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Integrability Conditions for n and t Dependent Dynamical Lattice Equations

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Abstract

Conditions necessary for the existence of local higher order generalized symmetries and conservation laws are derived for a class of dynamical lattice equations with explicit dependence on the spatial discrete variable and on time. We explain how to use the obtained conditions for checking a given equation. We apply those conditions to the study of a special class of differential difference equations interesting from the physical point of view.

1 Introduction

In this paper we consider lattice equations of the form:

$$\ddot{u}_n = f_n(t,\ \dot{u}_n,\ u_n,\ u_{n+1},\ u_{n-1}) \tag{1.1}$$

$$\frac{\partial f_n}{\partial u_{n+1}} \neq 0, \quad \frac{\partial f_n}{\partial u_{n-1}} \neq 0 \quad \forall\ n. \tag{1.2}$$

Here $u_n = u_n(t)$, dot denotes the derivative w.r.t. the continuous time variable t, and $\{f_n\}$ are an infinite set of a priori different functions of five variables: $f_n = f_n(t, w_n, x_n, y_n, z_n)$, such that all the functions depend on y_n and z_n. Classical representative of the class is the Toda like equation:

$$\ddot{u}_n + \alpha\dot{u}_n + 2\alpha^2 n = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}}, \tag{1.3}$$

which will be discussed in the applications.

Conditions (1.2) are necessary as in Theorems 1 and 2 below we will need to divide by the functions $\frac{\partial f_n}{\partial u_{n+1}}$ and $\frac{\partial f_n}{\partial u_{n-1}}$. This request means that our results do not apply to equations like

$$\ddot{u}_n = u_n^2 + (1 + (-1)^n)\, u_{n+1} + (1 - (-1)^n)\, u_{n-1}, \tag{1.4}$$

which are the compact representation of systems of coupled equations. In fact, by the change of variable,

$$v_n = u_{2n+1}, \qquad w_n = u_{2n}, \tag{1.5}$$

eq.(1.4) reads

$$\ddot{v}_n = v_n^2 + 2w_n, \qquad \ddot{w}_n = w_n^2 + 2v_n, \tag{1.6}$$

and can be written as a nonlinear ODE of fourth order for v_n.

The main aim of this paper is to derive five conditions, necessary to prove the existence of local generalized symmetries and conservation laws of high enough order. The existence of infinite hierarchies of local generalized symmetries and conservation laws is a very common property of 1+1 dimensional equations integrable by inverse scattering method.

We present in Section 2 the necessary conditions for the integrability of equations of the class (1.1, 1.2). These conditions are very convenient for testing the integrability of a given equation, and in Section 3 we will explain how to apply them to some differential difference equation. In Section 4 we present some conclusive remarks.

In our work we follow the standard scheme of the Generalized Symmetry Approach (GSA) (for a review, see the articles [1–6]; the application of the method to discrete-differential equations was developed in [7–9]). By the GSA method we require that an equation (1.1) possesses local generalized symmetries and conservation laws of sufficently high order. This is always the case for integrable equations, even on the lattice [10–13]. The case when an equation together with its generalized symmetries and conservation laws do not depend explicitly on t and n has been considered in [8], where an exhaustive list of lattice equations of the form

$$\ddot{u}_n = f(\dot{u}_n,\ u_n,\ u_{n+1},\ u_{n-1}) \tag{1.7}$$

has been presented (see also [6], [14]). In [9] we can find the discussion of the explicit dependence on the discrete variable n for Volterra type equations

$$\dot{u}_n = f_n(u_n,\ u_{n+1},\ u_{n-1}), \tag{1.8}$$

satisfying conditions (1.2).

Explicit dependence on time in the framework of the GSA (both in the discrete and in the continuous cases), as far as we know, has never been considered in the literature.

We apply the obtained integrability conditions to few particular cases. The results explain why some interesting classes of equations cannot have hierarchies of local generalized symmetries and conservation laws. For example, the class of equations

$$\ddot{u}_n = A_n(t,\ u_n,\ u_{n+1}) + B_n(t,\ u_n,\ u_{n-1}), \tag{1.9}$$

with A_n, B_n satisfying conditions (1.2), cannot satisfy the obtained integrability conditions. By the change of variables (1.5), eq.(1.9) can be rewritten as

$$\begin{aligned} \ddot{v}_n &= a_n(t,\ v_n,\ w_{n+1}) + b_n(t,\ v_n,\ w_n), \\ \ddot{w}_n &= c_n(t,\ w_n,\ v_n) + d_n(t,\ w_n,\ v_{n-1}), \end{aligned} \tag{1.10}$$

with obvious conditions for $a_n,\ b_n, c_n$ and d_n derived from eq.(1.2). The transformation (1.5) preserves the local structure of the generalized symmetries and conservation laws (see the details below and in [9]) and thus the integrability conditions obtained here can be applied to eq.(1.10). Then also the class of equations (1.10), which contains some physically interesting equations [17], contains no integrable lattice systems. Comparing the results obtained here with those of paper [9], where some examples with an essential dependence on the discrete spatial variable n have been found in the case (1.8), we see that the case (1.1, 1.2) seems to be more restrictive.

2 Integrability conditions

Here we derive the necessary conditions for the integrability of equations of the class (1.1,1.2). At first those which follow from the existence of local generalized symmetries and then the additional ones following from the existence of conservation laws. At the end of the Section we discuss how to use those conditions for checking if a given equation is integrable and for classifying integrable cases.

2.1 Definitions

We are interested in considering local lattice equations with local symmetry structure. This means that the equation, its symmetries and conservation laws are expressed in every point n in terms of functions of many variables with no integrals or sums. Moreover we consider symmetries and conservation laws described only by restricted functions, i.e. functions g_n such that:

$$g_n = g_n(t, u_{n+i_1}, u_{n+i_1-1}, \ldots u_{n+i_2}, \dot{u}_{n+j_1}, \dot{u}_{n+j_1-1}, \ldots \dot{u}_{n+j_2}), \tag{2.1}$$

where $i_k = i_k(g_n)$, $j_k = j_k(g_n)$ are some fixed finite integers for any given g_n ($i_1 \geq i_2$, $j_1 \geq j_2$), and

$$\frac{\partial g_n}{\partial u_{n+i_1}} \neq 0, \quad \frac{\partial g_n}{\partial u_{n+i_2}} \neq 0, \quad \frac{\partial g_n}{\partial \dot{u}_{n+j_1}} \neq 0, \quad \frac{\partial g_n}{\partial \dot{u}_{n+j_2}} \neq 0$$

for at least some n. For example,

$$g_n = (1 + (-1)^n)u_{n+1} + (1 - (-1)^n)(\dot{u}_n + \dot{u}_{n-1})$$

is a restricted function with $i_1 = i_2 = 1$, $j_1 = 0$, $j_2 = -1$.

A local generalized symmetry of eq.(1.1) is an equation

$$u_{n,\tau} = g_n, \tag{2.2}$$

with g_n a restricted function (2.1), compatible with eq.(1.1). Moreover we assume that, both in eq.(1.1) and eq.(2.2), $u_n = u_n(t, \tau)$. The compatibility condition between eq.(1.1) and eq.(2.2) implies

$$\frac{\partial^3 u_n}{\partial\tau\partial t^2} = \frac{\partial^3 u_n}{\partial t^2 \partial\tau} \tag{2.3}$$

and means that eq.(1.1) and eq.(2.2) have a set of common solutions. Eq.(2.3) can be rewritten as:

$$D_\tau f_n - D_{tt} g_n = 0, \tag{2.4}$$

where D_t and D_τ are the total derivative operators with respect to t and τ respectively and the differentiation of the functions f_n and g_n is carried out taking into account eqs.(2.2, 1.1). For example

$$D_t g_n = \frac{\partial g_n}{\partial t} + \sum_k \frac{\partial g_n}{\partial u_{n+k}} \dot{u}_{n+k} + \sum_k \frac{\partial g_n}{\partial \dot{u}_{n+k}} f_{n+k}. \tag{2.5}$$

The variables u_{n+k}, $\dot{u}_{n+k}$ are considered to be independent (here and everywhere below) and then eq.(2.4) is a constraint for the functions f_n and g_n.

In accordance with the ideology of the GSA, we rewrite eq.(1.1) as a systems of two first order (w.r.t. time t) equations, introducing the new variable $v_n = \dot{u}_n$. Instead of eqs.(1.1, 2.2) we thus have the following compatible system of two vector equations

$$\dot{U}_n = F_n, \qquad U_{n,\tau} = G_n, \tag{2.6}$$

where

$$U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad F_n = \begin{pmatrix} v_n \\ f_n \end{pmatrix}, \quad G_n = \begin{pmatrix} g_n \\ h_n \end{pmatrix},$$

with $h_n = D_t g_n$. Condition (2.4) implies

$$D_t D_\tau U_n - D_\tau D_t U_n = D_t G_n - D_\tau F_n = 0, \tag{2.7}$$

and this is the compatibility condition for the system (2.6).

