

# Replicator Dynamics and Mathematical Description of Multi-Agent Interaction in Complex Systems

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## Abstract

We consider the general properties of the replicator dynamical system from the standpoint of its evolution and stability. Vector field analysis as well as spectral properties of such system has been studied. A Lyapunov function for the investigation of the evolution of the system has been proposed. The generalization of replicator dynamics to the case of multi-agent systems is introduced. We propose a new mathematical model to describe the multi-agent interaction in complex system.

## 1 Introduction

Replicator equations were introduced by Fisher to capture Darwin's notion of the survival of the fittest [1]. Replicator dynamics is one of the most important dynamic models arising in biology and ecology [2, 3], evolutionary game theory [4] and economics [5, 6], traffic simulation systems and distributed computing [7] etc. It is derived from the strategies that fare better than average and thriving is on the cost of others at the expense of others (see e.g. [4]). This leads to the fundamental problem of how complex multi-agent systems widely met in nature can adapt to changes in the environment when there is no centralized control in the system. For a complex "living" system such a problem has been considered in [8]. By the term complex multi-agent systems being considered we mean one that comes into being, provides for itself and develops, pursuing its own goals [8]. In the same time replicator dynamics arises if the agents have to deal with conflicting goals and the behavior of such systems is quite different from the adaptation problem considered in [8]. In this article we develop a new dynamic model to describe the multi-agent interaction in complex systems of interacting agents sharing common but limited resources.

Based on this model we consider a population composed of  $n \in \mathbb{Z}_+$  distinct competing “varieties” with associated fitnesses  $f_i(v)$ ,  $i = \overline{1, n}$ , where  $v \in [0, 1]^n$  is a vector of relative frequencies of the varieties  $(v_1, v_2, \dots, v_n)$ . The evolution of relative frequencies is described by the following equations:

$$dv_i/dt = v_i(f_i(v) - \langle f(v) \rangle), \quad i = \overline{1, n}, \quad (1.1)$$

where

$$\langle f(v) \rangle = \sum_{i=1}^n v_i f_i(v). \quad (1.2)$$

The essence of (1.1) and (1.2) is simple: varieties with an above-average fitness expand, those with a below-average fitness contract.

Since  $v_i \in [0, 1]$  has to be nonnegative for all time the system (1.1),(1.2) is defined on the nonnegative orthant

$$\mathbb{R}_+^n = \{v \in \mathbb{R}^n : v_i \geq 0\}. \quad (1.3)$$

The replicator equation describes the relative share dynamics and thus holds on the unit  $(n - 1)$  dimensional space simplex

$$S_n = \left\{ v \in \mathbb{R}_+^n : \sum_{i=1}^n v_i = 1 \right\}. \quad (1.4)$$

We write Fisher’s model in the following form

$$dv_i/dt = v_i \left( \sum_{j=1}^n a_{ij} v_j - \sum_{j,k=1}^n a_{jk} v_j v_k \right), \quad (1.5)$$

where  $a_{ij} \in \mathbb{R}$ ,  $v_i \in [0, 1]$ ,  $i, j = \overline{1, n}$ ,  $n \in \mathbb{Z}_+$ .

One can check that the system (1.5) can be written in the matrix commutative form:

$$dP/dt = [[D; P], P]. \quad (1.6)$$

Here by definition

$$P = \left\{ (v_i v_j)^{\frac{1}{2}} : i, j = \overline{1, n} \right\}, \quad (1.7)$$

$$D = \frac{1}{2} \text{diag} \left\{ \sum_{k=1}^n a_{jk} v_k : j = \overline{1, n} \right\}. \quad (1.8)$$

It can be checked easily that the matrix  $P \in \text{End} \mathbb{R}^n$  is a projector, that is  $P^2 = P$  for  $t \in \mathbb{R}$ . This is very important for our further studying the structure of the vector field (1.6) on the corresponding projector matrix manifold  $\mathcal{P}$  [9, 10].

