_i^j\delta^2=a(u-u_0)^m+\cdots$ and $c\left(1-\delta^2\right)=b(u-u_0)^k+\cdots$ with $a,b\neq 0$, we assume that for any i and j, k is not equal to m or a is not equal to $-b$ if $k=m$. Then (2.23) becomes

$$\begin{aligned} &f_i^{\,j+1}-2f_i^{\,j}+f_i^{\,j-1}\\ &\qquad=\max\left(0,f_{i+1}^{\,j}+d\right)-2\max\left(0,f_i^{\,j}+d\right)+\max\left(0,f_{i-1}^{\,j}+d\right). \end{aligned} \tag{2.35}$$

This is identified with the ultradiscrete Toda equation in [15].

Let us consider solutions of the ultradiscrete Toda equation (2.35). By letting $g_n:=\mathrm{val}(\psi_n)$,

$$f_i^{\,j}=g_{i+1}^{\,j}-2g_i^{\,j}+g_{i-1}^{\,j}. \tag{2.36}$$

For the curve $y^2=x^3+1/4$, and u_0 at $x(u_0)=(-1/4)^{1/3}$, we have g_i as in Table 4:

Table 4. g_n at $y=0$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
g_n	∞	0	1	0	1	0	1	0	1	0	1	0	1	$\cdots$

Then we have a solution of (2.35) for $p=3$, $q=2$ and $n_0=0$: $d=-2$, $\mathrm{val}\left(c\left(1-\delta^2\right)\right)=0$,

Table 5. f_i^j ($p=3$, $q=2$ case)

$j\backslash i$	1	2	3	4	5	$\cdots$
0	∞	-2	2	-2	2	$\cdots$
1	2	-2	2	-2	2	$\cdots$
2	2	-2	2	-2	2	$\cdots$
3	2	-2	2	-2	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Next we deal with a curve $y^2=x^3-x$ and a point $u_0(x=0)$. The values of g_i are given in Table 6.

Table 6. g_n at $x=0$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
g_n	∞	0	1	4	5	8	13	16	21	28	33	40	49	$\cdots$

When $(p, q, n_0) = (5, 2, 0)$, we have $\mathrm{val}(c(1 - \delta^2)) = 0$, $d = 14$ and Table 7.

Table 7. f_i^j ($p = 5$, $q = 2$ case)

$j\backslash i$	1	2	3	4	$\cdots$
0	∞	18	14	18	$\cdots$
1	18	14	18	18	$\cdots$
2	14	18	18	14	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

In this case, $|\delta|_{\mathrm{val}} > 1$.

3 Genus two case

In this section we investigate genus two solutions of the Toda equations using the hyperelliptic σ functions and ψ functions.

The hyperelliptic σ function was defined by Klein after the prototype had been discovered by Weierstrass [1, 10, 19]. Let us fix a hyperelliptic curve with genus two,

$$C_2: \ y^2 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0, \tag{3.1}$$

where λ_i, $i = 0, 1, \ldots, 4$ are complex numbers. For a point in the symmetric product space of the curve C_2, $((x_1, y_1), (x_2, y_2)) \in \mathrm{Sym}^2 C_2$, its corresponding point $u \equiv (u_1, u_2)$ in the Jacobi variety J_2 is given by

$$u_1 := \int_\infty^{(x_1,y_1)} \frac{dx}{y} + \int_\infty^{(x_2,y_2)} \frac{dx}{y}, \qquad u_2 := \int_\infty^{(x_1,y_1)} \frac{xdx}{y} + \int_\infty^{(x_2,y_2)} \frac{xdx}{y}. \tag{3.2}$$

Here ∞ means the infinity point of the curve C_2. At the point, $\wp$ functions of genus two are defined as

$$\wp_{11} = \frac{f(x_1, x_2) - 2y_1 y_2}{(x_1 - x_2)^2}, \qquad \wp_{12} = x_1 x_2, \qquad \wp_{22} = x_1 + x_2, \tag{3.3}$$

where $f(x, z) := \sum_{j=0}^{2} x^j z^j (\lambda_{2j+1}(x + z) + 2\lambda_{2j})$. It is known that there is a global function over $\mathbf{C}^2$ such that

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma, \tag{3.4}$$

which is the σ function of genus two.

3.1 Continuous Toda equation

Though there were found solutions of the continuous Toda equation in terms of the θ function related to a hyperelliptic curve of genus g in [5], in this article we give another type of expression of solutions in terms of σ functions related to a curve with genus two.

