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Soliton Solutions of the $N = 2$ Supersymmetric KP Equation

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Abstract

The $N = 2$ super-KP equation associated with nonstandard flows is bilinearized using the Hirota method and soliton solutions are obtained. The bilinearization has been done for component fields and its KdV limit is discussed by comparing the soliton solutions obtained by this procedure with those found from the $N = 1$ superspace formalism. The equivalence of these two procedures in the KdV limit is observed.

1 Introduction

Supersymmetry is mathematically formulated by extending ordinary space to include anticommuting or Grassmann variables in order that bosons and fermions may be treated in a unified way. Interestingly a number of bosonic integrable hierarchies can be extended to the supersymmetric framework. The connection of integrable models with supersymmetry was established with the supersymmetrization of bosonic integrable models such as the sine-Gordon equation [1], the KP hierarchy [2], the KdV hierarchy [2], the Boussinesq equation [3] and a number of other systems. Moreover many of the methods developed in the study of integrable models were applied to the supersymmetric regime among them being the Bäcklund transformation [4], the Painlevé analysis [5], τ functions [6] and Darboux transformation [7], to mention some of them. In this context the pioneering work was that of Manin and Radul [2], who formulated the supersymmetric version of the KP hierarchy in $N = 1$ superspace employing the supersymmetric extension of the Lax formalism and this was based on the odd parity superLax operator. Later an even parity superLax operator associated with the $N = 2$ super-KP was obtained and the equations of motion were obtained from the nonstandard flow [8]. The bihamiltonian structures of the $N = 2$ super-KP hierarchies have been obtained by suitably formulating the Gelfand–Dikii method for supersymmetric systems [8, 9]. While much work has been done in the field of supersymmetric integrable hierarchies specially from the point of view of their algebraic structure, attempts to obtain the soliton solutions of such systems, in particular systems with $N = 2$ supersymmetry, have been few in number. In this paper an attempt is made to obtain the soliton solutions of the $N = 2$ super-KP equation associated with nonstandard flows applying the Hirota method.

The dynamical equations for the lowest spin superfields u_{-1} and u_0 in the nonstandard flow $N = 2$ super-KP hierarchy constitute the $N = 2$ KP equation. They are obtained from the even parity superLax operator [8]

$$L = D^2 + \sum_{i=0}^{\infty} u_{i-1}(X) D^{-i} \tag{1.1}$$

in conjunction with the nonstandard flow equation

$$\frac{dL}{dt_n} = [L^n_{>0}, L] \tag{1.2}$$

and have the explicit forms [8]

$$\begin{aligned} &\partial_t u_{-1} - \frac{1}{4} u_{-1}^{[6]} + \frac{1}{2}\left(u_{-1}^3\right)^{[2]} - \frac{3}{2}\left(u_0 u_{-1}^{[1]}\right)^{[2]} - \frac{3}{4}\partial_y^2 u_{-1}^{[-2]} \\ &\quad - \frac{3}{2} u_{-1}^{[2]} \partial_y u_{-1}^{[-2]} + \frac{3}{2} u_{-1}^{[1]} \partial_y u_0^{[-2]} + \frac{3}{2} u_0 \partial_y u_{-1}^{[-1]} - 3 u_0 \partial_y u_0^{[-2]} = 0, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\partial_t u_0 - \frac{1}{4} u_0^{[6]} - \frac{3}{2}\left(u_0 u_0^{[1]}\right)^{[2]} + \frac{3}{2}\left(u_0 u_{-1}^{[2]}\right)^{[2]} + \frac{3}{2}\left(u_0 u_{-1}^2\right)^{[2]} \\ &\quad - \frac{3}{4}\partial_y^2 u_0^{[-2]} - \frac{3}{2}\left(u_0 \partial_y u_0^{[-2]}\right)^{[1]} - \frac{3}{2} u_0^{[2]} \partial_y u_{-1}^{[-2]} - \frac{3}{2} u_0 \partial_y u_{-1} = 0, \end{aligned} \tag{1.4}$$

where $[\,n\,]$ denotes the nth derivative with respect to the superderivative D defined by

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}. \tag{1.5}$$

Equations (1.3) and (1.4) are nonlocal, supersymmetric evolution equations. Here the superfield u_{-1} is a bosonic superfield, while u_0 is a fermionic one, having conformal spins 1 and 3/2 respectively.

Among the various techniques which have been applied to generate soliton solutions of bosonic integrable systems, several have found extensions in the supersymmetric framework. Of these the Hirota method [10] has also been used extensively to find soliton solutions of a number of supersymmetric integrable equations. In bosonic integrable systems, this formalism has been widely applied to obtain the soliton solutions of a large number of nonlinear evolution equations (see for example [11–20]). However, in its supersymmetric extension the Hirota method has been applied mainly to $N = 1$ supersymmetric integrable systems [21–25]. In the context of integrable systems with $N = 2$ supersymmetry, the $N = 2$ KdV [28–30] in $N = 1$ superspace [26] has been bilinearized using the supersymmetric extension of the Hirota formalism [27].

