On a Certain Fractional $q$-Difference and its Eigen Function

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Abstract

A fractional $q$-difference operator is presented and its properties are investigated. Especially, it is shown that this operator possesses an eigen function, which is regarded as a $q$-discrete analogue of the Mittag-Leffler function. An integrable nonlinear mapping with fractional $q$-difference is also presented.

1 Introduction

Fractional derivative goes back to the Leipniz's note in his list to L'Hospital in 1695 and we now have many definitions of fractional derivatives [9]. In the last few decades, many authors pointed out that derivatives and integrals of fractional order, especially $1/2$-derivative, are very suitable for the description of physical phenomena.

We first define a fractional integral operator $I^a$ as follows.

Definition 1. Let $a$ be a nonnegative real number. For a given function $u(t) (t > 0)$, its integral of order $a$ is defined as follows.

\begin{align*}
I^a u(t) &= \int_0^t K(a; t - s)u(s)\,ds \\
I^0 u(t) &= u(t)
\end{align*}

(1.1) (1.2)

where $K(a; t)$ is a monomial given by

\[ K(a; t) := \frac{t^{a-1}}{\Gamma(a)} \quad (t > 0, \ a > 0). \]

(1.3)

Fractional derivatives of order $a > 0$ are defined by a combination of normal derivative and fractional integral in the following two manners.

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Definition 2. Let \( m \) be a positive integer such as \( m - 1 < a \leq m \). Then for a given \( m \) times continuously differentiable function \( u(t) \), its derivative of order \( a \) is defined by

\[
D^a u(t) \equiv (I^{m-a} D^m u)(t) = \int_0^t K(m-a; t-s) u^{(m)}(s) ds
\]

(1.4)

Definition 3. For the same \( a, m, u(t) \) in the previous definition, a derivative of order \( a \) is defined by

\[
D^a u(t) \equiv (D^m I^{m-a} u)(t) = \left( \frac{d}{dt} \right)^m \int_0^t K(m-a; t-s) u(s) ds.
\]

(1.5)

These two definitions are called Caputo and Riemann-Liouville fractional derivatives, respectively. We here adopt Caputo’s definition 3.

The Mittag-Leffler function,

\[
E_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(a j + 1)} \quad (a > 0, z \in \mathbb{C})
\]

(1.6)

was proposed by Mittag-Leffler [6] in 1903 as an entire function whose order can be calculated exactly. Afterwards, it was clarified that the Mittag-Leffler function also plays an important role in fractional calculus (See refs. [5, 8] for example). In other words, the Mittag-Leffler function,

\[
u(t) = \sum_{j=0}^{\infty} \lambda^j K(a j + 1; t) = E_a(\lambda t^a)
\]

(1.7)

is an eigen function of Caputo’s fractional derivative [5],

\[D^a u(t) = \lambda u(t) \quad (t > 0)\]

(1.8)

In a present paper [7], a discrete analogue of Mittag-Leffler function is presented, together with its relation with a certain fractional difference and a nonlinear integrable mapping with fractional difference has been proposed. The main purpose of this paper is a \( q \)-discretization of the above result. In section 2, we present a certain fractional \( q \)-difference operator, which is a slight modification of Al-Salam’s fractional \( q \)-difference operator [1], and investigate its properties. Section 3 is devoted to \( q \)-discretization of the Mittag-Leffler function. We also show that \( q \)-Mittag-Leffler function serves as an eigen function of the fractional \( q \)-difference operator. Finally in section 4, a new type of nonlinear integrable mapping equipped with fractional \( q \)-difference is presented.

2 Fractional \( q \)-difference

In this section, we present fractional \( q \)-addition and \( q \)-difference operators and investigate their properties.