The additive formula of σ function of genus two is given by [2],

$$\frac{\sigma(v+u)\sigma(v-u)}{[\sigma(v)\sigma(u)]^2} = -(\wp_{11}(u) - \wp_{11}(v) + \wp_{12}(u)\wp_{22}(v) - \wp_{12}(v)\wp_{22}(u)). \tag{3.5}$$

By letting

$$Q(u,v) := -(\wp_{11}(u) - \wp_{11}(v) + \wp_{12}(u)\wp_{22}(v) - \wp_{12}(v)\wp_{22}(u)), \tag{3.6}$$

we have

$$-\frac{\partial^2}{\partial u_i \partial u_j} \log(Q(u,v)) = \wp_{ij}(u+v) - 2\wp_{ij}(u) + \wp_{ij}(u-v). \tag{3.7}$$

Let us fix $u = nu_0 + t$, $v = u_0$, constant numbers $b_{ij} := \wp_{ij}(u_0)$ and

$$\Delta := \frac{\partial^2}{\partial t_1 \partial t_1} + b_{22}\frac{\partial^2}{\partial t_1 \partial t_2} + b_{12}\frac{\partial^2}{\partial t_2 \partial t_2}. \tag{3.8}$$

Then we have a relation,

$$\begin{aligned} &Q((n+1)u_0 + t, u_0) - 2Q(nu_0 + t, u_0) + Q((n-1)u_0 + t, u_0) \\ &\qquad = -\Delta \log Q(nu_0 + t, u_0). \end{aligned} \tag{3.9}$$

By considering the relations (3.3) we let u_0 correspond to a point $((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)) \in \mathrm{Sym}^2 C_2$ and then have

$$b_{22} = \bar{x}_1 + \bar{x}_2, \qquad b_{12} = \bar{x}_1 \bar{x}_2. \tag{3.10}$$

If the points are mutually conjugate or identical, *i.e.*, $\bar{x}_1 \equiv \bar{x}_2$,

$$\Delta = \left(\frac{\partial}{\partial t_1} + \bar{x}_1 \frac{\partial}{\partial t_2}\right)^2. \tag{3.11}$$

Hence for $t := t_1 + t_2/\bar{x}_1$,

$$q_n := \log Q(nc + t, c), \tag{3.12}$$

obeys the continuous Toda equation,

$$-\frac{d^2}{dt^2} q_n = \mathrm{e}^{q_{n+1}} - 2\mathrm{e}^{q_n} + \mathrm{e}^{q_{n-1}}. \tag{3.13}$$

As we showed in the genus one case, this genus two solution also comes from the addition formula (3.5).

3.2 Discrete Toda equation

We give relations between the discrete Toda equation and ψ functions of genus two as follows. Generalizations of the ψ function in (2.8) to genus two curves are given by two different definitions; one is defined over the Jacobi variety J_2 and another is defined over the curve C_2. The former is studied by Kanayama [8] and the latter is investigated by Grant, Cantor, Ônishi and this author (see the references in [13]). The definition by Kanayama is [8]

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma(u)^{n^2}}. \tag{3.14}$$

Further he showed that ψ_n obeys the same recursion relation as (2.9), viz

$$\psi_{n+m}\psi_{m-n} = \begin{vmatrix} \psi_{m-1}\psi_n & \psi_m\psi_{n+1} \\ \psi_m\psi_{n-1} & \psi_{m+1}\psi_n \end{vmatrix}, \tag{3.15}$$

basically using the the addition formula (3.5). Hence ψ_k obeys a relation which has the same form as (2.12). Kanayama gave the explicit forms of ψ_1, ψ_2, ψ_3 and ψ_4 in terms of $\wp$ functions (3.3) in [8]. We can compute an explicit form of any ψ_n as a rational function of the affine coordinates (x_1, y_2) and (x_2, y_2) of the curves $\mathrm{Sym}^2 C_2$ even though it is too large to give its explicit form here.