From eqs.(1.1, 2.2) we get that

$$F_n = F_n(t,\ U_n,\ U_{n+1},\ U_{n-1}),$$

$$G_n = G_n(t,\ U_{n+N},\ U_{n+N-1},\ \dots\ U_{n+M}), \tag{2.8}$$

where

$$\frac{\partial G_n}{\partial U_{n+N}} \neq 0, \qquad \frac{\partial G_n}{\partial U_{n+M}} \neq 0 \qquad \text{for some } n, \tag{2.9}$$

with $N = N(G_n)$ and $M = M(G_n)$ functions of $i_k(g_n)$ and $j_k(g_n)$. For instance $N = \max\ (i_1(g_n),\ j_1(g_n)+1)$. The partial derivative of a vector with respect to a vector, considered in eq.(2.9), is given by the matrix

$$\frac{\partial G_n}{\partial U_k} = \begin{pmatrix} \partial g_n/\partial u_k & \partial g_n/\partial v_k \\ \partial h_n/\partial u_k & \partial h_n/\partial v_k \end{pmatrix} . \tag{2.10}$$

The integers N and M are called left and right orders of the symmetry. In the case of the Toda model

$$\ddot{u}_n = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}}. \tag{2.11}$$

$M = -N$ and for any $N \geq 1$ we can find two different local generalized symmetries. If $N = 1$, one has an obvious symmetry with $G_n = F_n$, and a symmetry defined by

$$\begin{aligned} u_{n,\tau} &= g_n = e^{u_{n+1}-u_n} + e^{u_n-u_{n-1}} + v_n^2, \\ v_{n,\tau} &= u_{n,t\tau} = h_n = e^{u_{n+1}-u_n}(v_{n+1}+v_n) - e^{u_n-u_{n-1}}(v_n+v_{n-1}). \end{aligned} \tag{2.12}$$

In this case no explicit n and t dependence is present, $g_n = g(u_n, u_{n+1}, u_{n-1}, v_n)$ and $h_n = h(u_n, u_{n+1}, u_{n-1}, v_n, v_{n+1}, v_{n-1})$. An explicit n and t dependent symmetry for eq.(2.11) has been presented in [13] and is given by

$$g_n = t\left(\dot{u}_n^2 + e^{u_{n+1}-u_n} + e^{u_n-u_{n-1}} - 2\right) + (2n-1)\dot{u}_n + 2w_n(t)\ , \tag{2.13}$$

where $w_n(t)$ is defined by the following system of equations

$$w_{n+1}(t) - w_n(t) = \dot{u}_{n+1}\ , \qquad \dot{w}_n(t) = e^{u_{n+1}-u_n} - 1\ . \tag{2.14}$$

However, due to the presence of $w_n(t)$, the symmetry (2.13) is nonlocal. A local n and t dependent symmetry is presented in [13] for the Volterra equation

$$\dot{a}_n(t) = a_n(t)[a_{n-1}(t) - a_{n+1}(t)] \tag{2.15}$$

and reads:

$$\begin{aligned} g_n = a_n\{t[a_{n-1}(a_{n-2}+a_{n-1}+a_n-4) - a_{n+1}(a_{n+2}+a_{n+1}+a_n-4)] \\ +a_n - (n-1)a_{n-1} + (n+2)a_{n+1} - 4\}. \end{aligned} \tag{2.16}$$

A local conservation law of eq.(1.1) is a relation of the form

$$\dot{p}_n = q_{n+1} - q_n, \tag{2.17}$$

where p_n and q_n are restricted functions. The notion of conservation law is closely connected to that of constant of motion. In fact, if p_n has no explicit dependence on n, and one imposes the periodicity condition $u_{n+s} = u_n$, then the function $C_n = \sum_{k=1}^{s} p_{n+k}$ does not depend explicitely on time.

In the vector case, p_n and q_n are scalar functions of t and of the vectors U_{n+i}. If

$$p_n = p_n(t,\ U_{n+k_1},\ U_{n+k_1-1},\ \ldots\ U_{n+k_2}), \tag{2.18}$$

we can introduce its formal variational derivative

$$H_n = \frac{\delta p_n}{\delta U_n} = \sum_{i=-k_1}^{-k_2} \frac{\partial p_{n+i}}{\partial U_n}, \qquad \frac{\partial p_{n+i}}{\partial U_n} = \begin{pmatrix} \partial p_{n+i}/\partial u_n \\ \partial p_{n+i}/\partial v_n \end{pmatrix}. \tag{2.19}$$

In the case of nontrivial conservation law, the vector function $H_n \neq 0$ and has the form:

$$H_n = H_n(t,\ U_{n+m},\ U_{n+m-1},\ \ldots\ U_{n-m}), \tag{2.20}$$

where the matrices $\frac{\partial H_n}{\partial U_{n+m}}$, $\frac{\partial H_n}{\partial U_{n-m}}$ are different from zero for at least some n and $m \leq k_1 - k_2$[1].

The number m is the order of the conservation law (2.17). The Toda model (2.11) possesses two different conservation laws for any $m \geq 1$. In the simplest case $m = 1$, conservation laws are defined by the following conserved densities:

$$p_n = e^{u_{n+1}-u_n} + \frac{1}{2}v_n^2, \qquad \hat{p}_n = e^{u_{n+1}-u_n}(v_{n+1} + v_n) + \frac{1}{3}v_n^3. \tag{2.21}$$

2.2 Derivation of integrability conditions from formal symmetries

We are going to derive here three integrability conditions assuming the existence of local generalized symmetries. The theory of GSA for differential-difference equations is given in detail in [9]. Consequently, we will mainly pay attention to the peculiarity connected with the fact that we are considering a system of DDE's.

First of all, we introduce such formal symmetries that the integrability conditions do not depend on the order of the symmetry.

The Frechet derivative for the vector function (2.8) is defined by the matrix

$$G_n^* = \sum_{i=M}^{N} \frac{\partial G_n}{\partial U_{n+i}} T^i, \tag{2.22}$$

where T is the shift operator $Tf_n = f_{n+1}$. F_n^*, instead, is given by the following matrix operator

$$F_n^* = F_n^{(1)}T + F_n^{(0)} + F_n^{(-1)}T^{-1}, \tag{2.23}$$

with

$$F_n^{(1)} = \begin{pmatrix} 0 & 0 \\ \alpha_n & 0 \end{pmatrix}, \quad F_n^{(0)} = \begin{pmatrix} 0 & 1 \\ \beta_n & \gamma_n \end{pmatrix}, \quad F_n^{(-1)} = \begin{pmatrix} 0 & 0 \\ \delta_n & 0 \end{pmatrix}, \tag{2.24}$$

with

$$\alpha_n = \frac{\partial f_n}{\partial u_{n+1}}, \quad \beta_n = \frac{\partial f_n}{\partial u_n}, \quad \gamma_n = \frac{\partial f_n}{\partial v_n}, \quad \delta_n = \frac{\partial f_n}{\partial u_{n-1}}. \tag{2.25}$$

[1]In the case of (2.18), one may have $H_n = 0$, and m may be any number from 0 to $k_1 - k_2$ if $H_n \neq 0$

A formal symmetry is defined by the operator

$$A(L_n) = L_{n,t} - [F_n^*, L_n] = \sum_{i=-\infty}^{m+1} A_n^{(i)} T^i, \tag{2.26}$$

in terms of the *formal series*

$$L_n = L_n^m = \sum_{i=-\infty}^{m} l_n^{(i)} T^i \qquad l_n^{(m)} \neq 0 \ \text{ for some } \ n. \tag{2.27}$$

$l_n^{(i)}$ are 2×2 matrix coefficients and $L_{n,t} = \sum_{i=-\infty}^{m} \dot{l}_n^{(i)} T^i$ and $[F_n^*, L_n] = F_n^* L_n - L_n F_n^*$. Such series can be multiplied, using the standard rule $(l_n^{(i)} T^i)(l_n^{(j)} T^j) = l_n^{(i)} l_{n+i}^{(j)} T^{i+j}$. A series (2.27) is called a *formal symmetry* if it satisfies the equation $A(L_n) = 0$. Thus a formal symmetry satisfies the Lax equation:

$$L_{n,t} = [F_n^*, L_n]. \tag{2.28}$$

It can be proved (using a rather long calculation, see [1]) that the existence of an infinite hierarchy of generalized symmetries implies the existence of a formal symmetry. But, more simply we can look for approximate solutions of eq.(2.28). We will call a series (2.27) the Approximate Formal Symmetry (AFS) of order m and length k if the first k coefficients of $A(L_n)$ vanish, i.e.

$$A_n^{(i)} = 0, \qquad m+1 \geq i \geq m+2-k.$$

Applying the Frechet derivative to the compatibility condition (2.7), we are led to

$$D_t G_n^* - [F_n^*, G_n^*] = D_\tau F_n^* \ . \tag{2.29}$$

As by eq.(2.23) F_n^* contains only the shifts T^i with $i = -1, 0, 1$, eq.(2.29) shows that if there is a generalized symmetry (2.8) with $N \geq 1$, then the series

$$L_n = G_n^* + \ 0 \ T^{M-1} + \ 0 \ T^{M-2} + \ \ldots \tag{2.30}$$

is an AFS of the order and length $m = k = N$.

Let us denote $l_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix}$. The following Lemma shows that (as in the case of the Toda model) we can write down generalized symmetries of two types.

Lemma 1. *If the length of an AFS (2.27) of order m is $k \geq 2$, then*

$$b_n^{(m)} = 0, \tag{2.31}$$
$$d_n^{(m)} = a_n^{(m)}, \tag{2.32}$$

and $\forall\ n$ either $a_n^{(m)} \neq 0$ or $a_n^{(m)} = 0,\ c_n^{(m)} b_n^{(m-1)} \neq 0$.

