H^k Metrics on the Diffeomorphism Group of the Circle

Adrian CONSTANTIN † *and B KOLEV* ‡

† *Department of Mathematics, Lund University, P.O. Box 118, SE-221 00 Lund, Sweden*
E-mail: adrian.constantin@math.lu.se

‡ *CMI, 39, rue F. Joliot-Curie, 13453 Marseille cedex 13, France*
E-mail: kolev@cmi.univ-mrs.fr

Received February 13, 2003; Accepted May 07, 2003

Abstract

Each H^k inner product, $k \in \mathbb{N}$, endows the diffeomorphism group of the circle with a Riemannian structure. For $k \geq 1$ the Riemannian exponential map is a smooth local diffeomorphism and the length-minimizing property of geodesics holds.

1 Introduction

Some equations of mathematical physics arise as geodesic equations for certain right-invariant Riemannian metrics on diffeomorphism groups [1]. These groups have an infinite dimensional Lie group structure. Since their differentiable structure is modelled on a Fréchet space, the analysis is intricate and few rigourous results are available. The aim of this work is to report on a special case in which infinite-dimensional counterparts of results from classical Riemannian geometry can be established.

The family $\mathcal{D}$ of increasing diffeomorphisms of the unit circle $\mathbb{S} \subset \mathbb{C}$ is an infinite-dimensional Lie group. Its Lie algebra $T_{\mathrm{Id}}\mathcal{D}$ is the space $C^\infty(\mathbb{S})$ of real smooth periodic maps of period one. A right-invariant Riemannian metric on the diffeomorphism group $\mathcal{D}$ is determined by its value on $T_{\mathrm{Id}}\mathcal{D} = C^\infty(\mathbb{S})$. That is, there is a one-to-one correspondence between right-invariant Riemannian metrics on $\mathcal{D}$ and inner products on $C^\infty(\mathbb{S})$. We study Riemannian structures associated with the family of H^k inner products. Here $H^k(\mathbb{S})$ is the Hilbert space of all $L^2(\mathbb{S})$-functions f (square integrable periodic functions) with distributional derivatives $\partial_x^i f$ in $L^2(\mathbb{S})$ for $i = 0, \dots, k$, endowed with the inner product $\langle f, g\rangle_k = \sum_{i=0}^{k} \int_{\mathbb{S}} \partial_x^i f(x)\, \partial_x^i g(x)\, dx,\ f, g \in H^k(\mathbb{S})$.

In the next section we highlight some aspects of the diffeomorphism group $\mathcal{D}$ while in Section 3/Section 4 we present some results about the geodesic flow of H^k right-invariant metrics. The considerations presented here are detailed and developed in [10].

2 The diffeomorphism group

The group $\mathcal{D}$ is an open subset of $C^\infty(\mathbb{S},\mathbb{S})$, which is itself a closed subset of $C^\infty(\mathbb{S},\mathbb{C})$. We will describe the Fréchet manifold structure of $\mathcal{D}$.

The tangent space

For a C^1-path $t \mapsto \varphi(t)$ in $\mathcal{D}$ with $\varphi(0) = \mathrm{Id}$, we have $\varphi'(0)(x) \in T_x\mathbb{S}$ at $x \in \mathbb{S} \subset \mathbb{C}$. Therefore $\varphi'(0)$ is a vector field on $\mathbb{S}$ and we can identify $T_{\mathrm{Id}}\mathcal{D}$ with $\mathrm{Vect}(\mathbb{S})$, the space of smooth vector fields on $\mathbb{S}$. If $\xi(x)$ is a tangent vector to $\mathbb{S}$ at $x \in \mathbb{S} \subset \mathbb{C}$, then $\Re e\,[\overline{x}\,\xi(x)] = 0$ and $u(x) = \frac{1}{2\pi i}\,\overline{x}\,\xi(x) \in \mathbb{R}$. This allows us to identify the space of smooth vector fields on the circle with $C^\infty(\mathbb{S})$. The latter may be thought of as the space of real smooth periodic maps of period one and will be used as a model for the construction of local charts on $\mathcal{D}$. Note that $C^\infty(\mathbb{S})$ is a Fréchet space, its topology being defined by the countable collection of $C^n(\mathbb{S})$-seminorms: a sequence $u_j \to u$ as $j \to \infty$ if and only if for all $n \geq 0$ we have $u_j \to u$ in $C^n(\mathbb{S})$ as $j \to \infty$.