Due to its form, it is obvious that (3.15) is also related to the discrete Toda equation. For mutually prime integers p, q and an integer n_0, we define quantities,

$$\phi_i^{\,j} := \psi_{n_0+pi+qj}, \tag{3.16}$$

$\delta := \psi_q/\psi_p$ and $c\left(1-\delta^2\right) = \psi_{p+q}\psi_{p-q}/\psi_p^2$. Then (3.15) becomes

$$\delta^{-2}\phi_i^{\,j+1}\phi_i^{\,j-1} + c\left(1-\delta^{-2}\right)\phi_i^{\,j}\phi_i^{\,j} - \phi_{i+1}^{\,j}\phi_{i-1}^{\,j} = 0. \tag{3.17}$$

Hence we have a solution of the discrete Toda equation (2.21) in [7] as shown in Section 2.2.

As a simple Abel variety has a division field as its endomorphism in the category of the Abel variety as it is known as Poincaré's complete reducibility theorem [11]. Hence, even though the Jacobi variety J_2 is two-dimensional, an isometry $\varphi : J_2 \to J_2$ is characterized by an integer. The zeros of ψ_n belonging to $\mathbf{Z}/n\mathbf{Z}$ determine the isometry. Thus, as long as we deal with isometries of Jacobi variety, an extension of the ψ_n functions to functions with double-index must fail. It implies that (3.16) is a natural in the sense of a realization of the discrete equation in category of the Abel variety.

3.3 Ultradiscrete Toda equation

We consider the Jacobi variety J_2 as a commutative ring and its localization ring R_{u_0} and/or a field K_{u_0} related to R_{u_0}. Similar to the case of genus one, we deal with a point of curve satisfying

$$c\left(1-\delta^2\right) = \frac{\psi_{p+q}(u)\psi_{p-q}(u)}{\psi_p(u)^2} \in R_u^\times. \tag{3.18}$$

By letting

$$f_i^{\ j} := -\mathrm{val}\left(\frac{\phi_i^{\ j+1}\phi_i^{\ j-1}}{\phi_i^{\ j}\phi_i^{\ j}}\right), \qquad d := -\mathrm{val}\left(\delta^2\right), \tag{3.19}$$

and being supposed that for all of i and j, the valuations of the additions of f's expressed by the minimal functions like the equality case in the second relation of (2.28), we find a solution of the ultradiscrete Toda equation in [15],

$$\begin{aligned} &f_i^{\ j+1} - 2f_i^{\ j} + f_i^{\ j-1} \\ &\quad = \max\left(0, f_{i+1}^{\ j} + d\right) - 2\max\left(0, f_i^{\ j} + d\right) + \max\left(0, f_{i-1}^{\ j} + d\right). \end{aligned} \tag{3.20}$$

Even though the case of genus one gives us trivial solutions, genus two case is expected to provide us nontrivial solutions because it has larger degree of freedom than the elliptic curve case.

4 Discussion

In this article we have considered the relations between the Toda equations in the continuous, discrete and ultradiscrete levels and σ functions of genera one and two. We showed that these solutions, in principle, come from the addition formulae of the algebraic functions of algebraic curves of genera one and two.

As we started from the curves, all of the solutions are expressed by points at curves (2.1) and (3.1). As we gave some explicit solutions related to elliptic curves, we can basically find explicit forms of the other solutions in terms of points of the curves even of genus two, though they might be slightly complicated. As a next step, we should give more explicit computations of the ψ functions on the genus two case by finding a nicer strategy to handle the huge polynomials. However, it is remarkable that in our solutions, there do not appear excess parameters except the coefficients λ_i, $i = 0, 1, \ldots, 4$ in (2.1) and (3.1). In other words we have no ambiguity for the parameters even in genus two case and it means that we are free from the so-called Schottky problem. This contrasts with the solutions in terms of the θ functions over the Jacobi variety [5, 17]. Of course as it might be difficult to deal with the hyperelliptic integrals, thus we should select the methods according to the circumstances.

Further it is interesting that the ultradiscrete equations can be defined on the Jacobi varieties associated with nondegenerate algebraic curves over a field with character zero using the concept of discrete valuation. (In [14] we show that the ultradiscrete equations can be defined over fields with nonvanishing character.) It means that, if we find a recursion relation over an algebraic variety, we might have its ultradiscrete version by taking its discrete valuation.

Finally we comment on the higher genus case. Unfortunately since the addition formula is not simple [1, 4], we could not deal with σ functions associated with a curve with a higher genus as mentioned above. We hope that we can obtain such solutions in future. We note that the paper [4] may have some effects on the study.

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Appendix

Let us deal with $y^2 = x^3 + 1/4$, $y^2 = x^3 - x$ and $y^2 = x^2(x + 1/4)$ and show explicit function forms of their ψ functions.