Proof. From eq.(2.26) it follows that the condition $A_n^{(m+1)} = 0$ implies $F_n^{(1)} l_{n+1}^{(m)} = l_n^{(m)} F_{n+m}^{(1)}$ and gives two relations. The first of them is

$$\alpha_n a_{n+1}^{(m)} = d_n^{(m)} \alpha_{n+m} \ , \tag{2.33}$$

where α_n, defined in eq.(2.25), due to condition (1.2), cannot be zero for any n. The second is $\alpha_n b_{n+1}^{(m)} = 0$ and implies eq.(2.31). The condition $A_n^{(m)} = 0$ gives the matrix equation

$$l_{n,t}^{(m)} - F_n^{(1)} l_{n+1}^{(m-1)} - F_n^{(0)} l_n^{(m)} + l_n^{(m)} F_{n+m}^{(0)} + l_n^{(m-1)} F_{n+m-1}^{(1)} = 0. \tag{2.34}$$

The elements of the right upper corner of the matrix equation (2.34) provide us with eq.(2.32), and eq.(2.33) takes the form:

$$\alpha_n a_{n+1}^{(m)} = a_n^{(m)} \alpha_{n+m} \ . \tag{2.35}$$

As α_n cannot be zero, one has only two possibilities: $a_n^{(m)} \neq 0$ for any n and in this case $a_n^{(m)}$, solution of eq.(2.35), is written in terms of α_{n+j}, and the AFS will be denoted as an *AFS of the 1st type* or $a_n^{(m)} = 0$. In the second case, we use the diagonal elements of eq.(2.34) and get

$$c_n^{(m)} = \alpha_n b_{n+1}^{(m-1)} = b_n^{(m-1)} \alpha_{n+m-1} \ . \tag{2.36}$$

One can see that again there are only two possibilities $b_n^{(m-1)} \neq 0$ or $b_n^{(m-1)} = 0$ for all n. The first case corresponds to what we will denote as *an AFS of the 2nd type*. The last case is impossible because in this case we would have also $c_n^{(m)} = 0$ for any n, but this is in contradiction with the condition (2.27) for $l_n^{(m)}$. ∎

So, as in the case of the Toda lattice model (see for example eq.(2.12)), all known integrable lattice equations of the form (1.1) are such that their generalized symmetries generate for any order $m \geq 1$ two AFS, one of each type. In order to make theory simpler, we will assume that our equations (1.1, 1.2) have this symmetry structure. It should be remarked that the same integrability conditions we are going to obtain under this hypothesis, could be derived using only one generalized symmetry of high enough order. However, in this case the calculation would be more complicate (cf. [9]).

Integrability conditions are obtained, calculating coefficients of 1st type AFS starting from the Lax equation (2.28). A 2nd type AFS leads to the same conditions, as will be shown in Proposition 1. Generalized symmetries provide us with AFS such that $m = k$, and we can formulate the following Proposition for an AFS of this kind.

Proposition 1. *If L_n is a 2nd type AFS such that $m = k \geq 2$, then $(L_n)^2$ is an AFS of the 1st type of order $2m - 1$ and length $m - 1$.*

Proof. Let us consider the first two coefficients of the series

$$(L_n)^2 = l_n^{(m)} l_{n+m}^{(m)} T^{2m} + (l_n^{(m)} l_{n+m}^{(m-1)} + l_n^{(m-1)} l_{n+m-1}^{(m)}) T^{2m-1} + \ \ldots . \tag{2.37}$$

Using eq.(2.36), one can see that the first coefficient in eq.(2.37) vanishes, and the second one has the form of the leading coefficient of a 1st type AFS. The formula

$$A(L_n\hat{L}_n) = A(L_n)\hat{L}_n + L_n A(\hat{L}_n) \tag{2.38}$$

shows that $(L_n)^2$ will be an AFS of the order $2m-1$. As $m = k \geq 2$, the series $A(L_n)$ is of the first order, i.e. $A(L_n) = \sum_{i=-\infty}^{1} A_n^{(i)} T^i$. Then the right hand side of eq.(2.38) with $\hat{L}_n = L_n$ has the order $m+1$, but the order of the left hand side equals $2m$. The difference of those orders gives us the length $m-1$. ∎

With no loss of generality, we can derive integrability conditions using only 1st type AFS of the first order with arbitrary long length.

The inverse series $(L_n)^{-1}$ is found using the standard definition

$$L_n(L_n)^{-1} = (L_n)^{-1}L_n = E, \qquad (L_n)^{-1} = \sum_{i=-\infty}^{\widetilde{m}} \tilde{l}_n^{(i)} T^i,$$

where E is the unit matrix operator. In the case of 1st type AFS (2.27), $\widetilde{m} = -m$ and

$$\tilde{l}_n^{(-m)} = (l_{n-m}^{(m)})^{-1}, \qquad \tilde{l}_n^{(-m-1)} = -(l_{n-m}^{(m)})^{-1} l_{n-m}^{(m-1)} (l_{n-m-1}^{(m)})^{-1}, \quad \dots .$$

Inverse series will be of 1st type AFS again with the order $\widetilde{m} = -m$ and the same length $\tilde{k} = k$. This can be easily checked starting from $A(L_n^{-1}) = -L_n^{-1} A(L_n) L_n^{-1}$. Using the invertibility and formula (2.38), one easily proves the following Proposition, formulated for an AFS with $m = k$.

Proposition 2. *If L_n and $\hat{L}_n$ are 1st type AFS with orders and lengths such that $m = k \geq 1$ and $\widehat{m} = \hat{k} = k+1$, respectively, then the series $\tilde{L}_n = (L_n)^{-1}\hat{L}_n$ is a 1st type AFS with order $\widetilde{m} = 1$ and length $\tilde{k} = k$.*

Starting from generalized symmetries, we have obtained AFS of the first order. Its length depends on the order of the symmetries and can be arbitrarily long. Such AFS symplify the calculations necessary to derive integrability conditions. Moreover, in this case the resulting conditions will not depend on the order of the generalized symmetries.

Theorem 1. *The existence of 1st type AFS of order $m = 1$ and length $k \geq 3$ implies the following conditions:*

$$\dot{p}_n^{(i)} = (T-1)\, q_n^{(i)}, \qquad i = 1, 2, 3, \tag{C1}$$

$$p_n^{(1)} = \log \frac{\partial f_n}{\partial u_{n+1}}\,, \qquad p_n^{(2)} = \mu \left(2q_n^{(1)} + \frac{\partial f_n}{\partial \dot{u}_n} \right),$$

$$p_n^{(3)} = 2\mu q_n^{(2)} + 2n\dot{\mu} p_n^{(2)} + \frac{1}{4}(p_n^{(2)})^2 + \mu^2 \left(\frac{1}{4}\left(\frac{\partial f_n}{\partial \dot{u}_n}\right)^2 - \frac{1}{2}\left(\frac{\partial f_n}{\partial \dot{u}_n}\right)_t + \frac{\partial f_n}{\partial u_n} \right),$$

where $q_n^{(i)}$ are some restricted functions, and $\mu = \mu(t) \neq 0$.

Proof. The existence of a 1st type AFS, L_n of order 1 and lenght $k \geq 3$, enables one to use the first three conditions of $A(L_n) = 0$, i.e. we can require that $A_n^{(2)} = A_n^{(1)} = A_n^{(0)} = 0$. We already know that the leading term of L_n (as L_n is of the 1st type) is given by

$$l_n^{(1)} = \begin{pmatrix} a_n^{(1)} & 0 \\ c_n^{(1)} & a_n^{(1)} \end{pmatrix}, \qquad a_n^{(1)} \neq 0 \ \ \forall\, n \ .$$

We also may use eq.(2.35) with $m = 1$ and the results presented in Appendix A to show that

$$\frac{a_{n+1}^{(1)}}{\alpha_{n+1}} = \frac{a_n^{(1)}}{\alpha_n} = \mu^2(t) \neq 0. \tag{2.39}$$

Redefining $L_n \to \mu^2 L_n$, eq.(2.26) reads

$$\hat{A}(L_n) = L_{n,t} + \theta(t) L_n - [F_n^*, L_n] = \sum_{i=-\infty}^{2} \hat{A}_n^{(i)} T^i, \qquad \theta = 2\mu'/\mu. \tag{2.40}$$

The three first coefficients $\hat{A}_n^{(i)}$ are equal to zero, and instead of eq.(2.39) we have $a_n^{(1)} = \alpha_n$.

The condition $\hat{A}_n^{(1)} = 0$ gives:

$$\dot{\alpha}_n - c_n^{(1)} + \alpha_n(\theta + b_n^{(0)}) = 0, \tag{2.41}$$

$$\dot{\alpha}_n + c_n^{(1)} + \alpha_n(\theta - b_{n+1}^{(0)} + \gamma_{n+1} - \gamma_n) = 0, \tag{2.42}$$

$$\dot{c}_n^{(1)} + c_n^{(1)}(\theta - \gamma_n) + \alpha_n(\beta_{n+1} - \beta_n + d_n^{(0)} - a_{n+1}^{(0)}) = 0, \tag{2.43}$$

with the functions α_n, β_n, γ_n defined in eq.(2.25). The sum of eqs.(2.41, 2.42) divided by α_n read

$$2(\log \alpha_n)_t = (T - 1)(b_n^{(0)} - \gamma_n - 2n\theta).$$

So condition (C1) with $i = 1$ is satisfied, and

$$b_n^{(0)} = 2q_n^{(1)} + \gamma_n + 2n\theta. \tag{2.44}$$

$c_n^{(1)}$ follows from eq.(2.41):

$$\frac{c_n^{(1)}}{\alpha_n} = (T + 1)q_n^{(1)} + \gamma_n + (2n + 1)\theta. \tag{2.45}$$

The elements of the right upper corner of the equation $\hat{A}_n^{(0)} = 0$ provide us with the relation:

$$\dot{b}_n^{(0)} + b_n^{(0)}(\theta + \gamma_n) + a_n^{(0)} - d_n^{(0)} = 0. \tag{2.46}$$

Let us rewrite eq.(2.43) divided by α_n as

$$\left(\frac{c_n^{(1)}}{\alpha_n}\right)_t + \frac{c_n^{(1)}}{\alpha_n}(\theta - \gamma_n + (\log \alpha_n)_t) + d_n^{(0)} - a_{n+1}^{(0)} + (T-1)\beta_n = 0. \tag{2.47}$$

From eqs.(2.44, 2.45) and condition (C1) with $i = 1$, we find that the sum of eqs.(2.46, 2.47) reads

$$2\dot{\varphi}_n + \theta\varphi_n = (T-1)(a_n^{(0)} - q_{n,t}^{(1)} - (q_n^{(1)})^2 - 2n\theta q_n^{(1)} - \beta_n - c_n(t)), \tag{2.48}$$

with

$$\varphi_n = 2q_n^{(1)} + \gamma_n \ , \qquad (T-1)c_n = (4n+1)(\theta' + \theta^2).$$

Multiplying eq.(2.48) by μ, we see that also the condition (C1) with $i = 2$ is satisfied. From eqs.(2.46, 2.48), one gets $a_n^{(0)}$ and $d_n^{(0)}$ in terms of $q_n^{(2)}$. In such a way we will get formulae analogous to eqs.(2.44, 2.45), but more complicated.