## 2 Vector field analysis

In order to study the structure of the flow (1.6) on the projector matrix manifold  $\mathcal{P} \ni P$  we consider a functional  $\Psi : \mathcal{P} \rightarrow \mathbb{R}$ , where by definition the usual variation

$$\delta\Psi(P) := Sp(D(P)\delta P), \quad (2.1)$$

with  $D : \mathcal{P} \rightarrow End\mathbb{R}^n$ ,  $Sp : End\mathbb{R}^n \rightarrow \mathbb{R}$  being the standard matrix trace. Taking into account the natural metrics on  $\mathcal{P}$ , we consider the projection of the usual gradient vector field  $\nabla\Psi$  to the tangent space  $T(\mathcal{P})$  under the following conditions:

$$\varphi(X; P) := Sp(P^2 - P, X) = 0, \quad Sp(\nabla\varphi, \nabla_\varphi\Psi) |_{\mathcal{P}} = 0, \quad (2.2)$$

holding on  $\mathcal{P}$  for all  $X \in End\mathbb{R}^n$ . The first condition is evidently equivalent to  $P^2 - P = 0$ , that is  $P \in \mathcal{P}$ . Thereby we can formulate the lemma.

**Lemma 1.** *The functional gradient  $\nabla_\varphi\Psi(P)$ ,  $P \in \mathcal{P}$ , at condition (2.2) has the following commutator representation:*

$$\nabla_\varphi\Psi(P) = [[D(P), P], P]. \quad (2.3)$$

**Proof.** Consider the projection of the usual gradient  $\nabla\Psi(P)$  to the tangent space  $T(\mathcal{P})$  on the manifold  $\mathcal{P}$  having assumed that  $P \in End\mathbb{R}^n$  :

$$\nabla_\varphi\Psi(P) = \nabla\Psi(P) - \nabla_\varphi(\Lambda, P), \quad (2.4)$$

where  $\Lambda \in End\mathbb{R}^n$  is some unknown matrix. Taking into account the conditions (2.2) we find that

$$\nabla_\varphi\Psi(P) = D - \Lambda - P(D - \Lambda) - (D - \Lambda)P + PD + DP \quad (2.5)$$

$$= PD + DP + 2P\Lambda P, \quad (2.6)$$

where we have made use of the conditions

$$\nabla_\varphi\Psi(P) = D - \Lambda + P\Lambda + \Lambda P$$

and

$$P(D - \Lambda) + (D - \Lambda)P + 2P\Lambda P = D - \Lambda.$$

Now one can see from (2.6) and the second condition in (2.4) that

$$P\Lambda P = -PDP \quad (2.7)$$

for all  $P \in \mathcal{P}$ , giving rise to the final result

$$\nabla_{\varphi}\Psi(P) = PD + DP - 2PDP, \quad (2.8)$$

coinciding exactly with the commutator (2.3). ■

It should be noted that the manifold  $\mathcal{P}$  is also a symplectic Grassmann manifold ([9, 10]), the canonical symplectic structure of which is given by the expression:

$$\omega^{(2)}(P) := \text{Sp}(PdP \wedge dPP), \quad (2.9)$$

where  $d\omega^{(2)}(P) = 0$  for all  $P \in \mathcal{P}$ , and the differential form (2.9) is non-degenerate [9, 11] on the tangent space  $T(\mathcal{P})$ .

We assume that  $\xi : \mathcal{P} \rightarrow \mathbb{R}$  is an arbitrary smooth function on  $\mathcal{P}$ . Then the Hamiltonian vector field  $X_{\xi} : \mathcal{P} \rightarrow T(\mathcal{P})$  on  $\mathcal{P}$  generated by this function relative to the symplectic structure (2.9) is given as follows:

$$X_{\xi} = [[D_{\xi}, P], P], \quad (2.10)$$

where  $D_{\xi} \in \text{End}\mathbb{R}^n$  is some matrix. The vector field  $X_{\xi} : \mathcal{P} \rightarrow T(\mathcal{P})$  generates on  $\mathcal{P}$  the flow

$$dP/dt = X_{\xi}(P) \quad (2.11)$$

which is defined globally for all  $t \in \mathbb{R}$ . This flow by construction is evidently compatible with the condition  $P^2 = P$ . This means in particular that

$$-X_{\xi} + PX_{\xi} + X_{\xi}P = 0. \quad (2.12)$$

Thus we state that the dynamical system (1.6) being considered on the Grassmann manifold  $\mathcal{P}$  is Hamiltonian which makes it possible to formulate the following statement.