Before getting onto the main subject, we first give definitions of \( q \)-number, \( q \)-binomial coefficient and \( q \)-difference operator, together with their properties, which are required in
Let \( q \) be a given complex number. Throughout this paper, we impose the assumption,
\[
|q| > 1. \tag{2.1}
\]
We introduce \( q \)-number \([a]_q\) defined by
\[
[a]_q = \frac{q^a - 1}{q - 1}, \tag{2.2}
\]
we here rewrite \([a]_q\) as \([a]\) for the sake of simplicity. By making use of the \( q \)-number, \( q \)-binomial coefficient is given as follows.
\[
\begin{align*}
\begin{bmatrix} x \\ n \end{bmatrix} &= \frac{[x][x-1] \cdots [x-n+1]}{[n]!} = \frac{[x][x-1] \cdots [x-n+1]}{[n][n-1] \cdots [1]} \tag{2.3}
\end{align*}
\]
We here list some important properties of \( q \)-number and \( q \)-binomial coefficient used in future.
\[
\begin{align*}
-x &= -q^{-x}[x] \tag{2.4} \\
\begin{bmatrix} -x \\ n \end{bmatrix} &= (-1)^n q^{-nx - \frac{1}{2} n(n-1)} \begin{bmatrix} x + n - 1 \\ n \end{bmatrix} \tag{2.5} \\
\begin{bmatrix} x \\ n \end{bmatrix} - \begin{bmatrix} x - 1 \\ n \end{bmatrix} &= q^{x-n} \begin{bmatrix} x - 1 \\ n - 1 \end{bmatrix} \tag{2.6} \\
\begin{bmatrix} x \\ n \end{bmatrix} - \begin{bmatrix} x - 1 \\ n - 1 \end{bmatrix} &= q^n \begin{bmatrix} x - 1 \\ n \end{bmatrix} \tag{2.7} \\
\sum_{k=0}^{n} \begin{bmatrix} x \\ n-k \end{bmatrix} \begin{bmatrix} y \\ k \end{bmatrix} q^{k^2 - nk + k} x &= \begin{bmatrix} x + y \\ n \end{bmatrix} \tag{2.8}
\end{align*}
\]
We here adopt backward \( q \)-difference operator \( \Delta_q \) defined by
\[
\Delta_q f(x) = \frac{f(x) - f(q^{-1}x)}{(1 - q^{-1})x} \tag{2.9}
\]
Through dependent and independent variable transformations
\[
x = q^n, f(x) = f(q^n) = f_n, \tag{2.10}
\]
the \( q \)-difference operator in eq. (2.9) is rewritten equivalently as
\[
\Delta_q f_n = \frac{f_n - f_{n-1}}{q^n - q^{n-1}}. \tag{2.11}
\]
We next introduce a fractional \( q \)-addition operator \( I^\alpha_q \) defined as follows.

**Definition 4.** Let \( \alpha \) be a non-negative real number and \( \{f_n\} \) is a given complex sequence. Then a \( q \)-addition operator of fractional order \( \alpha \) for \( \{f_n\} \) is defined by
\[
I^\alpha_q f_n = q^{(n-1)\alpha} (q - 1)^\alpha \sum_{k=0}^{(n-1)} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2} k(k-1)} f_{n-k} \quad (\alpha > 0, n \geq 1) \tag{2.12}
\]
\[
I^0_q f_n = f_n \quad (n \geq 1) \tag{2.13}
\]

\(^1\)For details of \( q \)-analysis, see ref. [2] for example.
Substitution of $\alpha = 1$ into eq. (2.12) gives

\[
I_q f_n = q^{n-1} (q - 1) \sum_{k=0}^{n-1} \left[ \frac{-1}{k} \right] q^{\frac{1}{2} k(1-k)} f_{n-k}
\]

\[
= q^{n-1} (q - 1) \sum_{k=0}^{n-1} (-1)^k (-1)^k q^{-\frac{1}{2} k(k-1)} q^{\frac{1}{2} k(k-1)} f_{n-k}
\]

\[
= (q - 1) \sum_{k=0}^{n-1} q^{n-1-k} f_{n-k}
\]

\[
= (q - 1) \sum_{k=1}^{n} q^{k-1} f_k,
\]

which is a finite version of Jackson integral. This fractional $q$-addition operator satisfies the following lemma.