Let us consider the sum of the diagonal elements of the equation $\hat{A}_n^{(0)} = 0$. It can be expressed in the form:

$$(a_n^{(0)} + d_n^{(0)})_t + \theta(a_n^{(0)} + d_n^{(0)}) = (T-1)(b_n^{(-1)}\alpha_{n-1}) \sim 0 \ ,$$

the equivalence relation being defined in Appendix A. Multiplying by μ^2, we see that $(\mu^2(a_n^{(0)} + d_n^{(0)}))_t \sim 0$. This conserved density $\mu^2(a_n^{(0)} + d_n^{(0)})$ is equivalent to $2p_n^{(3)}$. ∎

2.3 Formal conserved densities

Now we derive some additional integrability conditions by requiring the existence of local conservation laws. The theory is very similar to the one of Section 2.2, and we briefly discuss the main points of it, comparing the results with those of that Section.

In Section 2.1 we gave the definition of local conservation laws (2.17) and of their order. Applying the operator $\delta/\delta U_n$ introduced in eq.(2.19) to eq.(2.17), we obtain

$$\delta\dot{p}_n/\delta U_n = 0, \tag{2.49}$$

and eq.(2.49) can be written in the form:

$$(D_t + F_n^{*\dagger})H_n = 0 \ . \tag{2.50}$$

Here D_t is the operator of total time differentiation (2.5), H_n is the formal variational derivative (2.19) of the conserved density p_n, and $F_n^{*\dagger}$ is the transposed of the Frechet derivative F_n^*, defined by

$$F_n^{*\dagger} = \sum_{i=-1}^{1}\left(\frac{\partial F_{n+i}}{\partial U_n}\right)^{\dagger} T^i = (F_{n+1}^{(-1)})^{\dagger}T + (F_n^{(0)})^{\dagger} + (F_{n-1}^{(1)})^{\dagger}T^{-1},$$

where $F^{(j)}_{n+i}$ have been defined in eq.(2.23), and † denotes matrix transposition. One gets

$$F_n^{*\dagger} = \tilde{F}_n^{(1)} T + \tilde{F}_n^{(0)} + \tilde{F}_n^{(-1)} T^{-1},$$

where the coefficients $\tilde{F}_n^i$ are given by

$$\tilde{F}_n^{(1)} = \begin{pmatrix} 0 & \delta_{n+1} \\ 0 & 0 \end{pmatrix}, \quad \tilde{F}_n^{(0)} = \begin{pmatrix} 0 & \beta_n \\ 1 & \gamma_n \end{pmatrix}, \quad \tilde{F}_n^{(-1)} = \begin{pmatrix} 0 & \alpha_{n-1} \\ 0 & 0 \end{pmatrix}$$

with α_n, β_n, γ_n, δ_n given by eq.(2.25).

Let us note that the compatibility condition (2.7) can be rewritten as an equation for the right hand side G_n of the generalized symmetry:

$$(D_t - F_n^*) G_n = 0 \ , \tag{2.51}$$

and this equation is very similar to eq.(2.50).

In Section 2.2, instead of solving eq.(2.7) for the symmetries, we considered the formal symmetry L_n, defined by the equation $A(L_n) = 0$, where $A(L_n)$ is defined by eq.(2.26)). Here, instead of solving eq.(2.50) for the variational derivative of a conserved density, we consider the formal conserved density. A formal conserved density will be a solution S_n of the equation $B(S_n) = 0$, where

$$B(S_n) = S_{n,t} + S_n F_n^* + F_n^{*\dagger} S_n = \sum_{i=-\infty}^{m+1} B_n^{(i)} T^i, \tag{2.52}$$

with

$$S_n = \sum_{i=-\infty}^{m} S_n^{(i)} T^i, \qquad S_n^{(m)} \neq 0 \ \text{ for some } \ n, \tag{2.53}$$

and $B_n^{(i)}$, $S_n^{(i)}$ 2×2 matrices. We will consider approximate solutions of the equation $B(S_n) = 0$ which will be called Approximate Conserved Densities (ACD). The series S_n, given by eq.(2.53), is an ACD of order m and length k if the first k coefficients $B_n^{(i)}$ of the series $B(S_n)$ vanish.

After applying the operation * to eq.(2.50), one obtains

$$B(H_n^*) = \sum_{i=-2}^{2} \widetilde{H}_n^{(i)} T^i, \tag{2.54}$$

where $\widetilde{H}_n^{(i)}$ are some matrix coefficients which form is not essential. The series

$$S_n = H_n^* + \ 0\ T^{-m-1} + \ 0\ T^{-m-2} + \ \dots \tag{2.55}$$

is of the order m (see formula (2.20) and the definition of the Frechet derivative (2.22, 2.8)). For this reason, eq.(2.54) shows that, if p_n is a conserved density of order $m \geq 2$, then the series (2.55) is an ACD of order m and length $k = m - 1$. So, starting from a local conservation law (2.17), using the formal variational derivative (2.19) we get an ACD (2.55) (cf. with eq.(2.30)).

Introducing the following notation for the coefficients of an ACD (2.53):

$$S_n^{(i)} = \begin{pmatrix} \alpha_n^{(i)} & \beta_n^{(i)} \\ \gamma_n^{(i)} & \delta_n^{(i)} \end{pmatrix},$$

one can prove, as in Lemma 1, that if the length of an ACD is $k \geq 2$, then

$$\delta_n^{(m)} = 0, \qquad \gamma_n^{(m)} = -\beta_n^{(m)}, \tag{2.56}$$

and two cases only are possible:

$$\begin{array}{ll} \text{1st type ACD:} & \beta_n^{(m)} \neq 0 \quad \forall\, n\ , \\ \text{2nd type ACD:} & \beta_n^{(m)} = 0, \quad \alpha_n^{(m)} \delta_n^{(m-1)} \neq 0 \quad \forall\, n\ . \end{array}$$

For all known integrable equations of the form (1.1) and for any order $m \geq 1$, the existence of local conservation laws generate two ACD of length $m-1$, one of each types (see example eq.(2.21)). In order to obtain here additional integrability conditions, we assume the same should be true for the conservation laws.

As it has been already said, the integrability conditions (C1) can be derived, using only one generalized symmetry of high enough order. To get all five conditions, we could use a generalized symmetry and a conservation law of high enough order (with no connection between the simmetry and the conservation law and no restriction for the type and order) or a pair of local conservation laws. However, in such a case, the calculations would be more complicate. So we use the following scheme.

Given an ACD S_n and an AFS L_n, one obtains another ACD by considering their product $S_n L_n$, as

$$B(S_n L_n) = B(S_n) L_n + S_n A(L_n)\ . \tag{2.57}$$

If the ACD S_n is of the 2nd type of order m_1 and length k_1, and the AFS L_n is also of 2nd type of order m_2 and length $k_2 > k_1$, then the new ACD $S_n L_n$ will be of the 1st type, of order $m = m_1 + m_2 - 1$ and length $k = k_1 - 1$. This shows that any ACD of 2nd type can be reduced to one of the 1st type.

Let us consider a 1st type ACD S_n of order $m \geq 2$ and length $m - 1$ and an AFS L_n of the 1st type of order 1 and length greater than $m - 1$ (see Proposition 2). It is easy to verify that $S_n L_n^{1-m}$ is a new ACD of the 1st type of order 1 and length $m-1$ (see eqs.(2.38, 2.57)). We will use such an ACD for deriving additional integrability conditions. As the ACD is of 1st order, those conditions will not depend on orders of local conservation laws.

Theorem 2. *If there exists an ACD of the 1st type of order $m = 1$ and length $k \geq 2$, then we must have*

$$r_n^{(i)} = (T - 1) s_n^{(i)}, \qquad i = 1, 2, \tag{C2}$$

$$r_n^{(1)} = \log\left(\frac{\partial f_n}{\partial u_{n+1}} / \frac{\partial f_n}{\partial u_{n-1}}\right), \qquad r_n^{(2)} = \dot{s}_n^{(1)} + \frac{\partial f_n}{\partial \dot{u}_n} \ ,$$

where $s_n^{(i)}$ *are some restricted functions.*

Proof. As S_n is an ACD of the 1st type with $m = 1$, after defining $\beta_n^{(1)} = -\gamma_n^{(1)} = \varphi_n \neq 0$, we have

$$S_n^{(1)} = \begin{pmatrix} \alpha_n^{(1)} & \varphi_n \\ -\varphi_n & 0 \end{pmatrix}.$$

The equation:

$$B_n^{(2)} = S_n^{(1)} F_{n+1}^{(1)} + \tilde{F}_n^{(1)} S_{n+1}^{(1)} = 0$$

obtained from eq.(2.52) with $m = 1$, gives the relation

$$\varphi_{n-1}\alpha_n = \delta_n \varphi_n \ , \qquad (2.58)$$

with α_n, δ_n defined by eq.(2.25). Than one can see that

$$r_n^{(1)} = \log(\frac{\alpha_n}{\delta_n}) = (T-1)(\log \varphi_{n-1}) \sim 0,$$

i.e the first condition of (C2) is satisfied. We can define $s_n^{(1)} = \log \varphi_{n-1}$, where $s_n^{(1)}$ is the restricted function defined by (C2) with $i = 1$.