Unlike the $N = 2$ KdV equation, the $N = 2$ super-KP equation could not be bilinearized using the super analogue of the Hirota operator. This difficulty with the $N = 2$ super-KP equation, in contrast to the $N = 2$ super-KdV equation, may arise due to nonlocality and the involvement of space-time dependence in the dynamical equation in a complicated way. However, in the form of evolution equations of the component fields, the $N = 2$ super-KP is in fact bilinearizable and soliton solutions can be found. In achieving this goal, we were guided by the bilinearization of the $N = 2$ super-KdV equation (characterized by the parameter $a = -2$), for which identical soliton solutions are obtained irrespective of the fact whether the Hirota method or its $N = 1$ superspace analogue is used.

In this paper the $N = 2$ super KP equation is bilinearized following the Hirota method and N soliton solutions are obtained. In the next section, *i.e.* Section 2, the bilinearization of the $N = 2$ super-KP is discussed. In Section 3 we discuss its soliton solutions. In Section 4 the soliton solutions of the $N = 2$ KdV equation ($a = -2$), obtained by the application of the Hirota method to its component equations is compared with those obtained from the Hirota formalism in $N = 1$ superspace [27]. Section 5 is the concluding one.

2 Bilinear forms of $N = 2$ KP equation

The $N = 2$ KP equation can be rewritten in a convenient form after a rescaling of $t = -4t$, $y = -\frac{1}{2}y$, $u_{-1} = -\frac{1}{2}u_{-1}$ and $u_0 = -\frac{1}{2}u_0$ so that from (1.3) and (1.4), we have

$$\begin{aligned}
&\partial_t u_{-1} + u_{-1}^{[6]} - \frac{1}{2}\left(u_{-1}^3\right)^{[2]} + 3\left(u_{-1}^{[1]}u_0\right)^{[2]} + 12\partial_y^2 u_{-1}^{[-2]} \\
&\quad + 6u_{-1}^{[2]}\partial_y u_{-1}^{[-2]} - 6u_{-1}^{[1]}\partial_y u_0^{[-2]} - 6u_0\partial_y u_{-1}^{[-1]} + 12u_0\partial_y u_0^{[-2]} = 0, \qquad (2.1)\\
&\partial_t u_0 + u_0^{[6]} - 3\left(u_0 u_0^{[1]}\right)^{[2]} + 3\left(u_0 u_{-1}^{[2]}\right)^{[2]} - \frac{3}{2}\left(u_0 u_{-1}^2\right)^{[2]} \\
&\quad + 12\partial_y^2 u_0^{[-2]} + 6\left(u_0\partial_y u_0^{[-2]}\right)^{[1]} + 6u_0^{[2]}\partial_y u_{-1}^{[-2]} + 6u_0\partial_y u_{-1} = 0. \qquad (2.2)
\end{aligned}$$

The superfields u_{-1} and u_0 have the following component form

$$u_{-1} = u_{-1}^b + \theta u_{-1}^f \qquad (2.3)$$

and

$$u_0 = u_0^f + \theta u_0^b. \qquad (2.4)$$

The time evolution of the component fields u_{-1}^b, u_{-1}^f, u_0^f and u_0^b consequently can be obtained from the dynamical equations for the superfields (2.1) and (2.2) to give a set of four coupled partial differential equations, explicitly

$$\begin{aligned}
&\partial_t u_{-1}^b + \partial_x^3 u_{-1}^b - \frac{1}{2}\partial_x\left(u_{-1}^b\right)^3 + 3\partial_x\left(u_{-1}^f u_0^f\right) + 12\partial_y^2\partial_x^{-1}u_{-1}^b - 6u_{-1}^f\partial_y\partial_x^{-1}u_0^f \\
&\quad + 6\left(\partial_x u_{-1}^b\right)\partial_y\partial_x^{-1}u_{-1}^b - 6u_0^f\partial_y\partial_x^{-1}u_{-1}^f + 12u_0^f\partial_y\partial_x^{-1}u_0^f = 0, \qquad (2.5)\\
&\partial_t u_{-1}^f + \partial_x^3 u_{-1}^f - \frac{3}{2}\partial_x\left(u_{-1}^{b^2}u_{-1}^f\right) + 3\partial_x\left[(\partial_x u_{-1}^b)u_0^f\right] - 3\partial_x\left(u_0^b u_{-1}^f\right) \\
&\quad + 12\partial_y^2\partial_x^{-1}u_{-1}^f + 6(\partial_x u_{-1}^b)\partial_y\partial_x^{-1}u_{-1}^f + 6(\partial_x u_{-1}^f)\partial_y\partial_x^{-1}u_{-1}^b + 6u_0^f\partial_y u_{-1}^b \\
&\quad + 6u_{-1}^f\partial_y\partial_x^{-1}u_0^b - 6u_0^b\partial_y\partial_x^{-1}u_{-1}^f - 6(\partial_x u_{-1}^b)\partial_y\partial_x^{-1}u_0^f + 12u_0^b\partial_y\partial_x^{-1}u_0^f \\
&\quad - 12u_0^f\partial_y\partial_x^{-1}u_0^b = 0, \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
&\partial_t u_0^f + \partial_x^3 u_0^f - 3\partial_x\left(u_0^b u_0^f\right) + 3\partial_x\left(u_0^f \partial_x u_{-1}^b\right) - \frac{3}{2}\partial_x\left(u_{-1}^{b2} u_0^f\right) \\
&\quad + 12\partial_y^2\partial_x^{-1}u_0^f + 6u_0^b\partial_y\partial_x^{-1}u_0^f - 6u_0^f\partial_y\partial_x^{-1}u_0^b + 6\left(\partial_x u_0^f\right)\partial_y\partial_x^{-1}u_{-1}^b \\
&\quad + 6u_0^f\partial_y u_{-1}^b = 0,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\partial_t u_0^b + \partial_x^3 u_0^b - 3\partial_x u_0^{b2} + 3\partial_x\left(u_0^f\partial_x u_0^f\right) + 3\partial_x\left(u_0^b\partial_x u_{-1}^b\right) \\
&\quad - 3\partial_x\left(u_0^f\partial_x u_{-1}^f\right) - \frac{3}{2}\partial_x\left(u_0^b u_{-1}^{b2}\right) - 3\partial_x\left(u_{-1}^b u_{-1}^f u_0^f\right) + 12\partial_y^2\partial_x^{-1}u_0^b \\
&\quad + 6(\partial_x u_0^f)\partial_y\partial_x^{-1}u_0^f + 6u_0^f\partial_y u_0^f + 6(\partial_x u_0^b)\partial_y\partial_x^{-1}u_{-1}^b - 6(\partial_x u_0^f)\partial_y\partial_x^{-1}u_{-1}^f \\
&\quad + 6u_0^b\partial_y u_{-1}^b - 6u_0^f\partial_y u_{-1}^f = 0.
\end{aligned} \tag{2.8}$$

