Journal of Nonlinear Mathematical Physics Volume 11, Number 1 (2004), 102–112 Article

Solution of the Goldfish N-Body Problem in the Plane with (Only) Nearest-Neighbor Coupling Constants All Equal to Minus One Half

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Received April 26, 2003; Accepted June 03, 2003

Abstract

The (Hamiltonian, rotation- and translation-invariant) "goldfish" N-body problem in the plane is characterized by the Newtonian equations of motion

$$\ddot{z}_n - i\,\omega\,\dot{z}_n = 2\sum_{m=1,m\neq n}^{N} a_{n,m}\;\dot{z}_n\;\dot{z}_m\;\left(z_n - z_m\right)^{-1},$$

written here in their complex version, entailing the identification of the *real* "physical" plane with the *complex* plane. In this paper we exhibit in completely explicit form the solution of the initial-value problem for this N-body model in the special case in which the two-body interaction only acts among "nearest neighbors" (namely, only among particles whose labels differ by one unit: $a_{n,m} = 0$ unless $|n-m| = 1$) and the corresponding coupling constants all equal minus one half, $a_{n,n+1} = a_{n+1,n} = -1/2$, $n = 1, 2, ..., N-1$. This result implies that, if ω is a real nonvanishing constant, say, without loss of generality, $\omega > 0$, then *all* the solutions of this N-body model are completely periodic indeed *isochronous* with period $T = 2\pi\,/\,\omega$. An analogous conclusion holds as well for the model in which also the first and last particle interact with the same coupling constant, namely $a_{1,N} = a_{N,1} = -1/2$ (rather than vanishing).

1 Introduction and main result

Recently the N-body problem in the plane characterized by the Newtonian equations of motion

$$\begin{aligned}\ddot{\vec{r}}_n &= \omega\,\hat{k}\wedge\dot{\vec{r}}_n + 2\sum_{m=1,m\neq n}^{N} r_{nm}^{-2}\left(\alpha_{n,m} + \tilde{\alpha}_{n,m}\,\hat{k}\wedge\right)\cdot \\ &\quad\cdot\left[\dot{\vec{r}}_n(\dot{\vec{r}}_m\cdot\vec{r}_{nm}) + \dot{\vec{r}}_m(\dot{\vec{r}}_n\cdot\vec{r}_{nm}) - \vec{r}_{nm}(\dot{\vec{r}}_m\cdot\dot{\vec{r}}_m)\right]\end{aligned} \tag{1.1}$$

has been much investigated ([1] – [6]). Here the N two-vectors $\vec{r}_n \equiv \vec{r}_n(t)$ identify the positions, as funtions of the (real) time variable t, of the moving point-particles in a plane which for notational convenience is immersed in three-dimensional space, so that $\vec{r}_n \equiv (x_n, y_n, 0)$; $\hat{k}$ is the unit three-vector orthogonal to that plane, $\hat{k} \equiv (0, 0, 1)$, so that $\hat{k} \wedge \vec{r}_n \equiv (-y_n, x_n, 0)$; $\vec{r}_{nm} \equiv \vec{r}_n - \vec{r}_m$, hence $r^2_{nm} \equiv \vec{r}_{nm} \cdot \vec{r}_{nm} \equiv (x_n - x_m)^2 + (y_n - y_m)^2$; superimposed dots denote of course time derivatives; ω is a real constant (without loss of generality, *nonnegative*), that, whenever it does not vanish, sets the time scale and to which we then associate the period

$$T = \frac{2\pi}{\omega} \; ; \tag{1.2}$$

and the "coupling constants" $\alpha_{n,m}, \tilde{\alpha}_{n,m}$ are *a priori* arbitrary (of course *real*; a sufficient condition for this system to be Hamiltonian [1] [3] is the requirement that these constants be symmetrical in their two indices, $\alpha_{n,m} = \alpha_{m,n}, \tilde{\alpha}_{n,m} = \tilde{\alpha}_{m,n}$, as we hereafter assume). Note that, in the special case without two-body forces ($\alpha_{n,m} = \tilde{\alpha}_{n,m} = 0$) this N-body problem describes N (equal) charged particles, not interacting among themselves, moving on a plane in the presence of a constant magnetic field orthogonal to that plane ("cyclotron"); all solutions are then *completely periodic* with period T, see (1.2), since each particle moves with such a period on its circular trajectory (the center and radius of which are determined by its initial position and velocity).