**Lemma 1.** Let $\alpha, \beta$ be non-negative real numbers, $a, b$ be complex numbers and $\{f_n\}, \{g_n\}$ be given complex sequences. Then $q$-addition operators satisfy the following linearity and commutation rules.

\[
I_0^\alpha (af_n + bg_n) = a(I_0^\alpha f_n) + b(I_0^\alpha g_n) \tag{2.14}
\]

\[
I_0^\alpha I_0^\beta f_n = I_0^\beta I_0^\alpha f_n = I_0^{\alpha + \beta} f_n \tag{2.15}
\]

**Proof of Lemma 1.** Equation (2.14) is obvious. We prove a commutation rule (2.15) by employing some properties of a $q$-binomial coefficient.

\[
I_0^\alpha I_0^\beta f_n
\]

\[
= q^{(n-1)\alpha} (q - 1)^{\alpha} \sum_{k=0}^{n-1} (-1)^k \left[ \frac{-\alpha}{k} \right] q^{\frac{1}{2} k(k-1)/2} q^{(n-k-1)\beta} (q - 1)^{\beta}
\]

\[
\times \sum_{j=0}^{n-k-1} (-1)^j \left[ \frac{-\beta}{j} \right] q^{j(1-j)/2} f_{n-k-j}
\]

\[
= q^{(n-1)(\alpha+\beta)} (q - 1)^{\alpha+\beta} \sum_{k=0}^{n-1} (-1)^k \left[ \frac{-\alpha}{k} \right] q^{\frac{1}{2} k(k-1)/2} q^{-\beta k}
\]

\[
\times \sum_{j=0}^{n-k-1} (-1)^j \left[ \frac{-\beta}{j} \right] q^{j(1-j)/2} f_{n-k-j}
\]

\[
= q^{(n-1)(\alpha+\beta)} (q - 1)^{\alpha+\beta} \sum_{k=0}^{n-1} (-1)^k \left[ \frac{-\alpha}{k} \right] q^{\frac{1}{2} k(k-1)/2} q^{-\beta k}
\]

\[
\times \sum_{j=0}^{n-k-1} (-1)^{n-k-1-j} \left[ \frac{-\beta}{n-j-1-k} \right] q^{(n-k-1-j)(n-k-2-j)/2} f_{j+1}
\]
\[= q^{(n-1)(\alpha + \beta)} (q - 1)^{\alpha + \beta} \sum_{j=0}^{n-1} (-1)^{n-j-1} f_{j+1} \]
\[\times \sum_{k=0}^{n-j-1} \left[ -\frac{\alpha}{k} \right] \left[ -\frac{\beta}{n - j - 1 - k} \right] q^{k(k-1)/2+(n-k-1-j)(n-k-2-j)/2} q^{-\beta k} \]
\[= q^{(n-1)(\alpha + \beta)} (q - 1)^{\alpha + \beta} \sum_{j=0}^{n-1} (-1)^{n-j-1} q^{-\alpha(n-j-1)/2} f_{j+1} \]
\[\times \sum_{k=0}^{n-j-1} \left[ -\frac{\alpha}{k} \right] \left[ -\frac{\beta}{n - j - 1 - k} \right] q^{k^2-k(n-j-1)-\beta k} \]
\[= q^{(n-1)(\alpha + \beta)} (q - 1)^{\alpha + \beta} \sum_{j=0}^{n-1} (-1)^{n-j-1} q^{-\alpha(n-j-1)/2} f_{j+1} \left[ -\frac{\alpha - \beta}{n - j - 1} \right] \]
\[= q^{(n-1)(\alpha + \beta)} (q - 1)^{\alpha + \beta} \sum_{j=0}^{n-1} (-1)^{j} q^{j(j-1)/2} f_{n-j} \left[ -\frac{\alpha - \beta}{j} \right] \]
\[= q^{\alpha + \beta} f_n, \]

which completes the proof. \(\blacksquare\)

Next we present a fractional \(q\)-difference operator \(\Delta_q^\alpha\), which can be regarded as a \(q\)-discrete version of Caputo’s fractional derivative operator.