Let us consider the second equation, $B_n^{(1)} = 0$. The difference of the right upper and left lower corner elements divided by φ_n gives

$$\Omega_n = 2(\log \varphi_n)_t + \gamma_{n+1} + \gamma_n + \frac{\delta_{n+1}}{\varphi_n}\delta_{n+1}^{(0)} - \frac{\alpha_n}{\varphi_n}\delta_n^{(0)} = 0. \qquad (2.59)$$

Using eqs.(2.58, 2.59), Ω_n can be written as

$$\Omega_n = 2s_{n+1,t}^{(1)} + (T+1)\gamma_n + (T-1)\frac{\delta_n \delta_n^{(0)}}{\varphi_{n-1}} \sim 2(s_{n,t}^{(1)} + \gamma_n) = 2r_n^{(2)}.$$

This shows that $r_n^{(2)} \sim 0$, i.e. the second condition of (C2) is satisfied as well.∎

2.4 Discussion of the integrability conditions (C1, C2)

Let us discuss here some consequences of the integrability conditions (C1, C2) which will be useful for classifying equations and checking their integrability. In particular, we will present the explicit form of the integrability conditions, and at the end we will consider the particularly important case of equation (1.1) with f_n not depending on $\dot{u}_n$.

It is convenient to check the integrability conditions (C1, C2), taking into account eq.(A.3) of Appendix A, i.e. the fact that, if higher symmetries and conservation laws do exist, the following equalities must be valid

$$\frac{\delta}{\delta u_n}\dot{p}_n^{(i)} = \frac{\delta}{\delta \dot{u}_n}\dot{p}_n^{(i)} = 0 \qquad (i = 1, 2, 3), \qquad (2.60)$$

$$\frac{\delta}{\delta u_n} r_n^{(i)} = \frac{\delta}{\delta \dot{u}_n} r_n^{(i)} = 0 \qquad (i = 1, 2). \tag{2.61}$$

From (C1, C2), $p_n^{(1)}$ and $r_n^{(1)}$ are defined explicitly in terms of f_n of eq.(1.1). So the conditions (2.60, 2.61) with $i = 1$ are explicit. One only needs to apply partial derivatives and arithmetical operations, and than checks if a function is equal to zero. Such checking can be easily done by a computer.

To check eqs.(2.60, 2.61) with $i = 2$, we need first to find the functions $q_n^{(1)}$, $s_n^{(1)}$. It turns out that also these conditions can be written in explicit form. All partial derivatives of $q_n^{(1)}$, $s_n^{(1)}$ w.r.t. u_{n+k}, $\dot{u}_{n+k}$ are found by differentiating the relations (C1, C2) with $i = 1$. Taking into account the form of $p_n^{(2)}$ given in (C1), we see that only the following two terms: $2\dot{\mu} q_n^{(1)}$ and $2\mu \partial q_n^{(1)}/\partial t$ in $\dot{p}_n^{(2)}$ cannot be rewritten in an explicit form (cf. eq.(2.5)). However, after applying variational derivatives w.r.t. u_n or $\dot{u}_n$, we get an explicit expression for eq.(2.60) with $i = 2$. Exactly the same will happen for $r_n^{(2)}$ given by eq.(C2).

As an example, let us consider eq.(2.61) with $i = 2$. The function $r_n^{(1)}$ is a restricted function of the type of eq.(2.1) with $i_1 = 1$, $i_2 = -1$, $j_1 = j_2 = 0$. This means, see Appendix A, that $r_n^{(1)}$ and $s_n^{(1)}$ cannot depend on $\dot{u}_{n+k}$, and thus

$$s_n^{(1)} = s_n^{(1)}(t, u_n, u_{n-1}) \ . \tag{2.62}$$

Consequently

$$r_n^{(2)} = \frac{\partial s_n^{(1)}}{\partial t} + \frac{\partial s_n^{(1)}}{\partial u_n}\dot{u}_n + \frac{\partial s_n^{(1)}}{\partial u_{n-1}}\dot{u}_{n-1} + \frac{\partial f_n}{\partial \dot{u}_n} \ ,$$

and differentiating eq.(C2) with $i = 1$, one obtains

$$\frac{\partial s_n^{(1)}}{\partial u_n} = \frac{\partial r_{n-1}^{(1)}}{\partial u_n} \ , \qquad \frac{\partial s_n^{(1)}}{\partial u_{n-1}} = -\frac{\partial r_n^{(1)}}{\partial u_{n-1}} \ .$$

So we have

$$\frac{\delta r_n^{(2)}}{\delta \dot{u}_n} = \frac{\partial}{\partial u_n}(r_{n-1}^{(1)} - r_{n+1}^{(1)}) + \frac{\partial^2 f_n}{\partial \dot{u}_n^2} = 0 \ . \tag{2.63}$$

In this way we have obtained an explicit integrability condition. As $r_n^{(1)}$ has no dependence on $\dot{u}_{n+k}$, differentiating eq.(2.63) w.r.t. $\dot{u}_n$, we obtain

$$\partial^3 f_n / \partial \dot{u}_n^3 = 0 \quad \forall\ n \ . \tag{2.64}$$

This is a very simple and general necessary condition for the integrability which implies that any integrable lattice equation of the form (1.1), (1.2) may have only quadratic dependence on $\dot{u}_n$.

Let us study the particularly important case of equations (1.1, 1.2) with $\partial f_n/\partial \dot{u}_n = 0$. In this case the integrability conditions can be greatly simplified. From the form of $r_n^{(2)}$

and from eq.(2.62) it follows that conditions (C2) with $i = 2$ is equivalent to $s_n^{(1)} \sim 0$ (see eq.(A.8) of Appendix A). Also, the conserved density $p_n^{(1)}$ of conditions (C1) do not depend of $\dot{u}_{n+k}$, and thus it is trivial, i.e. is a total difference: $p_n^{(1)} = (T-1)\omega_n$. We have $q_n^{(1)} = \dot{\omega}_n$, as ω_n is defined up to an arbitrary function of t. By redefining: $p_n^{(2)} \mapsto 2p_n^{(2)}$ and $q_n^{(2)} \mapsto 2q_n^{(2)}$, we can express $p_n^{(2)}$ and $p_n^{(3)}$ of conditions (C1) in terms of ω_n, as stated in the following Theorem.

Theorem 3. *When*

$$\partial f_n / \partial \dot{u}_n = 0 \tag{2.65}$$

for any n, the integrability conditions for the lattice equations (1.1, 1.2) can be simplified and, instead of conditions (C2) with $i = 2$ and conditions (C1) with $i = 1$, we have

$$s_n^{(1)} \sim 0 \ , \qquad p_n^{(1)} = (T-1)\ \omega_n \tag{$C3$}$$

where ω_n is a restricted function. The conserved densities $p_n^{(2)}$ and $p_n^{(3)}$ of conditions (C1) are replaced by

$$p_n^{(2)} = \mu(t)\ \dot{\omega}_n \ , \qquad p_n^{(3)} = 4\mu\ (q_n^{(2)} + n\dot{\mu}\dot{\omega}_n) + \mu^2 \left(\dot{\omega}_n^2 + \frac{\partial f_n}{\partial u_n} \right). \tag{2.66}$$

Let us notice that we must interpret the functions $q_n^{(i)}$, $s_n^{(i)}$, ω_n and $\mu(t)$ in the same way as in all the previous integrability conditions: we have only to require the existence of those functions, such that they satisfy the conditions (C1, C2, C3). When studying a given equation, i.e. for a given function f_n, we at first define the functions $q_n^{(i)}$, $s_n^{(i)}$, ω_n up to an arbitrary t - dependent integration function, i.e. up to elements of $\mathrm{Ker}(T-1)$. Then we have to require that some concrete t-dependent functions exist (instead of arbitrary ones) such that the conditions are satisfied.

Let us notice that the integrability conditions (C1, C2) are the same as those of the previous papers [8] and [6], where the equations and their local generalized symmetries and conservation laws had no explicit dependence on n and t. The dependence on time introduce a new function $\mu(t)$ which, when no t dependence is allowed, reduces to a constant.

Equation (1.3) is an example of equation with $\dot{\mu} \neq 0$. In fact,

$$f_n = (T-1)e^{u_n - u_{n-1}} - \alpha\dot{u}_n - 2\alpha^2 n \sim -\alpha\dot{u}_n \ , \tag{2.67}$$

$p_n^{(1)}$ of condition (C1) has the form $p_n^{(1)} = (T-1)u_n$ and $q_n^{(1)} = \dot{u}_n + \beta(t)$, where β is an arbitrary integration function. $p_n^{(2)} \sim 2\mu\dot{u}_n$ and we can find its time derivative:

$$\dot{p}_n^{(2)} \sim 2\dot{\mu}\dot{u}_n + 2\mu f_n \sim 2(\dot{\mu} - \alpha\mu)\dot{u}_n$$

(see eqs.(A.4, 2.67)). Condition (C1) with $i = 2$ implies $\dot{\mu} = \alpha\mu$ which shows that $\dot{\mu} \neq 0$ if $\alpha \neq 0$. Equation (1.3) satisfies all conditions (C1, C2). This is not surprising, as the transformation

$$\tilde{u}_n = u_n + 2n(\alpha t - \log\alpha) \ , \qquad \tilde{t} = e^{-\alpha t} \tag{2.68}$$

reduces eq.(1.3) to the Toda lattice equation (2.11) [18]. Eq.(2.68) is a point transformations depending on n and t which do not change the integrability of eqs.(1.3, 2.11). The two equations related by the transformations(2.68) are equivalent.