To treat this N-body problem, (1.1), it is convenient to identify the *real* "physical" plane on which the N points $\vec{r}_n \equiv (x_n, y_n, 0)$ move, with the *complex* plane in which the complex numbers $z_n \equiv x_n + i\, y_n$ move. Indeed via this correspondence the equations of motion (1.1) take the following neater form:

$$\ddot{z}_n = i\ \omega\ \dot{z}_n + 2 \sum_{m=1, m\neq n}^{N} a_{n,m}\ \frac{\dot{z}_n\ \dot{z}_m}{z_n - z_m} \tag{1.3}$$

with

$$a_{n,m} = \alpha_{n,m} + i\ \tilde{\alpha}_{n,m}. \tag{1.4}$$

Hereafter we restrict our consideration to this complex version of the Newtonian equations of motion.

Then one notes [7] [3] that, via the change of independent variable

$$z_n(t) = \zeta_n(\tau), \tag{1.5a}$$

$$\tau \equiv \tau(t) = \frac{\exp(i\omega t) - 1}{i\,\omega}, \tag{1.5b}$$

the equations of motion (1.3) become

$$\zeta''_n = 2 \sum_{m=1, m\neq n}^{N} a_{n,m}\ \frac{\zeta'_n\ \zeta'_m}{\zeta_n - \zeta_m}. \tag{1.6}$$

Here and hereafter primes denote of course differentiations with respect to the (complex) independent variable τ. Note that this change of independent variable, (1.5), implies that the *complex* variable $\tau(t)$ is a periodic function of the *real* variable t (time) with period T (see (1.2)), and moreover it entails the following very simple relations among the initial data for z_n and ζ_n,

$$z_n(0) = \zeta_n(0), \;\; \dot{z}_n(0) = \zeta_n'(0). \tag{1.7}$$

Hence to obtain the solution of the equations of motion (1.3), namely of the Newtonian equations of motion (1.1), one can instead solve, with the *same* initial conditions (see (1.7)), the equations of motion (1.6), and then use the change of independent variable (1.5) to obtain the desired solution of the equations of motion (1.3).

Hereafter we mainly focus on the simpler version (1.6) of the equations of motion. Note that the discussion we just made implies that, if the solution $\zeta_n(\tau)$ of the equations of motion (1.6) is *meromorphic* in the complex variable τ, the corresponding solution $z_n(t)$ of the equations of motion (1.3), namely as well the solution $\overrightarrow{r}_n(t)$ of the Newtonian equations of motion (1.1), is either *singular* or *completely periodic* with period T, see (1.2); the latter alternative is the generic one, indeed the *only one* if the solution $\zeta_n(\tau)$ is *entire*.

In the special case in which all the coupling constants equal unity, $a_{n,m} = 1$, this N-body problem, (1.6), is *completely integrable* indeed *solvable*, as originally shown a quarter century ago [8] and recently discussed in much detail [3]. Indeed the solution of the initial-value problem for the equations of motion (1.6) is given in this case by the N roots of the following algebraic equation in ζ:

$$\sum_{n=1}^{N} \frac{\zeta_n'(0)}{\zeta - \zeta_n(0)} = \frac{1}{\tau}. \tag{1.8}$$

Hence in this special case the solutions $\zeta_n(\tau)$ of the equations of motion (1.6) are generally not meromorphic functions of the complex variable τ, but they only feature a *finite* number of *rational* branch points (which occur at the values of τ at which this algebraic equations features a *multiple* root).[2] Hence in this case *all* the solutions $z_n(t)$ of the equations of motion (1.3), namely as well *all* the solutions $\overrightarrow{r}_n(t)$ of the Newtonian equations of motion (1.1), are *completely periodic*, but some of them with a period which is a finite multiple of T.[2] [3] Because of the neatness of this model and its solution, this special case was declared [9] a "goldfish", and subsequently this name was attributed [2] to this entire class of models, see (1.1), (1.3) and (1.6). As indicated by the title of this paper, we persevere in this attribution.