Local charts

To define a local chart around the point $\varphi_0 \in \mathcal{D}$, we take the neighborhood $U_0 = \{\varphi \in \mathcal{D} : \|\varphi - \varphi_0\|_{C^0(\mathbb{S})} < 1/2\}$ of φ_0 and we define

$$u(x) = \Psi_0(\varphi) = \frac{1}{2\pi i}\log(\overline{\varphi_0(x)}\varphi(x)), \qquad x \in \mathbb{S}.$$

Note that $u(x)$ is a measure of the angle between $\varphi_0(x)$ and $\varphi(x)$. We obtain the local charts $\{U_0, \Psi_0\}$, with the change of charts given by

$$\Psi_2 \circ \Psi_1^{-1}(u_1) = u_1 + \frac{1}{2\pi i}\log(\overline{\varphi_2}\,\varphi_1).$$

The previous transformation being just a translation on the vector space $C^\infty(\mathbb{S})$, the structure described above endows $\mathcal{D}$ with a smooth manifold structure based on the Fréchet space $C^\infty(\mathbb{S})$.

Lie group structure

A direct computation (see [12]) shows that the composition and the inverse are both smooth maps from $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$, respectively $\mathcal{D} \to \mathcal{D}$, so that the group $\mathcal{D}$ is a Lie group.

The Lie bracket on the Lie algebra $T_{\mathrm{Id}}\mathcal{D} \equiv C^\infty(\mathbb{S})$ of $\mathcal{D}$ is given by

$$[u, v] = -(u_x v - u v_x), \qquad u, v \in C^\infty(\mathbb{S}).$$

Each $v \in T_{\mathrm{Id}}\mathcal{D}$ gives rise to a one-parameter group of diffeomorphisms $\{\eta(t,\cdot)\}$ obtained by solving $\eta_t = v(\eta)$ in $C^\infty(\mathbb{S})$ with initial data $\eta(0) = \mathrm{Id} \in \mathcal{D}$. Conversely, each one-parameter subgroup $t \mapsto \eta(t) \in \mathcal{D}$ is determined by its infinitesimal generator $v = \frac{\partial}{\partial t}\eta(t)\Big|_{t=0} \in T_{\mathrm{Id}}\mathcal{D}$. Evaluating the flow $t \mapsto \eta(t,\cdot)$ at $t = 1$ we obtain an element $\exp_L(v)$ of $\mathcal{D}$. The Lie-group exponential map $v \to \exp_L(v)$ is a smooth map of the Lie algebra to the Lie group [18]. Although the derivative of $\exp_L$ at $0 \in C^\infty(\mathbb{S})$ is the identity Id, $\exp_L$ is not locally surjective [18]. This failure, in contrast with the case of Hilbert manifolds [16], is due to the fact that the inverse function theorem does not necessarily hold in Fréchet spaces [14].

3 H^k metrics

The inertia operator

For $k \geq 0$ and $u, v \in T_{\mathrm{Id}}\mathcal{D} \equiv C^\infty(\mathbb{S})$, observe that

$$\langle u,\, v\rangle_k = \int_{\mathbb{S}} \sum_{i=0}^{k} (\partial_x^i u)\,(\partial_x^i v)\,dx = \int_{\mathbb{S}} A_k(u)\,v\,dx, \tag{3.1}$$

where $A_k : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ is the linear continuous isomorphism

$$A_k = 1 - \frac{d^2}{dx^2} + \cdots + (-1)^k \frac{d^{2k}}{dx^{2k}}.$$

Note that A_k is symmetric in the sense that

$$\int_{\mathbb{S}^1} A_k(u)\,v\,dx = \int_{\mathbb{S}^1} u\,A_k(v)\,dx, \qquad u, v \in C^\infty(\mathbb{S}).$$