A.1. $y^2 = x^3 + 1/4$

$$\psi_1 = 1, \tag{A.1}$$

$$\psi_2 = -2y, \tag{A.2}$$

$$\psi_3 = 3x(1+x)\left(1 - x + x^2\right) = 3x\left(1 + x^3\right), \tag{A.3}$$

$$\psi_4 = \psi_2\left(-1 + 10x^3 + 2x^6\right), \tag{A.4}$$

$$\psi_5 = -1 - 25x^3 - 15x^6 + 95x^9 + 5x^{12}, \tag{A.5}$$

$$\begin{aligned}\psi_6 &= \psi_2\psi_3\left(-2 + x^3\right)\left(1 - 3x + 3x^2 + x^3\right)\\ &\quad \times \left(1 + 3x + 6x^2 + 11x^3 + 12x^4 - 3x^5 + x^6\right),\end{aligned} \tag{A.6}$$

$$\begin{aligned}\psi_7 &= \left(1 - x^3 + 7x^6\right)\\ &\quad \times \left(1 - 48x^3 - 741x^6 - 1924x^9 - 363x^{12} + 141x^{15} + x^{18}\right),\end{aligned} \tag{A.7}$$

$$\begin{aligned}\psi_8 &= \psi_4\left(-1 - 104x^3 - 952x^6 - 4124x^9 - 3430x^{12}\right.\\ &\quad \left. - 1544x^{15} - 7336x^{18} + 616x^{21} + 2x^{24}\right),\end{aligned} \tag{A.8}$$

$$\begin{aligned}\psi_9 &= 3\psi_3\left(1 - 3x^2 + x^3\right)\left(1 + 3x^2 + 2x^3 + 9x^4 + 3x^5 + x^6\right)\\ &\quad \times \left(1 + 9x^2 + 3x^3 + 18x^5 - 24x^6 + 9x^8 + x^9\right)\\ &\quad \times \left(1 - 9x^2 + 6x^3 + 81x^4 - 45x^5 - 39x^6 + 324x^7\right.\\ &\quad + 153x^8 - 142x^9 + 486x^{10} + 396x^{11} + 582x^{12}\\ &\quad \left. + 324x^{13} + 198x^{14} - 48x^{15} + 81x^{16} - 9x^{17} + x^{18}\right),\end{aligned} \tag{A.9}$$

$$\begin{aligned}\psi_{10} &= \frac{1}{2}\psi_2\psi_5\left(1 - 177x^3 - 474x^6 - 7070x^9 - 104805x^{12}\right.\\ &\quad - 542232x^{15} - 862941x^{18} - 1404072x^{21} - 368055x^{24}\\ &\quad \left. + 29380x^{27} - 55284x^{30} + 1173x^{33} + x^{36}\right),\end{aligned} \tag{A.10}$$

$$\begin{aligned}\psi_{11} &= -1 - 242x^3 + 605x^6 + 102729x^9 + 2270301x^{12} + 17393277x^{15}\\ &\quad + 59389374x^{18} + 189881835x^{21} + 1106263389x^{24} + 4869514969x^{27}\\ &\quad + 10595519759x^{30} + 8054721004x^{33} - 22319781x^{36} - 4760052033x^{39}\\ &\quad - 8579472693x^{42} - 1596123771x^{45} + 66133914x^{48} - 62045313x^{51}\\ &\quad - 1153603x^{54} + 23221x^{57} + 11x^{60},\end{aligned} \tag{A.11}$$

$$\begin{aligned}\psi_{12} &= \psi_3\psi_4\left(-2 + x^3\right)\left(1 - 3x + 3x^2 + x^3\right)\\ &\quad \times \left(1 + 3x + 6x^2 + 11x^3 + 12x^4 - 3x^5 + x^6\right)\\ &\quad \times \left(-2 - 32x^3 - 84x^6 - 134x^9 + x^{12}\right)\end{aligned}$$