Because of the connection among the periodic character, as functions of the real time t, of the solutions $z_n(t)$ of the equations of motion (1.3), and the analytic character of the solutions $\zeta_n(\tau)$ of the equations of motion (1.6) as functions of the complex variable τ, it is of much interest to understand these analyticity properties. On the basis of the local analysis of the analytic structure of the solutions $\zeta_n(\tau)$ of the equations of motion (1.6) performed in Ref. [2] it was noted [4] that, in the 3-body case ($N = 3$, entailing the presence of 3 coupling constants, $a_{12} = a_{21}, a_{23} = a_{32}, a_{31} = a_{13}$), there are only 11 triplets (up, of course, to permutations) of coupling constant values for which *all* solutions $\zeta_n(\tau)$ of the equations of motion (1.6) might be *meromorphic* functions of the independent variable τ, and it was indeed conjectured [4] that this be the case, namely that in this 11 cases the equations of motion (1.6) have the "Painlevé property" to possess *only* meromorphic solutions (this is actually the simple version of the Painlevé property, applicable to *autonomous* ODEs). It was moreover conjectured [4] that, out of these 11 cases, there are three even more special ones, for which *all* solutions $\zeta_n(\tau)$ of the equations of motion (1.6) are *entire*. These three cases are characterized (up to permutations) by the following three triplets:

$$\text{case (i):}\quad a_{12} = a_{21} = 0,\ a_{23} = a_{32} = -\frac{1}{2},\ a_{31} = a_{13} = -\frac{1}{2}, \tag{1.9a}$$

$$\text{case (ii):}\quad a_{12} = a_{21} = -\frac{1}{2},\ a_{23} = a_{32} = -\frac{1}{2},\ a_{31} = a_{13} = -\frac{1}{2}, \tag{1.9b}$$

$$\text{case (iii):}\quad a_{12} = a_{21} = 0,\ a_{23} = a_{32} = -\frac{1}{2},\ a_{31} = a_{13} = -1. \tag{1.9c}$$

Case (i), see (1.9a), was actually solved [4], obtaining a polynomial general solution, which is of course *entire*. Case (ii) was then solved [5], obtaining a general solution which is as well *entire*, indeed just a combination of exponentials (up to degeneracies, giving rise to a polynomial behavior, and up to the center-of-mass motion, that is of course uniform, see (1.6)). And case (iii) was also solved [6], obtaining again a general solution which is as well *entire*. On the other hand it was shown [10] that these three are the *only* cases in which *all* the solutions $\zeta_n(\tau)$ of the equations of motion (1.6) with $N = 3$ are *meromorphic* (actually *entire*) functions of the independent variable τ.

The main result of this paper, proven in the following Section 2, is to obtain, in quite explicit form, the solution of the initial-value problem for the equations of motion (1.6) with arbitrary N, in the special case in which the two-body interaction only acts among "nearest neighbors" (namely, only among particles whose labels differ by one unit: $a_{n,m} = 0$ unless $|n - m| = 1$) and the corresponding coupling constants all equal minus one half,

$$a_{n,n+1} = a_{n+1,n} = -1/2, \quad n = 1, 2, ..., N - 1, \tag{1.10}$$

so that these equation of motion read

$$\zeta_1'' = -\,\zeta_1' \,\frac{\zeta_2'}{\zeta_1 - \zeta_2}, \tag{1.11a}$$

$$\zeta_n'' = -\,\zeta_n' \left(\frac{\zeta_{n-1}'}{\zeta_n - \zeta_{n-1}} + \frac{\zeta_{n+1}'}{\zeta_n - \zeta_{n+1}}\right), \quad n = 2, ..., N - 1\ , \tag{1.11b}$$

$$\zeta_N'' = -\,\zeta_N' \,\frac{\zeta_{N-1}'}{\zeta_N - \zeta_{N-1}}\ . \tag{1.11c}$$

Note that this model is the N-body generalization of the 3-body case (i), see (1.9a), to which it reduces for $N = 3$.