The following three lattice equations are other examples of equations which satisfy the integrability conditions (C1, C2):

$$\ddot{u}_n = P(\dot{u}_n)(\varphi(u_{n+1} - u_n) - \varphi(u_n - u_{n-1})) \ ,$$

$$\varphi'(z) = Q(\varphi(z)) \ , \qquad P''(z) = Q''(z) = \mathrm{const} \ ; \tag{2.69}$$

$$\ddot{u}_n = (R(u_n) - \dot{u}_n^2)\left(\frac{1}{u_{n+1} - u_n} - \frac{1}{u_n - u_{n-1}}\right) + \frac{R'(u_n)}{2} \ , \quad \frac{\partial^5 R(z)}{\partial z^5} = 0 \ ; \tag{2.70}$$

$$\ddot{u}_n = \exp(u_{n+1} - 2u_n + u_{n-1}) \ . \tag{2.71}$$

All the coefficients in the polynomials P, Q, R are arbitrary constants (which may be complex too), i.e. really eqs.(2.69, 2.70) are classes of equations with many constant parameters. The Toda lattice equation (2.11) belong to the class (2.69).

Equations (2.69)-(2.71) give a complete list of equations, up to point transformations

$$\tilde{u}_n = \sigma_n(t, u_n) \ , \qquad \tilde{t} = \theta(t), \tag{2.72}$$

satisfying the conditions (C1, C2) in the n- and t-independent case [8]. For any fixed constant coefficients, the lattice equations (2.69)-(2.71) are integrable in the sense that have infinite hierarchies of generalized symmetries and conservation laws [6].

As a further example, let us consider two known equations with an explicit dependence on the discrete variable n which are of the form (1.1, 1.2) and are related to the Toda lattice. The first of them was considered in the paper [15] and has the form:

$$\ddot{u}_n = a_n e^{\varrho_{n+2} u_{n+1} - \varrho_n u_n} + b_n e^{\varrho_{n+1} u_n - \varrho_{n-1} u_{n-1}} + A_n \dot{u}_n^2 + B_n \dot{u}_n + C_n \ , \tag{2.73}$$

where

$$\varrho_n = \xi n + \zeta \neq 0 \ , \qquad \xi \neq 0 \tag{2.74}$$

for all n, and the other coefficients are specific functions depending only on n. The second one has been found in [9]:

$$\frac{\ddot{u}_n}{\varrho_{n+1}\varrho_n} = \exp\frac{u_{n+1} - u_n}{\varrho_{n+1}} - \exp\frac{u_n - u_{n-1}}{\varrho_n} \tag{2.75}$$

with ϱ_n given by eq.(2.74). Both of them belong to the class

$$\ddot{u}_n = \alpha_n(t) e^{\beta_n(t) u_{n+1} + \gamma_n(t) u_n} + \phi_n(t, \dot{u}_n, u_n, u_{n-1}) \ , \tag{2.76}$$

with $\alpha_n \beta_n \neq 0$. Applying the conditions (C1) with $i = 1$ to eq.(2.76), one easily can check that

$$p_{n,t}^{(1)} \sim (\beta_{n-1} + \gamma_n)\dot{u}_n + (\beta_{n-1}' + \gamma_n')u_n \sim 0$$

and immediately obtains the following condition: $\beta_n = -\gamma_{n+1}$ for all n. Both equations (2.73) and (2.75) do not satisfy this condition and, consequently, cannot have the local symmetry structure. It is not surprising, as for example eq.(2.73) has in the corresponding spectral problem the spectral parameter with a dependence on the time (as a master symmetry) [15]. The equation (2.75) is closely related to the potential Toda lattice (2.71) by a transformation of the form:

$$\tilde{u}_n = \varrho_n u_{n+1} - \varrho_{n+1} u_n + \eta_n \ ,$$

where η_n is a specific function defined by ϱ_n (if u_n is a solution of eq.(2.71), then $\tilde{u}_n$ satisfies eq.(2.75)). After this transformation, standard generalized symmetries and conservation laws of eq.(2.71) become nonlocal (are not expressed in terms of restricted functions (2.1)).

3 Applications

We apply here the integrability conditions to some classes of equations characterized by the fact of either have many point symmetries or are of physical interest. They belong to the particular case of eq.(1.1, 1.2) when $\partial f_n / \partial \dot{u}_n = 0$ for any n. We are not able to classify such equations completely, but we can solve the problem in each of the two following subclasses:

$$\partial^2 f_n / \partial u_{n+1} \partial u_{n-1} \neq 0 \quad \forall \ n \ , \tag{3.1}$$

$$\partial^2 f_n / \partial u_{n+1} \partial u_{n-1} = 0 \quad \forall \ n \ . \tag{3.2}$$

A number of lattice equations contained in the papers [16] and [17] belong to those subclasses.

There is in [16] a classification of equations of the form (1.1), (2.65) according to their Lie point symmetry algebras. One can assume that the existence of many symmetries is an indication of integrability. Consequently we can ask ourselves if some of the equations of [16] are integrable in the sense we discuss here. The highest dimensions of the symmetry algebras are 7 and 6, and the corresponding equations (1.1, 2.65) are defined by three types of functions f_n:

$$f_n = a_n \xi_n^\gamma \ , \qquad f_n = e^{a_n \xi_n + b_n} \ , \qquad f_n = a_n \log \xi_n + b_n \ , \tag{3.3}$$

where

$$\xi_n = \alpha_n (u_{n+1} - u_n) - \alpha_{n+1} (u_n - u_{n-1}) \ , \tag{3.4}$$

$\gamma \neq 0, 1$ and the coefficients a_n, α_n do not vanish. Let us notice that also the potential Toda lattice (2.71) belong to this class. All such equations belong to the class (3.1). 5 dimensional symmetry algebras correspond to functions f_n which are of the form

$$f_n = \phi_n(t, \xi_n) + \sigma_n(t)(u_{n+1} - u_n) \ , \qquad \frac{\partial^2 \phi_n}{\partial \xi_n^2} \neq 0 \tag{3.5}$$

$$f_n = a_n u_{n+1}^{b_n} u_n^{c_n} u_{n-1}^{d_n} \ , \qquad a_n b_n d_n \neq 0 \ , \tag{3.6}$$

$$f_n = (u_{n+1} - u_n)^{-3} \psi_n \left(\frac{u_{n+1} - u_n}{u_n - u_{n-1}} \right) \ , \tag{3.7}$$

with ξ_n given by eq.(3.4)). Eq.(1.1), with f_n given by eqs.(3.5, 3.6) are always of the class (3.1), but eq.(3.7) belongs to this class only if $z\psi_n''(z) \neq 2\psi_n'(z)$ for any n. If $\psi_n(z) = a_n z^3 + b_n$, $a_n b_n \neq 0$, eq.(3.7) is of the class (3.2). In the following we will investigate the integrability for almost all the equations defined by the functions (3.3 - 3.7).

In the paper [17] the authors investigate the following system:

$$\begin{aligned} m_1 \ddot{v}_k &= \alpha_1(w_{k+1} - v_k) - \alpha_2(v_k - w_k) + \varepsilon[\beta_1(w_{k+1} - v_k)^2 - \beta_2(v_k - w_k)^2] , \\ m_2 \ddot{w}_k &= \alpha_2(v_k - w_k) - \alpha_1(w_k - v_{k-1}) + \varepsilon[\beta_2(v_k - w_k)^2 - \beta_1(w_k - v_{k-1})^2] \end{aligned} \tag{3.8}$$

where $(\alpha_i, \beta_i, m_i, \varepsilon)$, $i = 1, 2$, are nonzero constant coefficients. Eq.(3.8) describes the evolution of a perturbation on a diatomic chain. Eq.(3.8) belong to the class of equations (1.10) and can be written as

$$M_n \ddot{u}_n = \varphi_{n+1}(u_{n+1} - u_n) - \varphi_n(u_n - u_{n-1}) , \quad \varphi_n(z) = \xi_n z^2 + \zeta_n z , \tag{3.9}$$

where M_n, ξ_n and ζ_n are two-periodic functions of n, a subclass of eq.(1.9). Eq.(3.9) is of the class (3.2), and studying this class, we can look for an integrable approximation to system (3.8).

In the last part of this Section, we will present two Theorems which will provide results for lattice equations of the form (1.1, 1.2, 2.65) satisfying condition (3.1) or (3.2). First of all, let us present a preliminary calculation in the general case (2.65).