**Definition 5.** Let \(\alpha\) be a positive real number and \(m\) be a positive integer which satisfies \(m-1 < \alpha \leq m\). Then a fractional \(q\)-difference operator of order \(\alpha > 0\) is given by

\[\Delta_q^\alpha f_n = I_q^{m-\alpha} \Delta_q^m f_n = q^{-(n-1)(\alpha-m)} (q - 1)^{-(\alpha-m)} \sum_{k=0}^{n-1} (-1)^k \left[ \frac{\alpha - m}{k} \right] q^{\frac{k^2}{2} k(k-1) \Delta_q^m f_{n-k}} \quad (2.16)\]

**Remark 1.** Fractional \(q\)-difference operator was first proposed by Al-Salam [1] in 1966. Let \(f(x)\) be a given function and \(\alpha \in \mathbb{R}\setminus\{1, 2, 3, \cdots\}\). Then a \(q\)-difference operator \(K^\alpha_q\) is given by

\[K^\alpha_q f(x) = x^{-\alpha} (1 - q)^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \left[ \frac{\alpha}{k} \right] q^{k(k-1)/2-\alpha(a-1)/2} f(x q^{a-k}) \quad (2.17)\]

Fractional \(q\)-difference operator \(\Delta_q^\alpha\) presented here is a slight modification of Al-Salam’s operator \(K^\alpha_q\). The operator \(\Delta_q^\alpha\) satisfies the commutative rule,

\[K^\alpha_q \Delta_q^\beta = \Delta_q^\beta K^\alpha_q = \Delta_q^{\alpha + \beta} \quad (2.18)\]

for any \(\alpha, \beta\), whereas the commutation rule for \(\Delta_q^\alpha\) does not always hold. However, as is mentioned in the next section, the operator \(\Delta_q^\alpha\) possesses an eigen function, which is regarded as a \(q\)-discrete analogue of the Mittag-Leffler function.
3 $q$-Mittag-Leffler function

This section provides a $q$-discrete analogue of the Mittag-Leffler function and its relation with the fractional $q$-difference operator $\Delta_q^\alpha$. We first introduce a fundamental function $M_q(a; n)$ defined by

$$M_q(a; n) = (q - 1)^{a-1} \left[ \frac{n + a - 2}{n - 1} \right] \quad (a > 0, n \in \mathbb{Z}_{\geq 1}). \quad (3.1)$$

$$M_q(a; 0) = \begin{cases} 
1 & (a = 1) \\
0 & (a \neq 1) 
\end{cases} \quad (3.2)$$

**Remark 2.** In the limit $q \to 1$ and $n \to \infty$ with $t = (q - 1)n > 0$ fixed, the above function converges to a monomial,

$$M_q(a; n) \to K(a; t) = \frac{t^{a-1}}{\Gamma(a)}. \quad (3.3)$$

It is a well-known fact that this function $K(a; t)$ plays an essential role in the theory of fractional derivatives.

The above fundamental function $M_q(a; n)$ satisfies the following two lemmas which states the relation between $M_q(a; n)$ and $q$-difference (or fractional $q$-addition) operator.