**Statement 1.** *A gradient vector field of the form (2.10) on the Grassmann manifold  $\mathcal{P}$  is Hamiltonian with respect to the canonical symplectic structure (2.9) and a Hamiltonian function  $\xi : \mathcal{P} \rightarrow \mathbb{R}$ , satisfying*

$$\nabla\xi(P) = [D_{\xi}, P] + ZP + PZ - Z, \quad D_{\xi} = D, \quad (2.13)$$

*giving rise to the equality  $\nabla_{\varphi}\Psi(P) = X_{\xi}(P)$  for all  $P \in \mathcal{P}$  with  $Z : \mathcal{P} \rightarrow \text{End}\mathbb{R}^n$  being some still undetermined matrix.*

For the expression (2.13) to be the gradient of some function  $\xi : \mathcal{P} \rightarrow \mathbb{R}$  it must satisfy the Volterra condition [9, 11]

$$(\nabla \xi)' = (\nabla \xi)'*$$

The latter involves linear constraint equations on the mapping  $Z : \mathcal{P} \rightarrow \text{End} \mathbb{R}^n$  which are always solvable. Namely, the following relationship on the mapping  $Z : \mathcal{P} \rightarrow \text{End} \mathbb{R}^n$

$$2[\cdot, D] + D'^* [P, \cdot] + Z'^*(P) + Z'*(.P) - Z'^*(.) = [D'^* \cdot, P] + Z'*(.)P + PZ'*(.) - Z'*(.), \quad (2.14)$$

where the sign “ $'$ ” means the standard Frechet derivative, holds now for any  $P \in \text{End} \mathbb{R}^n$ . The linear matrix equation (2.14) is in reality a system of  $n^2 \in \mathbb{Z}_+$  equations on  $n^2 \in \mathbb{Z}_+$  unknown scalar functions. Concerning the matrix  $D : \mathcal{P} \rightarrow \text{End} \mathbb{R}^n$  (1.8) the following useful representation holds:

$$D = \text{diag}\{Sp(A_j P) \delta_{jk} : j, k = \overline{1, n}\}, \quad (2.15)$$

where diagonal matrices

$$A_j = \text{diag}\{a_{jk} \delta_{ks} : k, s = \overline{1, n}\}$$

for all  $j = \overline{1, n}$ .

In general, it is well known [9, 11] that all reductions of Lie-Poisson matrix flows the Grassmann (projector) manifolds are Hamiltonian subject to the canonical symplectic structure (2.9), being exactly the case under our consideration.

In the case the explicit form of the Hamiltonian function  $\xi : \mathcal{P} \rightarrow \mathbb{R}$  is obtained by means of some tedious and a bit cumbersome calculations which we shall deliver in our next report.

Consider now the  $(n-1)$ -dimensional Riemannian space

$$M_g^{n-1} = \left\{ v_i \in \mathbb{R}_+ : i = \overline{1, n}, \sum_{i=1}^n v_i = 1 \right\}$$

with the metric

$$ds^2(v) := d^2 \Psi |_{\mathcal{P}}(v) = \sum_{i,j=1}^n g_{ij}(v) dv_i dv_j |_{\mathcal{P}},$$

where

$$g_{ij}(v) = \frac{\partial^2 \Psi(v)}{\partial v_i \partial v_j}, \quad i, j = \overline{1, n}, \quad \sum_{i=1}^n v_i = 1.$$

Subject to the metrics on  $M_g^{n-1}$  we can calculate the gradient  $\nabla_{\varphi} \Psi$  of the function  $\Psi : \mathcal{P} \rightarrow \mathbb{R}$  and set on  $M_{\varphi}^{n-1}$  the gradient vector field

$$dv/dt = \nabla_{\psi}\Psi(v), \quad (2.16)$$

where  $v \in M_{\psi}^{n-1}$ , or  $\sum_{i=1}^n v_i = 1$  is satisfied. Having calculated (2.16) we can formulate the following statement.