For $\eta \in \mathcal{D}$, let $R_{\eta *} : T_{\mathrm{Id}}\mathcal{D} \to T_\eta\mathcal{D}$, $u \mapsto u \circ \eta$, be the derivative of the right-translation $R_\eta : \mathcal{D} \to \mathcal{D}$, $\varphi \mapsto \varphi \circ \eta$. We extend the inner product (3.1) to each tangent space $T_\eta\mathcal{D}$, $\eta \in \mathcal{D}$, by right-translation

$$\langle V, W\rangle_k := \left\langle R_{\eta^{-1}*}V,\, R_{\eta^{-1}*}W \right\rangle_k, \qquad V, W \in T_\eta\mathcal{D}.$$

This way we obtain a smooth right invariant metric on $\mathcal{D}$.

The connection

Since the previously defined right-invariant metric defines a weak topology on $\mathcal{D}$, the existence of an associated Levi–Civita connection is not certain. However, the existence of a connection is ensured [9] if there exists a bilinear operator $B : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ with the property that

$$\langle B(u, v),\, w\rangle = \langle u,\, [v, w]\rangle, \qquad u, v, w \in T_{\mathrm{Id}}\mathcal{D} = C^\infty(\mathbb{S}).$$

In the case of the H^k right-invariant metric, the operator is given by

$$B_k(u, v) = -\,A_k^{-1}\left(2v_x A_k(u) + v A_k(u_x)\right), \qquad u, v \in C^\infty(\mathbb{S}).$$

We have the following result.

Theorem 1. *Let $k \geq 0$. There exists a unique Riemannian connection ∇^k on $\mathcal{D}$ associated to the right-invariant metric defined on $T_{\mathrm{Id}}\mathcal{D}$ by* (3.1)*.*

4 The geodesic flow

The geodesic equation

The existence of the connection ∇^k enables us to define parallel translation along a curve on $\mathcal{D}$. Throughout the discussion, let $I \subset \mathbb{R}$ be an open interval with $0 \in I$. If $\alpha : I \to \mathcal{D}$ is a C^2-curve, a lift $\gamma : I \to T\mathcal{D}$ is called α-parallel if

$$v_t = \frac{1}{2}\left(vu_x - v_x u + B_k(u,v) + B_k(v,u)\right), \qquad t \in I,$$

where $u, v \in C^1(I, C^\infty(\mathbb{S}))$ are defined by $u = \alpha_t \circ \alpha^{-1}$, respectively $v = \gamma \circ \alpha^{-1}$. A C^2-curve $\varphi : I \to \mathcal{D}$ with the property that φ_t is φ-parallel is called a *geodesic*. That is, a curve $\varphi \in C^2(I, \mathcal{D})$ with $\varphi(0) = \mathrm{Id}$ is a geodesic if and only if

$$u_t = B_k(u,u), \qquad t \in I, \tag{4.1}$$

where $u = \varphi_t \circ \varphi^{-1} \in T_{\mathrm{Id}}\mathcal{D} \equiv C^\infty(\mathbb{S})$. Equation (4.1), called the *Euler equation*, is the geodesic equation transported by right-translation to the Lie algebra $T_{\mathrm{Id}}\mathcal{D}$. Problems of type (4.1) arise in fluid mechanics.

Example 1. For $k = 0$, that is, for the L^2 right-invariant metric, equation (4.1) becomes the inviscid Burgers equation

$$u_t + 3uu_x = 0. \tag{4.2}$$

Equation (4.2) can be studied quite explicitly [15]. All solutions of (4.2) but the constant functions have a finite life span and (4.2) is a simplified model for the occurence of shock waves in gas dynamics.

Example 2. For $k = 1$, that is, for the H^1 right-invariant metric, equation (4.1) becomes cf. [19] the Camassa–Holm equation

$$u_t + uu_x + \partial_x \left(1 - \partial_{x^2}\right)^{-1}\left(u^2 + \frac{1}{2}\,u_{x^2}\right) = 0. \tag{4.3}$$

Equation (4.3) is a model for the unidirectional propagation of shallow water waves [2]. It has a bi-Hamiltonian structure [13] and is completely integrable [4, 11]. Some solutions of (4.3) exist globally in time [3, 6], whereas others develop singularities in finite time [3, 7, 17]. The blowup phenomenon can be interpreted as a simplified model for wave breaking – the solution (representing the surface water wave) stays bounded while its slope becomes vertical in finite time [5].