To rewrite the condition (C2) with $i = 1$ in a simpler way we introduce the functions

$$\begin{aligned} &z_n(t, u_n, u_{n-1}), \qquad y_n = \frac{\partial^2 z_n}{\partial u_n \partial u_{n-1}} = \exp s_n^{(1)}, \\ &\psi_n = \frac{\delta z_n}{\delta u_n} = \frac{\partial}{\partial u_n}(z_{n+1} + z_n), \end{aligned} \tag{3.10}$$

with $s_n^{(1)}$ given by eq.(2.62). As $\partial\psi_n/\partial u_{n+1} = y_{n+1}$ and $\partial\psi_n/\partial u_{n-1} = y_n$, the exponent of condition (C2) with $i = 1$ reads:

$$\frac{\partial \psi_n}{\partial u_{n-1}} \frac{\partial f_n}{\partial u_{n+1}} = \frac{\partial \psi_n}{\partial u_{n+1}} \frac{\partial f_n}{\partial u_{n-1}} .$$

Consequently f_n and ψ_n are functionally dependent i.e.

$$f_n = \varphi_n(t, u_n, \psi_n) , \qquad \partial\varphi_n/\partial\psi_n \neq 0 . \tag{3.11}$$

So, instead of the first condition of (C2), we have the representation (3.11). As we are considering the case when f_n satisfies the condition (2.65), the first of the conditions (C3), contained in Theorem 3, implies $s_n^{(1)} = (T - 1)\hat{s}_n$, where $\hat{s}_n = \hat{s}_n(t, u_{n-1})$. Then y_n can be represented in the form

$$y_n = \frac{\rho_n}{\rho_{n-1}} , \qquad \rho_n = \rho_n(t, u_n) . \tag{3.12}$$

Two other integrability conditions will be used partially. From eqs.(C1, C3, 3.12) one has

$$p_n^{(1)} = \log \frac{\partial f_n}{\partial u_{n+1}} = \log \frac{\partial \varphi_n}{\partial \psi_n} + \log y_{n+1} \sim \log \frac{\partial \varphi_n}{\partial \psi_n} \sim 0. \tag{3.13}$$

As φ_n depends on the same variables as ψ_n, taking into account eq.(A.9) we get

$$\frac{\partial^2}{\partial u_{n+1}\partial u_{n-1}}\log\frac{\partial\varphi_n}{\partial\psi_n}=\frac{\partial}{\partial u_{n+1}}\left(y_n\frac{\partial}{\partial\psi_n}\log\frac{\partial\varphi_n}{\partial\psi_n}\right)=y_{n+1}y_n\frac{\partial^2}{\partial\psi_n^2}\log\frac{\partial\varphi_n}{\partial\psi_n}=0 \quad (3.14)$$

for all n. From eq.(3.14) we get

$$\frac{\partial\varphi_n}{\partial\psi_n}=e^{\Delta_n}, \qquad \Delta_n=a_n(t,u_n)\psi_n+b_n(t,u_n)\ . \tag{3.15}$$

From eq.(3.11), by differentiation

$$\frac{\partial^2 f_n}{\partial u_{n+1}\partial u_{n-1}}=a_n y_{n+1}y_n e^{\Delta_n}. \tag{3.16}$$

So, the conditions (3.1, 3.2) can be formulated in terms of a_n

$$\frac{\partial^2 f_n}{\partial u_{n+1}\partial u_{n-1}}=0 \qquad \Leftrightarrow \qquad a_n=0. \tag{3.17}$$

The following condition

$$\dot{p}_n^{(2)}=(\mu\dot{\omega}_n)_t\sim 0\ , \qquad \omega_n=\omega_n(t,u_n,u_{n-1})\ ,$$

takes place (see eqs.(C3, 2.66)). Consequently the second of eq.(2.60) with $i=2$ is equivalent to the following three conditions:

$$\frac{\partial^2\omega_n}{\partial u_n\partial u_{n-1}}=0\ , \qquad \frac{\partial\Omega_n}{\partial u_n}=0\ , \qquad \frac{\partial\mu^{1/2}\Omega_n}{\partial t}=0\ , \tag{3.18}$$

where

$$\Omega_n=\frac{\delta\omega_n}{\delta u_n}=\frac{\partial}{\partial u_n}(\omega_{n+1}+\omega_n)\ .$$

The first two conditions of eq.(3.18) require that $\Omega_n=\xi_n(t)$, the last one implies $\xi_n(t)=\nu_n\mu(t)^{-1/2}$, where ν_n depends only on n. Then the function ω_n satisfies the condition

$$\omega_n\sim\xi_n(t)u_n\ . \tag{3.19}$$

We can now prove the following theorems:

Theorem 4. *An integrable lattice equation of the form (1.1), (1.2), (2.65) satisfying the condition (3.1) is equivalent (up to a point transformation (2.72)) to the potential Toda lattice (2.71).*

Proof. In the case (3.1), it follows from eq.(3.17) that $a_n\neq 0$ for any n. Let us use the condition (C1) with $i=2$, where $p_n^{(2)}$ is given by eq.(2.66). Taking into account eqs.(3.19, A.4), one has

$$p_{n,t}^{(2)}\sim\mu\xi_n f_n+(\mu\xi_n')'u_n\sim 0\ .$$

The equality $\mu\xi_n\partial^2 f_n/\partial u_{n+1}\partial u_{n-1} = 0$, obtained taking into account eq.(A.9), gives $\xi_n = 0$, which implies $\omega_n \sim 0$.

We can use the second of conditions (C3) to find the partial derivatives of ω_n. It follows from eqs.(3.12, 3.13, 3.15) that $p_n^{(1)} = a_n\psi_n + b_n + \log\rho_{n+1} - \log\rho_n = \omega_{n+1} - \omega_n$, and thus we obtain

$$\frac{\partial\omega_n}{\partial u_{n-1}} = -a_n y_n \ , \qquad \frac{\partial\omega_{n+1}}{\partial u_{n+1}} = a_n y_{n+1} + \frac{\partial \log\rho_{n+1}}{\partial u_{n+1}}.$$

The first of conditions (3.18) implies $\frac{\partial(a_n\rho_n)}{\partial u_n} = \frac{\partial(a_n/\rho_n)}{\partial u_n} = 0$, and consequently $a_n = a_n(t)$, $\rho_n = \rho_n(t)$. As $\omega_n \sim 0$ (see eq.(A.3)), one can find

$$\Omega_n = \frac{\delta\omega_n}{\delta u_n} = a_{n-1}y_n - a_{n+1}y_{n+1} = 0.$$

Thus, as we have to do with the kernel of $T - 1$,

$$a_n a_{n-1} y_n = a_{n+1} a_n y_{n+1} = \kappa(t)$$

and it is possible to find an explicit expression for y_n

$$y_n = \frac{\kappa(t)}{a_n(t)a_{n-1}(t)} \ .$$

From eqs.(3.10, 3.15), we get

$$\Delta_n = \frac{\kappa}{a_{n+1}}u_{n+1} + \frac{\kappa}{a_{n-1}}u_{n-1} + c_n(t, u_n)$$

and eq.(3.13), i.e. $\Delta_n \sim 0$, provides us an equation for c_n:

$$\frac{\delta\Delta_n}{\delta u_n} = 2\frac{\kappa}{a_n} + \frac{\partial c_n}{\partial u_n} = 0 \ .$$

Taking into account the results so far obtained, we can rewrite eq.(1.1) as

$$\ddot{u}_n = \frac{1}{a_n}e^{\Delta_n} + d_n(t, u_n) \ , \tag{3.20}$$

where

$$\frac{\delta\Delta_n}{\delta u_n} = 2\frac{\kappa}{a_n} + \frac{\partial c_n}{\partial u_n} = 0 \ .$$

A point transformation of the form $\tilde{u}_n = u_n + \zeta_n(t)$ and a ridefinition of a_n, d_n allow one to rewrite eq.(3.20) as

$$\ddot{u}_n = \exp(\delta_{n+1}u_{n+1} - 2\delta_n u_n + \delta_{n-1}u_{n-1}) + d_n(t, u_n) \ , \tag{3.21}$$

with $\delta_n(t) \neq 0$ for any n.

Let us apply to eq.(3.21) the second of conditions (2.60) with $i = 3$. The coefficient of $\dot u_n$ gives $2\mu\delta_n' + \mu'\delta_n = 0$, and thus one can write $\delta_n(t)$ as $\delta_n(t) = \alpha_n\mu(t)^{-1/2}$ with α_n depending only on n.

The transformations

$$\tilde u_n = \eta(t)u_n \ , \qquad \tilde t = \theta(t) \ , \qquad \theta' = \eta^2 \neq 0 \tag{3.22}$$

$$\hat u_n = \varepsilon_n u_n + \lambda_n(t) \ , \qquad \varepsilon_n \neq 0 \tag{3.23}$$

do not introduce $\dot u_n$ in the equation. Using them we can reduce the equation to the form (3.21) with $\delta_n = 1$.

Now $\mu' = 0$ and the condition (2.60) with $i = 3$ give an equation for d_n:

$$2(d_{n+1} - 2d_n + d_{n-1}) = \frac{\partial^2 d_n}{\partial u_n^2} + 4\frac{\partial d_n}{\partial u_n} \ . \tag{3.24}$$

Differentiating eq.(3.24) w.r.t. u_{n+1}, one has $\partial d_n/\partial u_n = 0$, and thus

$$d_n = \beta_1(t)n + \beta_2(t). \tag{3.25}$$

Eq.(3.21) with d_n given by eq.(3.25) and $\delta_n = 1$ is reduced by a transformation of the form (3.23) to the potential Toda lattice (2.71). ∎

Theorem 5. *Any integrable lattice equation of the form (1.1, 1.2, 2.65) satisfying the condition (3.2) is reduced by a point transformation (2.72) to the Toda model (2.11) or an equation linear in u_{n+1}, u_n and u_{n-1}.*

Proof. In this case $a_n = 0$, and the conditions (3.13, 3.15) imply $b_n \sim 0$. Then $b_n = b_n(t)$, and using eqs.(3.10, 3.11, 3.12), we can express the right hand side f_n and $p_n^{(1)}$ of condition (C1) and ω_n of condition (C3) as

$$f_n = e^{b_n(t)}\psi_n + c_n(t, u_n) \ , \tag{3.26}$$

$$p_n^{(1)} = b_n + (T-1)\log\rho_n \ , \qquad \omega_n = \log\rho_n + \hat b_n(t) \ .$$

We now use eq.(3.18) with $\Omega_n = \frac{\partial \log\rho_n}{\partial u_n}$ and obtain

$$\log\rho_n = \xi_n(t)u_n + \zeta_n(t) \ , \qquad \xi_n(t) = \nu_n\mu(t)^{-1/2} \ . \tag{3.27}$$

Using the condition $\dot p_n^{(2)} \sim 0$ with $p_n^{(2)}$ given by eq.(2.66) we can check that the following equivalence relation takes place:

$$\mu^{-1/2}\dot p_n^{(2)} \sim \mu^{-1/2}\left(\mu\frac{\partial\log\rho_n}{\partial t}\right)_t + \nu_n c_n + \nu_n e^{b_n}\frac{\partial z_{n+1}}{\partial u_n} + \nu_{n+1}e^{b_{n+1}}\frac{\partial z_{n+1}}{\partial u_{n+1}} \sim 0.$$

In accordance with eq.(A.9), we apply the operator $\frac{\mu^{1/2}}{y_{n+1}}\frac{\partial^2}{\partial u_{n+1}\partial u_n}$ and obtain the condition (see also eq.(3.12)):

$$\nu_n^2 e^{b_n(t)} = \nu_{n+1}^2 e^{b_{n+1}(t)} = \tilde b(t) \ . \tag{3.28}$$

We can now split the proof into two different cases: $\tilde{b} = 0$ and $\tilde{b} \neq 0$.