$$\begin{aligned}
&\times \left(1 + 6x + 12x^2 + 4x^3 + 45x^4 + 36x^5 + 60x^6\right.\\
&\left.+ 72x^7 - 45x^8 + 58x^9 - 48x^{10} + 12x^{11} + x^{12}\right)\\
&\times \left(1 - 6x + 24x^2 - 64x^3 + 75x^4 + 456x^5 - 620x^6 + 252x^7 + 2070x^8\right.\\
&- 1618x^9 - 3072x^{10} + 3216x^{11} + 4003x^{12} - 9696x^{13} + 1416x^{14}\\
&+ 11396x^{15} + 1548x^{16} - 5058x^{17} + 460x^{18} + 1632x^{19} + 1653x^{20}\\
&\left.+ 692x^{21} + 192x^{22} - 12x^{23} + x^{24}\right), \qquad \text{(A.12)}
\end{aligned}$$

$$\begin{aligned}
\psi_{13} = &\left(1 + 16x^3 + 96x^6 + 13x^9 + 13x^{12}\right)\\
&\times \left(1 - 354x^3 - 17247x^6 + 92420x^9 - 6264417x^{12} - 91630974x^{15}\right.\\
&- 414038735x^{18} - 631690011x^{21} + 3596512338x^{24} + 43118516972x^{27}\\
&+ 215967505719x^{30} + 533661527514x^{33} + 582732421153x^{36}\\
&+ 284118813696x^{39} + 450924775284x^{42} + 1313707269872x^{45}\\
&+ 1846766455056x^{48} + 403474854555x^{51} - 263110973327x^{54}\\
&- 22534762701x^{57} + 685417938x^{60} - 111537892x^{63} - 798438x^{66}\\
&\left.+ 5748x^{69} + x^{72}\right), \qquad \text{(A.13)}
\end{aligned}$$

$$\begin{aligned}
\psi_{14} = &\,\psi_2\psi_7 \left(1 - 48x^3 - 741x^6 - 1924x^9 - 363x^{12} + 141x^{15} + x^{18}\right)\\
&\times \left(1 + 504x^3 + 2421x^6 + 5676x^9 + 166356x^{12} + 3098475x^{15} + 22597638x^{18}\right.\\
&+ 56826270x^{21} - 73281168x^{24} - 582904249x^{27} - 862862121x^{30}\\
&+ 133470252x^{33} + 317907519x^{36} - 632536713x^{39} - 77646699x^{42}\\
&\left.- 41502855x^{45} - 2997252x^{48} + 8847x^{51} + x^{54}\right), \qquad \text{(A.14)}
\end{aligned}$$

$$\begin{aligned}
\psi_{15} = &\,\psi_3\psi_5 \left(-5 + 65x^3 + 685x^6 + 3410x^9 + 11425x^{12}\right.\\
&\left.+ 5735x^{15} + 3145x^{18} - 520x^{21} + x^{24}\right)\\
&\times \left(1 - 6x + 6x^2 + 44x^3 + 21x^4 - 21x^5 + 676x^6 + 9x^7 - 9x^8 + 569x^9\right.\\
&+ 2841x^{10} - 2841x^{11} - 1694x^{12} + 13119x^{13} - 13119x^{14} + 10019x^{15}\\
&- 4284x^{16} + 4284x^{17} + 4591x^{18} - 1446x^{19} + 1446x^{20}\\
&\left.- 496x^{21} - 24x^{22} + 24x^{23} + x^{24}\right)\\
&\times \left(1 + 6x + 30x^2 + 124x^3 + 279x^4 - 495x^5 + 3036x^6 + 2871x^7 - 2790x^8\right.\\
&+ 60959x^9 - 13686x^{10} - 19695x^{11} + 469946x^{12} - 200034x^{13} + 128295x^{14}\\
&+ 602229x^{15} - 2440926x^{16} + 3056445x^{17} - 422129x^{18} - 9809094x^{19}\\
&+ 18607485x^{20} + 20165779x^{21} + 8262864x^{22} + 74286585x^{23} + 94839246x^{24}\\
&+ 74286585x^{23} + 94839246x^{24} + 71549460x^{25} + 150594579x^{26}\\
&+ 118349119x^{27} - 3156510x^{28} + 30275751x^{29} - 36357239x^{30} - 138954870x^{31}\\
&- 1389186x^{32} + 73952184x^{33} + 20894985x^{34} - 28859229x^{35} + 22894661x^{36}\\
&- 28859229x^{35} + 22894661x^{36} + 16500675x^{37} - 2444511x^{38} - 2237686x^{39}\\
&+ 1693800x^{40} + 672156x^{41} + 324606x^{42} + 58950x^{43} + 11034x^{44}\\
&\left.- 416x^{45} + 600x^{46} - 24x^{47} + x^{48}\right). \qquad \text{(A.15)}
\end{aligned}$$