Conservation of momentum

As a consequence of the right-invariance of the metric by the action of the group on itself, we obtain a particularly useful form of the conservation of momentum. If $\varphi \in C^2(I, \mathcal{D})$ with $\varphi(0) = \mathrm{Id}$ is a geodesic and $u = \varphi_t \circ \varphi^{-1}$, then

$$m_k(\varphi, t) = A_k(u) \circ \varphi \cdot \varphi_x^2, \tag{4.4}$$

satisfies $m_k(t) = m_k(0)$ as long as $m_k(t)$ is defined.

Existence of geodesics

Standard local existence theorems for differential equations with smooth right-hand side, valid for Hilbert spaces [16], do not hold in $C^\infty(\mathbb{S})$ cf. [14]. The strategy we develop to prove the existence of geodesics is the following. In a local chart the geodesic equation (4.1) can be expressed as the Cauchy problem

$$\begin{aligned} &\varphi_t = v, \\ &v_t = P_k(\varphi, v), \end{aligned} \tag{4.5}$$

with $\varphi(0) = \mathrm{Id}$, $v(0) = u(0)$. The operator P_k in (4.5) is specified by

$$P_k(\varphi, v) = \left[Q_k\left(v \circ \varphi^{-1}\right)\right] \circ \varphi,$$

where $Q_k : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ is the operator

$$Q_k(w) = B_k(w, w) + w w_x, \qquad w \in C^\infty(\mathbb{S}).$$

Since $C^\infty(\mathbb{S}) = \bigcap_{k \geq n} H^k(\mathbb{S})$ for all $n \geq 0$, we may consider the problem (4.5) on each Hilbert space $H^n(\mathbb{S})$. If $k \geq 1$ and $n \geq 3$, then P_k is a smooth map from $U^n \times H^n(\mathbb{S})$ to $H^n(\mathbb{S})$, where $U^n \subset H^n(\mathbb{S})$ is the open subset of all functions having a strictly positive derivative. The classical Cauchy–Lipschitz theorem in Hilbert spaces [16] yields the existence of a unique solution $\varphi_n(t) \in U^n$ of (4.5) for all $t \in [0, T_n)$ for some maximal $T_n > 0$. Relation (4.4) can be used to prove that $T_n = T_{n+1}$ for all $n \geq 3$. We obtain the following result.

Theorem 2. *Let $k \geq 1$. For every $u_0 \in C^\infty(\mathbb{S})$, there exists a maximal $T > 0$ and a unique geodesic $\varphi \in C^\infty([0, T), \mathcal{D})$ for the right-invariant metric* (3.1)*, starting at $\varphi(0) = \mathrm{Id} \in \mathcal{D}$ in the direction $u_0 = \varphi_t(0) \in T_{\mathrm{Id}}\mathcal{D}$. Moreover, the solution depends smoothly on the initial data $u_0 \in C^\infty(\mathbb{S})$.*

Remark. For $k = 0$, we have $P_0(\varphi, v) = -2\,\frac{v \cdot v_x}{\varphi_x}$, which is not an operator from $U^n \times H^n(\mathbb{S})$ into $H^n(\mathbb{S})$. Therefore the approach used for Theorem 2 is not suitable in this case. Nevertheless, the method of characteristics applied to the equation (4.2) can be used to show that even for $k = 0$ the statement of Theorem 2 holds [9].