**Statement 2.** *The gradient vector fields  $\nabla_{\varphi}\Psi$  on  $\mathcal{P}$  and  $\nabla_g\Psi$  on  $M_g^{n-1}$  are equivalent or in another words vector fields*

$$dv/dt = \nabla_g\Psi(v) \quad (2.17)$$

and

$$dP(v)/dt = [[D(v), P(v)], P(v)] \quad (2.18)$$

generates the same flow on  $M_g^{n-1}$ .

As a result of the Hamiltonian property of the vector field  $\nabla_{\varphi}\Psi$  on the Grassmann manifold  $\mathcal{P}$  we get a final statement.

**Statement 3.** *The gradient vector field  $\nabla_g\Psi$  (2.16) on the metric space  $M_g^{n-1}$  is Hamiltonian subject to the nondegenerate symplectic structure*

$$\omega_g^{(2)}(v) := \omega^{(2)}(P) |_{M_g^{n-1}} \quad (2.19)$$

for all  $v \in M_g^{2m}$  with the Hamiltonian function  $\xi_{\psi} : M_g^{2m} \rightarrow \mathbb{R}$ , where  $\xi_{\psi} := \xi |_{M_g^{n-1}}$ ,  $\xi : \mathcal{P} \rightarrow \mathbb{R}$  is the Hamiltonian function of the vector field  $X_{\xi}$  (2.10) on  $\mathcal{P}$ . Otherwise, if  $n \in Z_+$  is arbitrary, our two flows (2.17) and (2.18) are on  $\mathcal{P}$  only Poissonian.

### 3 Spectral properties

Consider the eigenvalue problem for a matrix  $P \in \mathcal{P}$ , depending on the evolution parameter  $t \in \mathbb{R}$ :

$$P(t)f = \lambda f, \quad (3.1)$$

where  $f \in \mathbb{R}^n$  is an eigenfunction,  $\lambda \in \mathbb{R}$  is a real eigenvalue  $P^* = P$ , i.e. matrix  $P \in \mathcal{P}$  is symmetric. It is seen from expression (2.17) that  $\text{spec}P(t) = \{0, 1\}$  for all  $t \in \mathbb{R}$ . Moreover, taking into account the invariance of  $SpP = 1$ , we can conclude that only one eigenvalue of the matrix  $P(t)$ ,  $t \in \mathbb{R}$ , is equal to 1, all others being zero. So we can formulate the next lemma.

**Lemma 2.** *The image  $ImP \subset \mathbb{R}^n$  of the matrix  $P(t) \in \mathcal{P}$  for all  $t \in \mathbb{R}$  is  $k$ -dimensional  $k = \text{rank}P$ , and the kernel  $KerP \subset \mathbb{R}^n$  is  $(n-k)$ -dimensional, where  $k \in Z_+$  is constant, not depending on  $t \in \mathbb{R}$ .*

As a consequence of the lemma we establish that at  $k = 1$  there exists a unique vector  $f_0 \in \mathbb{R}^n / (\text{Ker}P)$  for which

$$Pf_0 = f_0, \quad f_0 \simeq \mathbb{R}^n / (\text{Ker}P). \quad (3.2)$$

Due to the statement above for the projector  $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$  we can write down the following expansion in the direct sum of mutually orthogonal subspaces:

$$\mathbb{R}^n = \text{Ker}P \oplus \text{Im}P.$$

Take now  $f_0 \in \mathbb{R}^n$  satisfying condition (3.2). Then in accordance with (2.17) the next lemma holds.