**Lemma 2.** If $a > 0$ and $n \in \mathbb{Z}_{\geq 1}$, we have

$$\Delta_q M_q(a + 1; n) = M_q(a; n). \quad (3.4)$$

**Lemma 3.** If $\alpha \geq 0$, $a > 0$ and $n \in \mathbb{Z}_{\geq 1}$, we have

$$I_q^\alpha M_q(a; n) = M_q(a + \alpha; n). \quad (3.5)$$

**Proof of Lemma 2.** This is proved essentially by using an addition rule of $q$-binomial coefficient given by eq. (2.7).

$$\Delta_q M_q(a + 1; n) = \frac{M_q(a + 1; n) - M_q(a + 1; n - 1)}{q^n - q^{n-1}}$$

$$= (q - 1)^a \left( \left[ \frac{n + a - 1}{n - 1} \right] - \left[ \frac{n + a - 2}{n - 2} \right] \right) \frac{1}{q^{n-1}(q - 1)}$$

$$= (q - 1)^a q^{n-1} \left[ \frac{n + a - 2}{n - 1} \right] \frac{1}{q^{n-1}(q - 1)}$$

$$= (q - 1)^a q^{n-1} \left[ \frac{n + a - 2}{n - 1} \right]$$

$$= M_q(a; n)$$
which completes the proof.

**Proof of Lemma 3.** If \( \alpha = 0 \), it is obvious. We suppose \( \alpha > 0 \).

\[
I_q \alpha M_q(a; n) = q^{(n-1)\alpha} (q-1)^{\alpha} \sum_{k=0}^{n-1} (-1)^k \frac{(-\alpha)}{k} q^{\frac{1}{2}k(k-1)} M_q(a; n-k)
\]
\[
= q^{(n-1)\alpha} (q-1)^{a-1+\alpha} \sum_{k=0}^{n-1} (-1)^k \frac{(-\alpha)}{k} q^{\frac{1}{2}k(k-1)} \left[ \frac{n-k+a-2}{n-k-1} \right]
\]
\[
= q^{(n-1)\alpha} (q-1)^{a-1+\alpha} \left[ \sum_{k=0}^{n-1} (-1)^k \frac{(-\alpha)}{k} q^{\frac{1}{2}k(k-1)} \left[ \frac{n-k+a-2}{n-k-1} \right] \right]
\]
\[
= q^{(n-1)\alpha} (q-1)^{a-1+\alpha} q^{(n-1)\alpha+\frac{1}{2}(n-1)(-1)n-1} \left[ \frac{n+a+\alpha-2}{n-1} \right]
\]
\[
= (q-1)^{a+\alpha-1} \left[ \frac{n+a+\alpha-2}{n-1} \right] = M_q(a+\alpha; n),
\]
where we have employed an upper negation rule (2.5) twice and a Vandermonde convolution rule (2.8). This completes the proof.

We next introduce a \( q \)-analogue of the Mittag-Leffler function.

**Definition 6.** Let \( a \) be a positive real number. Then \( q \)-Mittag-Leffler function \( F_{a,q}(\lambda; n) \) is given by

\[
F_{a,q}(\lambda; n) = \sum_{j=0}^{\infty} \lambda^j M_q(a j + 1; n) = \sum_{j=0}^{\infty} \lambda^j (q-1)^{aj} \left[ \frac{n+aj-1}{n-1} \right]
\]

It can be verified easily from eq. (3.3) that the above function \( F_{a,q}(\lambda; n) \) converges to the Mittag-Leffler function \( E_a(\lambda t^a) \) in the limit \( q \to 1 \) and \( n \to \infty \) with \( t = (q-1)n \) fixed. The following main theorem states that \( q \)-Mittag-Leffler function serves as an eigen function of the fractional \( q \)-difference operator \( \Delta_q^a \).