The Riemannian exponential map

The previous results enable us to define the Riemannian exponential map $\mathfrak{exp}$ for the H^k right-invariant metric ($k \geq 0$). In fact, there exists $\delta > 0$ and $T > 0$ so that for all $u_0 \in \mathcal{D}$ with $\|u_0\|_{2k+1} < \delta$ the geodesic $\varphi(t; u_0)$ is defined on $[0, T]$. The homogeneity property $\varphi(t; s u_0) = \varphi(ts; u_0)$ of the geodesics, valid for all $t, s \geq 0$ for which both sides of the equality are well-defined, enables us to define $\mathfrak{exp}(u_0) = \varphi(1; u_0)$ on the open set $\{u_0 \in \mathcal{D} : \ \|u_0\|_{2k+1} < \frac{2\delta}{T}\}$ of $\mathcal{D}$. The map $u_0 \mapsto \mathfrak{exp}(u_0)$ is smooth and its Fréchet derivative at zero, $D\mathfrak{exp}_0$, is the identity operator. However, since we work on a Fréchet manifold, these facts do not necessarily ensure that $\mathfrak{exp}$ is a C^1 local diffeomorphism [14]. We proceed as follows. Working in $H^{k+3}(\mathbb{S})$, we deduce from the inverse function theorem in Hilbert spaces that $\mathfrak{exp}$ is a smooth diffeomorphism from an open neighborhood $\mathcal{O}_{k+3}$ of $0 \in H^{k+3}(\mathbb{S})$ to an open neighborhood Θ_{k+3} of $\mathrm{Id} \in U^{k+3}$. Moreover, we may choose $\mathcal{O}_{k+3}$

such that $D\mathfrak{exp}_{u_0}$ is a bijection of $H^{k+3}(\mathbb{S})$ for every $u_0 \in \mathcal{O}_{k+3}$. Given $n \geq k+3$, using (4.4) and the geodesic equation, we can show that there is no $u_0 \in H^n(\mathbb{S})$, $u_0 \notin H^{n+1}(\mathbb{S})$, with $\mathfrak{exp}(u_0) \in U^{n+1}$. Therefore for every $n \geq k+3$, $\mathfrak{exp}$ is a bijection from $\mathcal{O}_n = \mathcal{O}_{k+3} \cap H^n(\mathbb{S})$ to $\Theta_n = \Theta_{k+3} \cap H^n(\mathbb{S})$. Hence $\mathfrak{exp}$ is a bijection from $\mathcal{O} = \mathcal{O}_{k+3} \cap C^\infty(\mathbb{S})$ to $\Theta = \Theta_{k+3} \cap C^\infty(\mathbb{S})$. At this point, (4.4) and the geodesic equation can be used to prove that there is no $u_0 \in H^n(\mathbb{S})$, $u_0 \notin H^{n+1}(\mathbb{S})$, with $D\mathfrak{exp}_{u_0}(v) \in H^{n+1}(\mathbb{S})$ for some $u_0 \in \mathcal{O}$. This shows that for every $u_0 \in \mathcal{O}$ and $n \geq k+3$, the bounded linear operator $D\mathfrak{exp}_{u_0}$ is a bijection from $H^n(\mathbb{S})$ to $H^n(\mathbb{S})$. We obtain the following result.

Theorem 3. *The Riemannian exponential map for the H^k right-invariant metric on $\mathcal{D}$, $k \geq 1$, is a smooth local diffeomorphism from a neighborhood of zero on $T_{\mathrm{Id}}\mathcal{D}$ to a neighborhood of* Id *on $\mathcal{D}$.*

Remark. Note that for $k = 0$, $\mathfrak{exp}$ is not a C^1 local diffeomorphism from a neighborhood of $0 \in T_{\mathrm{Id}}\mathcal{D}$ to a neighborhood of $\mathrm{Id} \in \mathcal{D}$ cf. [9].

Length-minimizing property

Let $\mathcal{O}$ and Θ be the open neighborhoods of $0 \in C^\infty(\mathbb{S})$, respectively $\mathrm{Id} \in \mathcal{D}$, defined above. Then the map

$$G : \mathcal{D} \times \mathcal{O} \to \mathcal{D} \times \mathcal{D}, \qquad (\eta, u) \mapsto (\eta, R_\eta\, \mathfrak{exp}(u)),$$

is a smooth diffeomorphism onto its image. For $\eta \in \mathcal{D}$, let $\Theta(\eta) = R_\eta \Theta = R_\eta \mathfrak{exp}(\mathcal{O})$. We define the *polar coordinates* (r, w) of $\varphi \in \Theta(\eta)$ by setting $v = rw$ with $r \in \mathbb{R}_+$ and $\langle w, w \rangle_k = 1$, where $v \in \mathcal{O}$ is uniquely determined by $\varphi = \mathfrak{exp}(v) \circ \eta$. If $\gamma : [a, b] \to \Theta(\eta)$ is a piecewise C^1-curve, then

$$l(\gamma) \geq |r(b) - r(a)|,$$

where $l(\gamma)$ is the length of the curve and $(r(t), w(t))$ are the polar coordinates of $\gamma(t)$. Moreover, equality holds if and only if the function $t \mapsto r(t)$ is monotone and the map $t \mapsto w(t) \in \mathcal{O}$ is constant. This leads to