**Lemma 3.** *The vector  $f_0 \in \mathbb{R}^n$  satisfies the following linear evolution equation:*

$$df_0/dt = [D(v), P(v)] f_0 + C_0(t) f_0, \quad (3.3)$$

where  $C_0 : \mathbb{R} \rightarrow \mathbb{R}$  is a certain function depending on the choice of the vector  $f_0 \in \text{Im}P$ .

At some value of the vector  $f_0 \in \text{Im}P$  we can evidently ensure the condition that  $C_0 \equiv 0$  for all  $t \in \mathbb{R}^n$ . Moreover one easily observes that for the matrix  $P(t) \in \mathcal{P}$  one has [10] the representation  $P(t) = f_0 \otimes f_0$ ,  $\langle f_0, f_0 \rangle = 1$ , giving rise to the system (1.5) if  $f_0 := \{\sqrt{v_j} \in \mathbb{R}_+ : j = \overline{1, n}\} \in \mathbb{R}_+^n$ .

## 4 Lyapunov Function

Consider the gradient vector fields  $\nabla_\varphi \Psi$  on  $\mathcal{P}$  and  $\nabla_g \Psi$  on  $M_g^{n-1}$ . It is easy to state that the function  $\Psi : \mathcal{P} \rightarrow \mathbb{R}$  given by (2.1) and equal on  $M_g^{n-1}$  subject to the following expression

$$\Psi(v) = \frac{1}{4} \sum_{i,j=1}^n a_{ij} v_i v_j - c \sum_{i=1}^n v_i, \quad (4.1)$$

which has to be considered under the condition  $c \in \mathbb{R}_+$ ,  $\sum_{i=1}^n v_i = 1$ , being at the same time a Lyapunov function for the vector fields  $\nabla_\varphi \Psi$  on  $\mathcal{P}$  and  $\nabla_g \Psi$  on  $M_g^{n-1}$ . Indeed:

$$\begin{aligned} \frac{d\Psi}{dt} &= \langle \nabla_g \Psi, \frac{dv}{dt} \rangle_{T(M_g^{n-1})} \\ &= \langle \nabla_g \Psi, \nabla_g \Psi \rangle_{T(M_g^{n-1})} \geq 0, \end{aligned} \quad (4.2)$$

$$\begin{aligned} \frac{d\Psi}{dt} &= Sp(DdP)/dt = Sp\left(D \frac{dP}{dt}\right) \\ &= Sp\left(D, [[D, P], P]\right) = -Sp([P, D], [P, D]) \\ &= Sp([P, D], [P, D]^*) \geq 0 \end{aligned} \quad (4.3)$$

for all  $t \in \mathbb{R}$ , where  $\langle \cdot, \cdot \rangle_{T(M_g^{n-1})}$  is the scalar product on  $T(M_g^{n-1})$  obtained via the reduction of the scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^n$  upon  $T(M_g^{n-1})$  under the constraint  $\sum_{i=1}^n v_i = 1$ .

## 5 Multi-agent replicator system

In the above we have considered the case when  $\text{rank}P(t) = 1$ ,  $t \in \mathbb{R}$ . It is naturally to study now the case when  $n \geq \text{rank}P(t) = k > 1$  being evidently constant for all  $t \in \mathbb{R}$  too. This means therefore that there exists some orthonormal vectors  $f_\alpha \in \text{Im}P$ ,  $\alpha = \overline{1, k}$ , such that

$$P(t) = \sum_{\alpha=1}^k f_\alpha(t) \otimes f_\alpha(t) \quad (5.1)$$

for all  $t \in \mathbb{R}$ . Put now  $f_\alpha := \left\{ f_\alpha^{(i)} := \theta_i^{(\alpha)} \sqrt{v_i^{(\alpha)}} \in \mathbb{R}_+ : \theta_i^{(\alpha)} := \text{sign } f_\alpha^{(i)}, i = \overline{1, n} \right\} \in \text{Im}P$ ,