To cast the equations (2.5), (2.6), (2.7) and (2.8) into bilinear form consider the following transformations of the component fields:

$$u_{-1}^b = 2\partial_x \log\frac{\tilde\tau_1}{\tilde\tau_2}, \qquad u_{-1}^f = 2\xi\partial_x^2\log\frac{\tilde\tau_1}{\tilde\tau_2}, \tag{2.9}$$

$$u_0^f = 2\xi\partial_x^2\log\tilde\tau_1, \qquad u_0^b = 2\partial_x^2\log\tilde\tau_1, \tag{2.10}$$

where ξ is a spin $-\frac{1}{2}$ Grassmann odd parameter required in order that the fields have the correct conformal dimension. The τ functions in (2.9) and (2.10) are bosonic fields in contrast to those required to bilinearize the $N=2$ KdV in $N=1$ superspace where the τ functions are bosonic superfields [27]. The justification of choosing the fermionic fields as in (2.9) and (2.10) is discussed in detail in Section 4.

Substitution of (2.9) and (2.10) in (2.5), (2.6), (2.7) and (2.8) leads to the following bilinear forms

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4 + 12\mathbf{D}_y^2)(\tilde\tau_1\bullet\tilde\tau_1) = 0, \tag{2.11}$$

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4 + 12\mathbf{D}_y^2)(\tilde\tau_2\bullet\tilde\tau_2) = 0, \tag{2.12}$$

$$(\mathbf{D}_x^2 - 2\mathbf{D}_y)(\tilde\tau_1\bullet\tilde\tau_2) = 0, \tag{2.13}$$

where the Hirota operator $\mathbf{D}$ is defined by

$$\mathbf{D}_x^n(f\bullet g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_1=x_2=x}. \tag{2.14}$$

Note that (2.11) is the bilinear form for the bosonic KP equation [15]. The $N=2$ KP equation, in contrast, is represented by two additional bilinear equations.

3 Soliton solutions

The τ function for the N-soliton solutions for the $N=2$ KP equation can be written as

$$\tilde\tau_1 = \sum_{\mu_i=0,1}\exp\left(\sum_{i,j=1}^N\phi(i,j)\mu_i\mu_j + \sum_{i=1}^N\mu_i(\eta_i + \log\alpha_i)\right) \qquad (i<j), \tag{3.1}$$

where $\exp[\phi(i,j)]$ and α_i are the coefficients to be determined. For convenience we introduce $A_{ij} = \exp[\phi(i,j)]$. We may write the second τ function, namely $\tilde\tau_2$, by replacing α_i by β_i and A_{ij} by B_{ij}.

In particular the τ functions for one-soliton solution follow from (3.1) as

$$\tilde{\tau}_1 = 1 + \alpha e^{\eta} \tag{3.2}$$

and

$$\tilde{\tau}_2 = 1 + \beta e^{\eta} \tag{3.3}$$

with

$$\eta = k_x x + k_y y + \omega t \tag{3.4}$$

and α and β being constants which are determined from the initial conditions. It is easily shown that the substitution of (3.2) and (3.3) back in the first two equations, namely (2.11) and (2.12) of the Hirota form of the $N = 2$ super-KP, yields the dispersion relation

$$\omega k_x + k_x^4 + 12 k_y^2 = 0. \tag{3.5}$$

The dispersion relation is identical to that of the bosonic KP equation.