**Theorem 1.** If \( a > 0 \) and \( n \in \mathbb{Z}_{\geq 1} \), we have

\[
\Delta_q^a F_{a,q}(\lambda; n) = \lambda F_{a,q}(\lambda; n)
\]

**Proof of Theorem 1.** Let \( m \) be a positive integer such as \( m-1 < a \leq m \). Operating
\[ \Delta_q^m \text{ on } F_{a,q}(\lambda; n) \] and noticing \( \Delta_q M_q(1; n) = \Delta_q 1 = 0 \), we have from Lemma 2

\[ \Delta_q^m F_{a,q}(\lambda; n) = \sum_{j=0}^{\infty} \lambda^j \Delta_q^m M_q(aj + 1; n) \\
= \sum_{j=1}^{\infty} \lambda^j M_q(aj - m + 1; n). \tag{3.8} \]

Operating fractional \( q \)-addition operator \( I_q^{m-a} \) on both sides of the above equation and employing Lemma 3, we finally obtain

\[ \Delta_q a F_{a,q}(\lambda; n) = I_q^{m-a} \Delta_q^m F_{a,q}(\lambda; n) \\
= \sum_{j=1}^{\infty} \lambda^j I_q^{m-a} M_q(aj - m + 1; n) \\
= \sum_{j=1}^{\infty} \lambda^j M_q(aj - a + 1; n) \\
= \sum_{j=0}^{\infty} \lambda^{j+1} M_q(aj + 1; n) = \lambda F_{a,q}(\lambda; n), \tag{3.9} \]

which completes the proof.

\[ \Box \]

4 An integrable nonlinear mapping with fractional \( q \)-difference

We here give a new type of integrable nonlinear mapping which is equipped with fractional \( q \)-difference. We start with a linear mapping,

\[ \Delta_q^p g_n = -ag_n, \quad 0 < p \leq 1, 0 < a. \tag{4.1} \]

The above equation is rewritten equivalently as

\[ (1 + a(q^n - q^{n-1})^p)g_n = g_{n-1} + \sum_{k=1}^{n-1} (-1)^{k-1} \binom{p-1}{k} q^{k(k+1)/2} (g_{n-k} - g_{n-1-k}) \tag{4.2} \]

Through dependent variable transformation,

\[ u_n = \frac{1}{g_n + 1}, \tag{4.3} \]

we obtain the following nonlinear mapping with fractional \( q \)-difference.

\[ u_n = \frac{1 + a(q^n - q^{n-1})^p}{u_{n-1} + a(q^n - q^{n-1})^p + \sum_{k=1}^{n-1} (-1)^{k-1} \binom{p-1}{k} q^{k(k+1)/2} (u_{n-k} - u_{n-1-k})}. \tag{4.4} \]
The solution for eq. (4.4) is written as

\[ u_n = \frac{u_0}{u_0 + (1 - u_0) F_{p,q}(-a; n)}. \]  
(4.5)

Putting \( p = 1 \) in eq. (4.4), we have

\[ \frac{u_n - u_{n-1}}{q^n - q^{n-1}} = a u_{n-1} (1 - u_n) \]

which converges to the Riccati equation,

\[ \frac{du}{dt} = au(1 - u) \]  
(4.6)

in the continuum limit \( t = q^n, u(t) = u_n \) and \( q \to 1 \).

The following Figure 1 illustrates the time evolution of the fractional mapping with parameter \( p = n/4 (n = 1, 2, 3, 4) \) and \( u_0 = 0.2, a = 4, q = 2^{1/10} \).

![Figure 1. Time evolutions of the fractional mapping (4.4)](image)

5 Concluding Remarks

We have presented one definition of fractional \( q \)-difference operator. We have also shown that a \( q \)-discrete version of Mittag-Leffler function preserves the property that Mittag-Leffler function is an eigen function of a fractional derivative. It should be noted, however, that the Mittag-Leffler function possesses more abundant properties such as complex-integral expression, asymptotic behavior [10]. It is unknown whether its \( q \)-discrete version preserves such properties as well.

It is also an interesting problem to construct nonlinear integrable equations equipped with fractional derivative, difference or \( q \)-difference. Although it contains many difficult
problems, it is no doubt that the Mittag-Leffler function and its discrete analogues hold the key to this problem.

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References