$\sum_{i=1}^n v_i^{(\alpha)} = 1$  for all  $\alpha = \overline{1, k}$ . The corresponding gradient flow on  $\mathcal{P}$  then takes a modified commutator form with the Lyapunov function variation  $\delta\Psi : \mathcal{P} \rightarrow \mathbb{R}$  as follows

$$\delta\Psi = \sum_{\alpha=1}^k Sp(D_\alpha \delta P^{(\alpha)}), \quad D_\alpha := \text{diag}\{Sp(A_j^{(\alpha)} P^{(\alpha)}) : j = \overline{1, n}\} \quad (5.2)$$

where by definition,  $P^{(\alpha)} = f_\alpha(t) \otimes f_\alpha(t)$ ,  $P^{(\alpha)} P^{(\beta)} = P^{(\alpha)} \delta_{\alpha\beta}$ ,  $\alpha, \beta = \overline{1, k}$ ,  $\sum_{\alpha=1}^k P^{(\alpha)} = P$ .

The resulting flow on the projector manifold  $\mathcal{P}$  is given as

$$dP/dt = \nabla_\varphi \Psi(P), \quad (5.3)$$

where  $\nabla_\varphi : \mathcal{D}(\mathcal{P}) \rightarrow T(\mathcal{P})$  is calculated taking into account the set of natural constraints:

$$\begin{aligned} \varphi_{\alpha,\beta}(X; P) & : = Sp((P_\alpha P_\beta - P_\alpha \delta_{\alpha,\beta}) X_{\alpha,\beta}) = 0, \\ Sp(\nabla_\varphi \varphi_{\alpha,\beta}, \nabla_\varphi \Psi) & : = Sp[X_{\alpha,\beta} (\nabla_\varphi \Psi_\alpha P^{(\beta)} - \delta_{\alpha\beta} P_\beta)] = 0 \end{aligned} \quad (5.4)$$

for all  $X_{\alpha,\beta} \in \text{End}\mathbb{R}^n$  and every  $\alpha, \beta = \overline{1, k}$ . Assume now that  $k = \text{rank}P(t)$  for all  $t \in \mathbb{R}$ . Then the gradient flow brings about the following form:

$$\nabla_\varphi \Psi_\alpha = D_\alpha + \sum_{\beta=1}^k (P_\beta \Lambda_{\beta\alpha} + \Lambda_{\alpha\beta} P_\beta) - \Lambda_{\alpha\alpha}, \quad (5.5)$$

where  $\Lambda_{\beta\alpha} \in \text{End}\mathbb{R}^n$ ,  $\alpha, \beta = \overline{1, k}$ , are the corresponding matrix Lagrangian multipliers. To find them take into account that vector field (5.5) must satisfy the determining constraint

$$\nabla_\varphi \Psi_\alpha P_\beta + P_\alpha \nabla_\varphi \Psi_\beta = \delta_{\alpha,\beta} P_\beta \quad (5.6)$$

for all  $\alpha, \beta = \overline{1, k}$ . Substituting (5.5) and (5.6) into the second equation of (5.4) one finds easily that

$$\nabla_\varphi \Psi_\alpha = P_\alpha D_\alpha + D_\alpha P_\alpha + \sum_{\beta=1}^k (P_\beta \Lambda_{\beta\alpha} P_\alpha + P_\alpha \Lambda_{\alpha\beta} P_\beta). \quad (5.7)$$



Based now once more on the relationship (5.4) one can obtain that the expression

$$P_\beta \Lambda_{\beta\alpha} P_\alpha + P_\alpha \Lambda_{\alpha\beta} P_\beta = -(P_\beta D_\alpha P_\alpha + P_\alpha D_\alpha P_\beta) \quad (5.8)$$

holds for all  $\alpha, \beta = \overline{1, k}$ . As a result of (5.7) and (5.8) one finally obtains the following gradient vector field on the space of projectors  $\mathcal{P}$ :