From the third equation of this set, (2.13), we find that k_y is related to k_x through

$$k_y = \frac{1}{2}\frac{(\alpha+\beta)}{(\alpha-\beta)} k_x^2. \tag{3.6}$$

This imposes the following constraint between α and β parameters

$$\beta \neq \alpha \tag{3.7}$$

and reduces to the soliton solution of $N = 2$ KdV equation [27] if $\alpha = -\beta$. This is obvious from (3.6) as under this condition k_y becomes zero.

Despite the τ functions and the Hirota operators being bosonic, the supersymmetric structure is inherent in the soliton solutions because of the Grassmann odd factor ξ. This is clearly seen in the explicit forms of the one-soliton solution for the $N = 2$ KP which are found by straightforward substitution of the τ functions from (3.2) and (3.3) in the transformation equations (2.9) and (2.10). For the components of the bosonic superfield u_{-1}, (2.3), we have

$$u_{-1}^b = k_x \sinh\frac{1}{2}(\gamma_0 - \delta_0)\,\mathrm{sech}\frac{1}{2}(\phi+\gamma_0)\,\mathrm{sech}\frac{1}{2}(\phi+\delta_0), \tag{3.8}$$

$$\begin{aligned} u_{-1}^f = {} & -\frac{1}{2}k_x^2 \xi \sinh\frac{1}{2}(\gamma_0-\delta_0)\,\mathrm{sech}\frac{1}{2}(\phi+\gamma_0)\,\mathrm{sech}\frac{1}{2}(\phi+\delta_0) \\ & \times \left[\tanh\frac{1}{2}(\phi+\gamma_0) + \tanh\frac{1}{2}(\phi+\delta_0)\right]. \end{aligned} \tag{3.9}$$

Similarly for the fermionic superfield, u_0, the components are

$$u_0^f = \frac{1}{2}k_x^2 \xi \mathrm{sech}^2(\phi+\gamma_0) \tag{3.10}$$

and

$$u_0^b = \frac{1}{2}k_x^2 \mathrm{sech}^2(\phi+\gamma_0), \tag{3.11}$$

where we have chosen $\phi = k_x x + k_y y + \omega t$, $\alpha = \exp \gamma_0$ and $\beta = \exp \delta_0$. γ_0 and δ_0 are nonzero, real parameters. These solutions reduce to those of the $N = 2$ KdV under the restriction $\gamma_0 = \delta_0 \pm i\pi$ which is equivalent to $\beta = -\alpha$.

For two-soliton solutions we have the following τ functions

$$\tilde{\tau}_1 = 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_1\alpha_2 A_{12} e^{\eta_1+\eta_2} \tag{3.12}$$

which may be obtained by setting $N = 2$ in (3.1) and

$$\tilde{\tau}_2 = 1 + \beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + \beta_1\beta_2 B_{12} e^{\eta_1+\eta_2} \tag{3.13}$$

with $\eta_i = k_{ix}x + k_{iy}y + \omega_i t$ $(i = 1, 2)$ and α_i, β_i $(i = 1, 2)$ and A_{12}, B_{12} are unknown parameters. The parameters α_i and β_i satisfy the same relation as in the one-soliton solution, and the same dispersion relations $\omega k_{ix} + k_{ix}^4 + 12k_{iy}^2 = 0$ $(i = 1, 2)$ also follow from (2.11) and (2.12). The interaction terms A_{12}, B_{12} are additional parameters to be determined at the two-soliton level.

The two-soliton interaction terms A_{12} and B_{12} may be extracted by the substitution of the τ functions (3.12) and (3.13) in the bilinear equation (2.13) to yield

$$A_{12} = B_{12} = -\frac{(\alpha_1\beta_2 + \alpha_2\beta_1)(k_{1x} - k_{2x})^2 - 2(\alpha_1\beta_2 - \alpha_2\beta_1)(k_{1y} - k_{2y})}{(\alpha_1\alpha_2 + \beta_1\beta_2)(k_{1x} + k_{2x})^2 - 2(\alpha_1\alpha_2 - \beta_1\beta_2)(k_{1y} + k_{2y})}. \tag{3.14}$$

The vanishing of the coefficient of $e^{\eta_1+\eta_2}$ in (2.11) and (2.12), namely

$$\begin{aligned} &A_{12}[(\omega_1 + \omega_2)(k_{1x} + k_{2x}) + (k_{1x} + k_{2x})^4 + 12(k_{1y} + k_{2y})^2] \\ &\quad + [(\omega_1 - \omega_2)(k_{1x} - k_{2x}) + (k_{1x} - k_{2x})^4 + 12(k_{1y} - k_{2y})^2], \end{aligned} \tag{3.15}$$

also leads to the same interaction term A_{12} (3.14) ensuring the consistency of the ansatz for the two soliton solution.