Let us consider at first the case $\tilde{b} \neq 0$. Using formulae (3.10, 3.12, 3.26, 3.27, 3.28), we find the functions ρ_n, y_n, ψ_n, b_n, f_n and obtain as a result an equation of the form:

$$\nu_n \ddot{u}_n = e^{\xi_{n+1}u_{n+1}-\xi_n u_n+\alpha_{n+1}(t)} - e^{\xi_n u_n-\xi_{n-1}u_{n-1}+\alpha_n(t)} + \beta_n(t, u_n) \ . \tag{3.29}$$

The function ξ_n has the specific form (3.27) and ν_n depend only on n. Using eqs.(3.22, 3.23) we are able to transform eq.(3.29) into the simpler one:

$$\ddot{u}_n = e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}} + \gamma_n(t, u_n) \ . \tag{3.30}$$

Taking into account that $\omega_n = u_n + d(t)$, we have

$$\dot{p}_n^{(2)} \sim \dot{\mu}\dot{u}_n + \mu\gamma_n \sim 0 \ ,$$

and this condition implies: $\dot{\mu} = \partial\gamma_n/\partial u_n = 0$. From eq.(2.66) the last integrability condition reads:

$$\dot{p}_n^{(3)} \sim 2\mu^2(\gamma_n + \ddot{d})\dot{u}_n \sim 0 \ .$$

Then $\gamma_n = -\ddot{d}(t)$, and the point transformation $\hat{u}_n = u_n + d(t)$ turns the eq.(3.30) into the Toda model (2.11).

In the case $\tilde{b} = 0$, one can see from eqs.(3.12, 3.27, 3.28) that $y_n = y_n(t)$. Using eqs.(3.10, 3.26), eq.(1.1) takes the form:

$$\ddot{u}_n = \alpha_n(t)u_{n+1} + \beta_n(t)u_{n-1} + \gamma_n(t, u_n) \ , \tag{3.31}$$

with $\alpha_n\beta_n \neq 0$. For an equation of this kind, the functions $p_n^{(1)}$, ω_n, $p_n^{(2)}$ and $q_n^{(2)}$ may depend only on n and t. This implies that $p_n^{(3)} \sim \mu^2\partial\gamma_n/\partial u_n$. From eq.(A.8) we get that $p_{n,t}^{(3)} \sim 0$ can be replaced by the condition $\mu^2\partial\gamma_n/\partial u_n \sim 0$ which gives $\partial^2\gamma_n/\partial u_n^2 = 0$. Then the right hand side of eq.(3.31) is also linear in u_n. ∎

4 Conclusions

In this work we have presented 5 necessary conditions for the existence of local higher order generalized symmetries and conservation laws for equations explicitly depending on n and t of the form (1.1, 1.2). This conditions imply that any integrable equation of the form (1.1, 1.2) must satisfy eq.(2.64) and consequently may have, at most, quadratic dependence from $\dot{u}_n$.

We have used the obtained conditions to study a few classes of interesting equations. Unfortunately, as Theorems (4, 5) showed, the classes (3.1) and (3.2) (in the case (2.65)) contain only, up to point transformations of the form (2.72), the well known integrable equations (2.11) and (2.71). Moreover, we can prove that there is nothing new also when the function $f_n = f_n(t, w, x, y, z)$ defining eq.(1.1, 1.2, 2.65) is two-periodic w.r.t. n and satisfies condition (3.1) for one n and condition (3.2) for the other. In another words, among systems of the form

$$\ddot{v}_k = \Phi(t,\ v_k,\ w_{k+1},\ w_k)\ , \quad \Phi_{w_{k+1}}\Phi_{w_k} \neq 0\ ,$$
$$\ddot{w}_k = \Psi(t,\ w_k,\ v_k,\ v_{k-1})\ , \quad \Psi_{v_k}\Psi_{v_{k-1}} \neq 0$$

(see eq.(1.10)) which also contain eq.(3.8), there is no new integrable approximation of eq.(3.8).

The request of existence of local high order generalized symmetries seems to be too stringent. The next possibility is to allow for some nonlocality (as e.g. example of eq.(2.75) shows). Work on this is in progress.

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Appendix

A Properties of Restricted Functions

Let us discuss some properties of restricted functions. Most of properties can be proved very easily, and thus in that case we will only present the results.

Let us define an equivalence relation for restricted functions (2.1). Two restricted functions g_n and $\tilde{g}_n$ are equivalent ($g_n \sim \tilde{g}_n$) if their difference can be represented in the form $g_n - \tilde{g}_n = (T-1)\varphi_n$, with φ_n a restricted function. In particular, $g_n \sim 0$ if $g_n = (T-1)\sigma_n$. For any function $c_n = c_n(t)$ (i.e. depending on n and t only), one has $c_n \sim 0$, as the equation $\sigma_{n+1} - \sigma_n = c_n$ can always be solved for $\sigma_n = \sigma_n(t)$.

Let us discuss the case when a restricted function g_n can be represented as a total difference:

$$g_n = (T-1)h_n\ , \tag{A.1}$$

where h_n is another restricted function. If $g_n = 0$, then $h_n = h(t)$. If $g_n = g_n(t)$ (i.e. depends only on n and t), then h_n always can be found and is a function of the same kind: $h_n = h_n(t)$. In more general case of (2.1) we have two cases:

1. $i_1 = i_2$, then the function h_n cannot depend on variables u_{n+k} at all.

2. $i_1 > i_2$, then h_n may depend only on $\{u_{n+i_1-1}, \ldots, u_{n+i_2}\}$.

Moreover, for a function of the form (A.1), from formal variational derivative one has

$$\frac{\delta g_n}{\delta u_n} = \sum_{k=-i_1}^{-i_2} \frac{\partial g_{n+k}}{\partial u_n} = 0. \tag{A.2}$$

On the other hand, if $\delta g_n/\delta u_n = 0$, g_n is equivalent (up to a total difference) to a function $\hat{g}_n$ ($g_n \sim \hat{g}_n$) which has no dependence on $\{u_{n+k}\}_{k=i_1}^{i_2}$.

The same can be said about the dependence of g_n from variables $\{\dot{u}_{n+k}\}_{k=j_1}^{j_2}$. Summarizing, we can say that

$$g_n \sim 0 \qquad \text{iff} \qquad \frac{\delta g_n}{\delta u_n} = \frac{\delta g_n}{\delta \dot{u}_n} = 0 \ . \tag{A.3}$$

As we can easily see, when

$$g_n \sim 0 \qquad \Rightarrow \qquad D_t g_n \sim 0 \ , \quad \frac{\partial g_n}{\partial t} \sim 0 \ , \quad \eta(t) g_n \sim 0 \ , \tag{A.4}$$

where D_t is the operator of total differentiation and $\eta(t)$ is an arbitrary function of t. Moreover, for any restricted function g_n we have from eq.(2.5)

$$D_t g_n \sim \frac{\partial g_n}{\partial t} + \frac{\delta g_n}{\delta u_n} \dot{u}_n + \frac{\delta g_n}{\delta \dot{u}_n} f_n \ , \tag{A.5}$$

as for example $\frac{\partial g_n}{\partial u_{n+k}} \dot{u}_{n+k} \sim \frac{\partial g_{n-k}}{\partial u_n} \dot{u}_n$, and hence from eq.(A.2)

$$\sum_{k=i_2}^{i_1} \frac{\partial g_n}{\partial u_{n+k}} \dot{u}_{n+k} \sim \sum_{k=i_2}^{i_1} \frac{\partial g_{n-k}}{\partial u_n} \dot{u}_n = \sum_{l=-i_1}^{-i_2} \frac{\partial g_{n+l}}{\partial u_n} \dot{u}_n = \frac{\delta g_n}{\delta u_n} \dot{u}_n. \tag{A.6}$$

Let us consider a function g_n independent on $\dot{u}_{n+k}$ and such that $D_t g_n \sim 0$, then eq.(A.5) gives

$$\frac{\partial g_n}{\partial t} + \frac{\delta g_n}{\delta u_n} \dot{u}_n \sim 0 \ . \tag{A.7}$$

The left hand side of eq.(A.7) must be independent on $\dot{u}_n$, hence $\delta g_n/\delta u_n = 0$, and eq.(A.3) implies $g_n \sim 0$. Taking into account eq.(A.4), we get that if $\partial g_n/\partial \dot{u}_{n+k} = 0$ for all k, then

$$D_t g_n \sim 0 \qquad \Leftrightarrow \qquad g_n \sim 0 \ . \tag{A.8}$$

One simple, but very useful property of restricted function g_n is:

$$g_n \sim 0 \ , \quad i_1 > i_2 \quad \Rightarrow \quad \frac{\partial^2 g_n}{\partial u_{n+i_1} \partial u_{n+i_2}} = 0 \ \ \forall \ n \ . \tag{A.9}$$

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