Theorem 4. *Consider $\mathcal{D}$ endowed with the H^k right-invariant metric ($k \geq 1$). If $\eta, \varphi \in \mathcal{D}$ are close enough, more precisely, if $\varphi \circ \eta^{-1} \in \Theta$, then η and φ can be joined by a unique geodesic in $\Theta(\eta)$. Among all piecewise C^1-curves joining η to φ on $\mathcal{D}$, the geodesic is length minimizing.*

Specializing $k = 1$ in Theorem 4 we obtain that for the Camassa–Holm model for shallow water waves (Example 2) the Least Action Principle holds. That is, a state of the system is transformed to another nearby state through a uniquely determined flow of (4.3) that minimizes the kinetic energy cf. [8].

References

[1] Arnold V and Khesin B, Topological Methods in Hydrodynamics, Springer Verlag, New York, 1998.

[2] Camassa R and Holm D, An Integrable Shallow Water Equation with Peaked Solitons, *Phys. Rev. Letters* **71** (1993), 1661–1664.

[3] Constantin A, On the Cauchy Problem for the Periodic Camassa–Holm Equation, *J. Differential Equations* **141** (1997), 218–235.

[4] Constantin A, On the Inverse Spectral Problem for the Camassa–Holm Equation, *J. Funct. Anal.* **155** (1998), 352–363.

[5] Constantin A and Escher J, Wave Breaking for Nonlinear Nonlocal Shallow Water Equations, *Acta Mathematica* **181** (1998), 229–243.

[6] Constantin A and Escher J, Well-Posedness, Global Existence, and Blowup Phenomena for a Periodic Quasi-Linear Hyperbolic Equation, *Comm. Pure Appl. Math.* **51** (1998), 475–504.

[7] Constantin A and Escher J, On the Blow-Up Rate and the Blow-Up Set of Breaking Waves for a Shallow Water Equation, *Math. Z.* **233** (2000), 75–91.

[8] Constantin A and Kolev B, Least Action Principle for an Integrable Shallow Water Equation, *J. Nonlinear Math. Phys.* **8** (2001), 471–474.

[9] Constantin A and Kolev B, On the Geometric Approach to the Motion of Inertial Mechanical Systems, *J. Phys. A: Math. Gen.* **35** (2002), R51–R79.

[10] Constantin A and Kolev B, Geodesic Flow on the Diffeomorphism Group of the Circle, *Comment. Math. Helv.*, to appear.

[11] Constantin A and McKean HP, A Shallow Water Equation on the Circle, *Comm. Pure Appl. Math.* **52** (1999), 949–982.

[12] Ebin D and Marsden J, Groups of Diffeomorphisms and the Motion of an Incompressible Fluid, *Ann. Math.* **92** (1970), 102–163.

[13] Fuchssteiner B and Fokas A, Symplectic Structures, Their Bäcklund Transformation and Hereditary Symmetries, *Physica D* **4** (1981), 47–66.

[14] Hamilton R, The Inverse Function Theorem of Nash and Moser, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 65–222.

[15] Hörmander L, Lectures on Nonlinear Hyperbolic Differential Equations, Springer Verlag, Berlin, 1997.

[16] Lang S, Fundamentals of Differential Geometry, Springer Verlag, New York, 1999.

[17] McKean HP, Breakdown of a Shallow Water Equation, *Asian J. Math.* **2** (1998), 867–874.

[18] Milnor J, Remarks on Infinite-Dimensional Lie Groups, in Relativity, Groups and Topology, North-Holland, Amsterdam, 1983, 1009–1057.

[19] Misiolek G, A Shallow Water Equation as a Geodesic Flow on the Bott–Virasoro Group, *J. Geom. Phys.* **24** (1998), 203–208.