We consider a three-soliton solution to check the consistency of the parameters determined in the one and two-soliton solutions and to ensure the integrability of the system. The explicit forms of the τ functions for the three-soliton solution of the $N = 2$ KP may be obtained from (3.1) as

$$\begin{aligned} \tilde{\tau}_1 = {} & 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_3 e^{\eta_3} + \alpha_1\alpha_2 A_{12} e^{\eta_1+\eta_2} + \alpha_1\alpha_3 A_{13} e^{\eta_1+\eta_3} \\ & + \alpha_2\alpha_3 A_{23} e^{\eta_2+\eta_3} + \alpha_1\alpha_2\alpha_3 A_{12}A_{13}A_{23} e^{\eta_1+\eta_2+\eta_3} \end{aligned} \tag{3.16}$$

and a similar form for $\tilde{\tau}_2$.

From the bilinear form, (2.13), the only nontrivial condition at the three-soliton level is the coefficient of $\exp[\eta_1 + \eta_2 + \eta_3]$, namely

$$\begin{aligned} &A_{12}A_{13}A_{23}[(\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3)(k_{1x} + k_{2x} + k_{3x})^2 - 2(\alpha_1\alpha_2\alpha_3 - \beta_1\beta_2\beta_3) \\ &\qquad \times (k_{1y} + k_{2y} + k_{3y})] \\ &\quad + A_{23}[(\alpha_1\beta_2\beta_3 + \alpha_2\alpha_3\beta_1)(k_{1x} - k_{2x} - k_{3x})^2 - 2(\alpha_1\beta_2\beta_3 - \alpha_2\alpha_3\beta_1) \\ &\qquad \times (k_{1y} - k_{2y} - k_{3y})] \\ &\quad + A_{13}[(\alpha_2\beta_1\beta_3 + \alpha_1\alpha_3\beta_2)(k_{2x} - k_{1x} - k_{3x})^2 - 2(\alpha_2\beta_1\beta_3 - \alpha_1\alpha_3\beta_2) \\ &\qquad \times (k_{2y} - k_{1y} - k_{3y})] \end{aligned}$$

$$+ A_{12}[(\alpha_3\beta_1\beta_2 + \alpha_1\alpha_2\beta_3)(k_{3x} - k_{1x} - k_{2x})^2 - 2(\alpha_3\beta_1\beta_2 - \alpha_1\alpha_2\beta_3) \times (k_{3y} - k_{1y} - k_{2y})] = 0 \tag{3.17}$$

which may be shown to be satisfied straightforwardly by using the expression for interaction terms A_{ij}.

Similarly from the bilinear forms (2.11) and (2.12), the only nontrivial term, namely the coefficient of $\exp[\eta_1 + \eta_2 + \eta_3]$ which is

$$\begin{aligned} &A_{12}A_{13}A_{23}[(\omega_1 + \omega_2 + \omega_3)(k_{1x} + k_{2x} + k_{3x}) + (k_{1x} + k_{2x} + k_{3x})^4 \\ &\quad + 12(k_{1y} + k_{2y} + k_{3y})^2] \\ &\quad + A_{23}[(\omega_1 - \omega_2 - \omega_3)(k_{1x} - k_{2x} - k_{3x}) + (k_{1x} - k_{2x} - k_{3x})^4 \\ &\quad + 12(k_{1y} - k_{2y} - k_{3y})^2] \\ &\quad + A_{13}[(\omega_2 - \omega_1 - \omega_3)(k_{2x} - k_{1x} - k_{3x}) + (k_{2x} - k_{1x} - k_{3x})^4 \\ &\quad + 12(k_{2y} - k_{1y} - k_{3y})^2] \\ &\quad + A_{12}[(\omega_3 - \omega_1 - \omega_2)(k_{3x} - k_{1x} - k_{2x}) + (k_{3x} - k_{1x} - k_{2x})^4 \\ &\quad + 12(k_{3y} - k_{1y} - k_{2y})^2], \end{aligned} \tag{3.18}$$

vanishes and thereby ensures the existence of three-soliton solutions for the $N = 2$ super KP equation.

It is seen that at the level of the three-soliton solution, the only condition that is not trivially satisfied comes from the coefficient of $\exp\left[\sum_{i=1}^{3} \eta_i\right]$. The coefficients of all other exponents are zero identically by ensuring the existence of soliton solutions of lower orders. Extending this to N soliton solutions, we find that the vanishing of the coefficient of $\exp\left[\sum_{i=1}^{N} \eta_i\right]$ indeed confirms the existence of an N soliton solution. From the τ functions for the N soliton solution, (3.1), we find that in the product $\tilde{\tau}_1 \bullet \tilde{\tau}_2$, the terms that generate the nontrivial condition may be written in the following general form