A.2. $y^2 = x(x^2 - 1)$

$$\psi_1 = 1, \tag{A.16}$$
$$\psi_2 = -2y, \tag{A.17}$$
$$\psi_3 = 3(-2 + x)x^2(2 + x), \tag{A.18}$$
$$\psi_4 = -4yx^2\left(-6 - 15x^2 + x^4\right), \tag{A.19}$$
$$\psi_5 = x^4\left(-192 + 1632x^2 - 496x^4 - 220x^6 + 5x^8\right), \tag{A.20}$$
$$\begin{aligned}\psi_6 &= -6y(-2 + x)x^6(2 + x)\\ &\quad\times\left(-336 + 912x^2 - 1348x^4 - 100x^6 + x^8\right),\end{aligned} \tag{A.21}$$
$$\begin{aligned}\psi_7 &= x^8\left(27648 + 483840x^2 - 2951424x^4 + 2595456x^6 - 1101888x^8\right.\\ &\quad\left.+ 447840x^{10} - 31376x^{12} - 1544x^{14} + 7x^{16}\right),\end{aligned} \tag{A.22}$$
$$\begin{aligned}\psi_8 &= -8yx^{10}\left(-6 - 15x^2 + x^4\right)\\ &\quad\times\left(-18432 + 603648x^2 - 2432640x^4 + 2577312x^6\right.\\ &\quad\left.- 702392x^8 + 47744x^{10} - 23070x^{12} - 412x^{14} + x^{16}\right),\end{aligned} \tag{A.23}$$
$$\begin{aligned}\psi_9 &= -3(-2 + x)x^{14}(2 + x)\left(16367616 - 154607616x^2 + 1527054336x^4\right.\\ &\quad - 5301780480x^6 + 4162000896x^8 + 567207936x^{10} - 1938695936x^{12}\\ &\quad + 731321472x^{14} - 1489472x^{16} + 5367072x^{18} - 164000x^{20}\\ &\quad\left.- 2316x^{22} + 3x^{24}\right).\end{aligned} \tag{A.24}$$

A.3. $y^2 = x^2(x + 1/4)$

$$\psi_1 = 1, \tag{A.25}$$
$$\psi_2 = -2y, \tag{A.26}$$
$$\psi_3 = x^3(1 + 3x), \tag{A.27}$$
$$\psi_4 = -2yx^5(1 + 2x), \tag{A.28}$$
$$\psi_5 = x^{10}\left(1 + 5x + 5x^2\right), \tag{A.29}$$
$$\psi_6 = -2yx^{14}(1 + x)(1 + 3x), \tag{A.30}$$
$$\psi_7 = x^{21}\left(1 + 7x + 14x^2 + 7x^3\right), \tag{A.31}$$
$$\psi_8 = -2yx^{27}(1 + 2x)\left(1 + 4x + 2x^2\right), \tag{A.32}$$
$$\psi_9 = x^{36}(1 + 3x)\left(1 + 6x + 9x^2 + 3x^3\right), \tag{A.33}$$
$$\psi_{10} = -2yx^{44}\left(1 + 3x + x^2\right)\left(1 + 5x + 5x^2\right), \tag{A.34}$$
$$\psi_{11} = x^{55}\left(1 + 11x + 44x^2 + 77x^3 + 55x^4 + 11x^5\right), \tag{A.35}$$
$$\psi_{12} = -2yx^{65}(1 + x)(1 + 2x)(1 + 3x)\left(1 + 4x + x^2\right), \tag{A.36}$$
$$\psi_{13} = x^{78}\left(1 + 13x + 65x^2 + 156x^3 + 182x^4 + 91x^5 + 13x^6\right), \tag{A.37}$$
$$\psi_{14} = -2yx^{90}\left(1 + 5x + 6x^2 + x^3\right)\left(1 + 7x + 14x^2 + 7x^3\right), \tag{A.38}$$
$$\psi_{15} = x^{105}(1 + 3x)\left(1 + 5x + 5x^2\right)\left(1 + 7x + 14x^2 + 8x^3 + x^4\right), \tag{A.39}$$
$$\psi_{16} = -2yx^{119}(1 + 2x)\left(1 + 4x + 2x^2\right)\left(1 + 8x + 20x^2 + 16x^3 + 2x^4\right). \tag{A.40}$$

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