$$dP_\alpha/dt = [[D_\alpha, P_\alpha], P_\alpha] - \sum_{\beta=1, \beta \neq \alpha}^k (P_\alpha D_\alpha P_\beta + P_\beta D_\alpha P_\alpha), \quad (5.9)$$

or in the coordinate form

$$\begin{aligned} dv_i^{(\alpha)}/dt &= v_i^{(\alpha)} \left( \sum_{j=1}^n a_{ij}^{(\alpha)} v_j^{(\alpha)} - \sum_{j,k=1}^n a_{jk}^{(\alpha)} v_j^{(\alpha)} v_k^{(\alpha)} \right) \\ &\quad - \sum_{\beta \neq \alpha=1}^k \sum_{j,s=1}^n \theta_i^{(\alpha)} \theta_j^{(\alpha)} \theta_i^{(\beta)} \theta_j^{(\beta)} K_{ij,ij}^{(\alpha,\beta)} a_{js}^{(\alpha)} v_s^{(\alpha)}, \end{aligned} \quad (5.10)$$

where the two-species correlation tensor  $K_{ij,ls}^{\alpha,\beta} = 2\sqrt{v_i^{(\alpha)} v_j^{(\alpha)} v_l^{(\beta)} v_s^{(\beta)}}$ ,  $i, j, l, s = \overline{1, n}$ ,  $\alpha, \beta = \overline{1, k}$ , must satisfy the next invariant due to (5.4) sign conditions

$$\sum_{s=1}^n \theta_i^{(\alpha)} \theta_s^{(\alpha)} \theta_j^{(\beta)} \theta_s^{(\beta)} K_{is,js}^{\alpha,\beta} = 0 \quad (5.11)$$

for all  $\alpha \neq \beta = \overline{1, k}$  and  $i, j = \overline{1, n}$ . One can also obtain the flow (5.9) as a Hamiltonian flow with respect to the following canonical symplectic structure on the manifold  $\mathcal{P}$ :

$$\omega^{(2)}(P) = \sum_{\alpha=1}^k Sp(P_\alpha dP_\alpha \wedge dP_\alpha P_\alpha) \quad (5.12)$$

with the element  $P = \bigoplus_{\alpha=1}^k P_\alpha \in \mathcal{P}$  and some Hamiltonian function  $H \in D(P)$  which should be found making use of the expression

$$-i_{\nabla_\varphi \Psi} \omega^{(2)} = dH. \quad (5.13)$$

Straightforward calculations of (5.13) give rise to the same expression (5.9). The system of equations (5.10) is a natural generalization of the replicator dynamics for description of a model with multi-agent interaction. They can describe, for example, economic communities trying to adapt to changing environment. In this approach agents update their behavior in order to get maximum payoff under given matrices of strategies  $\{a_{ik}^{(\alpha)}\}$  in response to the information received from other agents. The structure of the system (5.4) is quite different from the system of equations (1.5). First term on the right hand side describes the individual evolution of each economic agent  $\alpha$  in according to its own independent replicator dynamics and the second term describes average payoffs of all others  $k$  agents except  $\alpha$ .

The obtained mathematical model has also a clear physical interpretation. In framework of this approach there are  $k$  species in the system with  $v_i^{(\alpha)}$  being the frequency of the pursuing strategy  $i < n$  for species  $\alpha$ . In this case each type of agents (species) may exhibit a number of possible strategies  $n$ . The state of the evolutionary system is then given by the vector  $\mathbf{v} = (v_1^1, \dots, v_n^1, v_1^{(\alpha)}, \dots, v_n^{(\alpha)}, \dots, v_1^k, \dots, v_n^k)^T$  being the solution of the  $k \times n$  equations (5.10). Multi-agent approach is also an appropriate concept for understanding behavior of living systems consisting from molecules and cells and characterizing their strong specialization, ecological and human societies as well as economic communities especially in the realm of agent-mediated electronic commerce. Some of applications of the multi-agent system approach have been also discussed in detail concerning another multi-agent models [12, 13], being of interest too concerning eventual imbedding them into the picture suggested in this work.

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