$$\begin{aligned} &1 \bullet \prod_{\substack{a,b,c=1 \\ a<b}}^{N} \beta_c A_{ab} + \sum_{i_1=1}^{N} \alpha_{i_1} e^{\eta_{i_1}} \bullet \prod_{\substack{a,b,c=1 \\ a<b \\ a,b,c \neq i_1}}^{N} \beta_c A_{ab} e^{\eta_c} \\ &\quad + \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{N} \alpha_{i_1}\alpha_{i_2} A_{i_1 i_2} e^{\eta_{i_1}+\eta_{i_2}} \bullet \prod_{\substack{a,b,c=1 \\ a<b \\ a,b,c \neq i_1,i_2}}^{N} \beta_c A_{ab} e^{\eta_c} \\ &\quad + \sum_{\substack{i_1,i_2,i_3=1 \\ i_1<i_2<i_3}}^{N} \alpha_{i_1}\alpha_{i_2}\alpha_{i_3} A_{i_1 i_2} A_{i_1 i_3} A_{i_2 i_3} e^{\eta_{i_1}+\eta_{i_2}+\eta_{i_3}} \bullet \prod_{\substack{a,b,c=1 \\ a<b \\ a,b,c \neq i_1,i_2,i_3}}^{N} \beta_c A_{ab} e^{\eta_c} \\ &\quad + \cdots + \sum_{\substack{i_1,i_2,i_3,\ldots,i_N=1 \\ i_1<i_2<i_3<\cdots<i_N}}^{N} \alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\cdots\alpha_{i_N} A_{i_1 i_2} A_{i_1 i_3} A_{i_2 i_3} \cdots A_{i_{N-1} i_N} \end{aligned}$$

$$\times\, e^{\eta_{i_1}+\eta_{i_2}+\eta_{i_3}+\cdots+\eta_{i_N}} \bullet 1. \tag{3.19}$$

Using (3.19) we find the coefficient of the term $\exp\left[\sum_{i=1}^{N}\eta_i\right]$ from the Hirota equation (2.13) to be

$$\begin{aligned}
&\prod_{\substack{a,b,c=1\\ a<b}}^{N} \beta_c A_{ab}[X^2-2Y] + \sum_{i_1=1}^{N} \prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1}}^{N} [\{2k_{i_1x}-X\}^2 - \{2k_{i_1y}-Y\}]\alpha_{i_1}A_{ab}\beta_c \\
&+ \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{N} \prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1,i_2}}^{N} [\{2(k_{i_1x}+k_{i_2x})-X\}^2 \\
&- 2\{2(k_{i_1y}+k_{i_2y})-Y\}]\alpha_{i_1}\alpha_{i_2}A_{i_1i_2}A_{ab}\beta_c \\
&+ \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N} \prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1,i_2,i_3}}^{N} [\{2(k_{i_1x}+k_{i_2x}+k_{i_3x})-X\}^2 \\
&- 2\{2(k_{i_1y}+k_{i_2y}+k_{i_3y})-Y\}]\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}A_{ab}\beta_c \\
&+\cdots+ \sum_{\substack{i_1,i_2,i_3,\ldots,i_N=1\\ i_1<i_2<i_3<\cdots<i_N}}^{N} [(k_{i_1x}+k_{i_2x}+k_{i_3x}+\cdots+k_{i_Nx})^2 \\
&- 2(k_{i_1y}+k_{i_2y}+k_{i_3y}+\cdots+k_{i_Ny})] \\
&\times \alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\cdots\alpha_{i_N}A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}\cdots A_{i_{N-1}i_N},
\end{aligned} \tag{3.20}$$

where $X = \sum_{m=1}^{N} k_{mx}$ and $Y = \sum_{n=1}^{N} k_{ny}$. In fact, at each order, the vanishing of (3.20) gives the nontrivial relation among the parameters. This identity has been verified explicitly up to four-soliton solutions.

For the other two bilinear forms for the $N = 2$ KP equation, namely (2.11) and (2.12), the expression equivalent to (3.20), *i.e.* the coefficient of the term $\exp\left[\sum_{i=1}^{N}\eta_i\right]$, is found to have the form

$$\begin{aligned}
&\prod_{\substack{a,b=1\\ a<b}}^{N} A_{ab}[X\Omega + X^4 + 12Y^2] \\
&+\sum_{i_1=1}^{N} \prod_{\substack{a,b=1\\ a<b\\ a,b\neq i_1}}^{N} [\{2k_{i_1x}-X\}\{2\omega_{i_1}-\Omega\} + \{2k_{i_1x}-X\}^4 + 12\{2k_{i_1y}-Y\}^2]A_{ab}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{i_1, i_2 = 1 \\ i_1 < i_2}}^{N} \prod_{\substack{a,b=1 \\ a<b \\ a,b \neq i_1, i_2}}^{N} [\{2(k_{i_1 x} + k_{i_2 x}) - X\}\{2(\omega_{i_1} + \omega_{i_2}) - \Omega\} \\
&+ \{2(k_{i_1 x} + k_{i_2 x}) - X\}^4 + 12\{2(k_{i_1 y} + k_{i_2 y}) - Y\}^2] A_{i_1 i_2} A_{ab} \\
&+ \sum_{\substack{i_1, i_2, i_3 = 1 \\ i_1 < i_2 < i_3}}^{N} \prod_{\substack{a,b=1 \\ a<b \\ a,b \neq i_1, i_2, i_3}}^{N} [\{2(k_{i_1 x} + k_{i_2 x} + k_{i_3 x}) - X\}\{2(\omega_{i_1} + \omega_{i_2} + \omega_{i_3}) - \Omega\} \\
&+ \{2(k_{i_1 x} + k_{i_2 x} + k_{i_3 x}) - X\}^4 + 12\{2(k_{i_1 y} + k_{i_2 y} + k_{i_3 y}) - Y\}^2] \\
&\times A_{i_1 i_2} A_{i_1 i_3} A_{i_2 i_3} A_{ab} \\
&+ \cdots + \sum_{\substack{i_1, i_2, i_3, \ldots, i_N = 1 \\ i_1 < i_2 < i_3 < \cdots < i_N}}^{N} [(k_{i_1 x} + k_{i_2 x} + k_{i_3 x} + \cdots + k_{i_N x}) \\
&\times (\omega_{i_1} + \omega_{i_2} + \omega_{i_3} + \cdots + \omega_{i_N}) + (k_{i_1 x} + k_{i_2 x} + k_{i_3 x} + \cdots + k_{i_N x})^4 \\
&+ 12(k_{i_1 y} + k_{i_2 y} + k_{i_3 y} + \cdots + k_{i_N y})^2] A_{i_1 i_2} A_{i_1 i_3} A_{i_2 i_3} \cdots A_{i_{N-1} i_N},
\end{aligned}
\tag{3.21}
$$

where $\Omega = \sum_{\lambda=1}^{N} \omega_\lambda$. (3.21) is zero at each level of soliton solution when one uses the interaction terms A_{ij} obtained from the two-soliton condition. This has been verified explicitly up to four solitons.

4 Comparison with $N = 2$ KdV equation

The bilinearization of the $N = 2$ super-KP in Section 2 in terms of τ functions which are spin 0 fields contrasts with the formalism adopted for the $N = 2$ super-KdV (with $a = -2$), which was bilinearized [27] in $N = 1$ superspace using τ functions which were spin 0 bosonic *superfields*. Bilinearization of the $N = 2$ KP in $N = 1$ superspace appears problematic due to the nonlocality as well as the y coordinate dependence present in the equations of motion (2.1) and (2.2) in an intricate way. It is interesting to note that, if the superfields are made independent of the y coordinate in (2.1), (2.2), the $N = 2$ super-KP equations reduce to the $N = 2$ supersymmetric KdV equation characterized by $a = -2$ [28–30, 26]. We find that soliton solutions of the $N = 2$ super KdV obtained from the Hirota formalism for the component equations or in $N = 1$ superspace lead to identical soliton solutions. This is clear from the discussion below.

The bilinear forms of the $N = 2$ KdV equation in $N = 1$ superspace are obtained by transforming the superfields as [27]

$$u_{-1} = 2D^2 \log \frac{\tau_1}{\tau_2} \tag{4.1}$$

for u_{-1} and

$$u_0 = 2D^3 \log \tau_1 \tag{4.2}$$

for u_0. In (4.1) and (4.2) τ_1 and τ_2 are bosonic superfields which can be written in component form as

$$\tau_i = \tau_i^b + \theta\tau_i^f \tag{4.3}$$

$(i = 1, 2)$, τ_i^b and τ_i^b being the bosonic and fermionic component fields. It is found that the $N = 2$ KdV equations have the following bilinear forms

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_1 \bullet \tau_1) = 0, \tag{4.4}$$
$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_2 \bullet \tau_2) = 0, \tag{4.5}$$
$$\mathbf{D}_x^2(\tau_1 \bullet \tau_2) = 0, \tag{4.6}$$

$\mathbf{S}$ being the supersymmetric generalization of the Hirota operator in $N = 1$ superspace and is defined by [21]

$$\mathbf{SD}_x^n f \bullet g = (D_{\theta_1} - D_{\theta_2})(\partial_{x_1} - \partial_{x_2})^n f(x_1, \theta_1) g(x_2, \theta_2)\Big|_{\substack{x_1 = x_2 = x \\ \theta_1 = \theta_2 = \theta}} . \tag{4.7}$$

To compare the bilinear forms for $N = 2$ KdV equation in ordinary space and $N = 1$ superspace, we observe that, if the Hirota equations for the $N = 2$ KP (2.11), (2.12) and (2.13) become y independent, they reduce to

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tilde{\tau}_1 \bullet \tilde{\tau}_1) = 0, \tag{4.8}$$
$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tilde{\tau}_2 \bullet \tilde{\tau}_2) = 0, \tag{4.9}$$
$$\mathbf{D}_x^2(\tilde{\tau}_1 \bullet \tilde{\tau}_2) = 0. \tag{4.10}$$

These are exactly the bilinear equations that would be obtained for the $N = 2$ KdV in component form with the same choice of transformation equations as for the $N = 2$ KP (2.9) and (2.10) with the obvious restriction that they be independent of the y coordinate. Note that the above bilinear forms of $N = 2$ KdV equation do not involve the super Hirota operator (4.7) in contrast to the forms (4.4), (4.5) and (4.6), but both forms for the $N = 2$ KdV equation give rise to identical soliton solutions.

This allows us to identify (2.9) and (2.10) with (4.1) and (4.2) provided

$$\tau_1^f = \xi\partial_x\tau_1^b, \qquad \tau_2^f = \xi\partial_x\tau_2^b \tag{4.11}$$

with the identification $\tau_i^b = \tilde{\tau}_i$. The relation between the fermionic and bosonic components, (4.11), however, becomes evident from the soliton solutions.

To compare the above soliton solutions with those which follow from the bilinear forms (2.11), (2.12) and (2.13), we write down the explicit expressions for the one soliton solutions [27] following (4.4), (4.5) and (4.6) as

$$u_{-1} = -(2k)\mathrm{cosech}(\psi + \gamma_0) - \theta(2k\zeta)\cosh(\psi + \gamma_0)\mathrm{cosech}^2(\psi + \gamma_0) \tag{4.12}$$

and

$$u_0 = -\frac{1}{2}k\zeta\mathrm{sech}^2\frac{1}{2}(\psi + \gamma_0) + \theta\frac{k^2}{2}\mathrm{sech}^2\frac{1}{2}(\psi + \gamma_0), \tag{4.13}$$

where $\psi = kx + \omega t$ and ζ is a Grassmann odd parameter of spin 1/2, the τ functions in (4.1), (4.2) being of the form

$$\tau_1 = 1 + \alpha e^{\psi+\zeta\theta}, \qquad \tau_2 = 1 - \alpha e^{\psi+\zeta\theta}. \tag{4.14}$$

It is straightforward that (3.8), (3.9), (3.10) and (3.11) reduce to (4.12) and (4.13) provided $\beta = -\alpha$ and if we identify

$$\xi = -\frac{\zeta}{k_x}. \tag{4.15}$$

The two-soliton solution of $N = 2$ KdV equation obtained from (4.4), (4.5) and (4.6) immediately follows from the two-solutions of the $N = 2$ KP obtained in Section 3 if $\beta_i = -\alpha_i$ with $(i = 1, 2)$. The dispersion relations, therefore, become

$$\omega_i + k_{ix}^3 = 0, \qquad i = 1, 2 \tag{4.16}$$

and the interaction terms reduce to

$$A_{12} = B_{12} = \frac{(k_{1x} - k_{2x})^2}{(k_{1x} + k_{2x})^2}. \tag{4.17}$$

To compare the above result, (4.16) and (4.17) with those obtained in [27] we observe that the bilinear forms (4.4), (4.5) and (4.6) add an additional condition to the fermionic parameters

$$k_i\zeta_j = k_j\zeta_i \qquad (i, j = 1, 2;\ i \neq j), \tag{4.18}$$

where the spin 1/2 parameter ζ is introduced in (4.12). The condition (4.18) makes the two-soliton solutions of $N = 2$ KdV equation obtained via the two procedures apparently inconsistent, but note that (4.18) ensures that the ratio ζ_i/k_{ix} be same for both the solitons and the explicit form of one-soliton dictates that the ratio is the same as $-\xi$ (4.15), *i.e.*

$$\xi = -\frac{\zeta_1}{k_1} = -\frac{\zeta_2}{k_2}. \tag{4.19}$$

This condition also follows from the expressions of u_{-1} and u_0 when written in terms of two-soliton solutions. Importantly, with this identification, the two-soliton solutions of $N = 2$ KdV equation obtained from both the bilinear forms namely (4.4), (4.5) and (4.6) and (4.8), (4.9) and (4.10) become consistent, although a condition such as (4.18) does not arise separately when the equations of motions are written in the component fields. This strongly indicates that the bilinearization of the $N = 2$ KP equation in terms of the bosonic Hirota operator by use of the τ functions defined in (2.9) and (2.10) is equivalent to bilinearization in terms of the super Hirota operator in $N = 1$ superspace.

In this context we mention that Carstea, Grammaticos and Ramani [25] have obtained the two and higher soliton solutions of the $N = 1$ KdV of Manin–Radul–Mathieu without constraint on the fermionic parameters. We have observed that this does not always appear to be possible for supersymmetric equations [24, 27]. For the $N = 2$ KdV the bilinear equation, (4.6), which involves only the bosonic Hirota operator forces one to choose two and higher soliton solutions with constraints on the fermionic parameters. In such a case bilinearization of the equation in $N = 1$ superspace or in terms of the evolution equations of the component fields leads to identical soliton solutions. Since the $N = 2$ KP reduces to the $N = 2$ KdV, it is reasonable to expect that, even if the $N = 2$ KP could be bilinearizable in $N = 1$ superspace, the soliton solutions obtained would be the constrained solutions only.

5 Conclusion

In this paper, we have shown that the $N = 2$ nonstandard flow KP equation is bilinearizable in component form and soliton solutions have been obtained for it. By comparison with the $N = 2$ KdV equation, it is seen that the right choice of τ functions ensures that this is equivalent to bilinearization via the super Hirota formalism in $N = 1$ superspace. Though this is not evident from the Hirota equations themselves, it becomes obvious from the soliton solutions that in fact the methods are equivalent and it is expected that this will also be valid for the $N = 2$ KP equation, although the exact comparison may be made only if it is bilinearized in $N = 1$ superspace.

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