The explicit solution of the initial-value problem for this model, (1.11), reads indeed as follows:

$$\zeta_1(\tau) = \zeta_1(0) + \tau\, \zeta_1'(0) - \sum_{m=2}^{N} \frac{\tau^m}{m!}\,(-)^m\, \varphi_1^{(m)}(0), \tag{1.12a}$$

$$\zeta_n(\tau) = \zeta_n(0) + \tau\,\zeta_n'(0) + \sum_{m=2}^{N} \frac{\tau^m}{m!}(-)^m \cdot$$
$$\cdot \left\{ \sum_{\ell=\max(0,\,m+n-N-2)}^{\min(m-2,\,n-2)} (-)^{m-\ell} \binom{m-2}{l} \varphi_{n-\ell-1}^{(m)}(0) - \sum_{\ell=\max(0,\,m+n-N-1)}^{\min(m-2,\,n-1)} (-)^{m-\ell} \binom{m-2}{l} \varphi_{n-\ell}^{(m)}(0) \right\},$$
$$n = 2, ..., N-1\ , \tag{1.12b}$$

$$\zeta_N(\tau) = \zeta_N(0) + \tau\,\zeta_N'(0) + \sum_{m=2}^{N} \frac{\tau^m}{m!}\,\varphi_{N+1-m}^{(m)}(0). \tag{1.12c}$$

Here and below we use of ocurse the standard definition of the binomial coefficients,

$$\binom{p}{\ell} = \frac{p!}{\ell!\,(-p-\ell)!}, \tag{1.13}$$

and the "auxiliary variables" $\varphi_n^{(m)}(\tau)$ are defined in terms of the dependent variables $\zeta_n(\tau)$ and their derivatives $\zeta_n'(\tau)$ by the explicit formula

$$\varphi_n^{(m)}(\tau) = \frac{\prod_{\ell=1}^{m} \zeta_{n+\ell-1}'(\tau)}{\prod_{\ell=1}^{m-1} \left[\zeta_{n+\ell-1}(\tau) - \zeta_{n+\ell}(\tau)\right]}, \quad m = 2, ..., N; \quad n = 1, ..., N+1-m. \tag{1.14}$$

Clearly *all* these solutions, (1.12), are polynomial, hence as well *entire*, in τ.

These results imply of course, via (1.5) and (1.7), an analogous formula providing the solution of the initial value-problem for the N-body model (1.1) or (1.3) with (1.10), clearly entailing that, if ω is a real nonvanishing constant, say, as we assumed without loss of generality, $\omega > 0$, then *all* the solutions of this N-body model, (1.1) or (1.3), are completely periodic indeed *isochronous* with period T, see (1.2). Hence this N-body model in the plane is a remarkable example of *nonlinear harmonic oscillators*. [11]

[As indicated above, we restrict attention here to the case with real, indeed positive, ω; but the results we report are as well valid for complex ω, in which case however the solutions $z_n(t)$ of the equations of motion (1.3) will not be periodic, but clearly, as $t \to \infty$, rather spiral out to infinity if $\mathrm{Im}(\omega) < 0$, and spiral instead to some fixed points (the values of which are easily evinced from the solution (1.12) via (1.5)), if $\mathrm{Im}(\omega) > 0$. This latter case is susceptible of a natural physical interpretation, as it correspond to the presence of a "friction" term acting on each moving particle and represented by a (negative) force proportional to its speed: see (1.1), and keep in mind that, in the right-hand side of these real Newtonian equations of motion, if ω is complex the one-body "Lorentzian" force $\omega\,\widehat{k} \wedge \dot{\vec{r}}_n$ gets replaced by $\mathrm{Re}(\omega)\,\widehat{k} \wedge \dot{\vec{r}}_n - \mathrm{Im}(\omega)\,\dot{\vec{r}}_n$].

Analogous results are obtained in Section 3 for the N-body model which differs from the one detailed above only because the constant coupling the first and last particles also equals minus one half,

$$a_{1N} = a_{N1} = -1/2, \tag{1.15}$$

rather than vanishing. In this case we show that the *general* solution of the equations of motion (1.6) is as well *entire* (indeed *exponential*, up to degeneracies yielding polynomial terms, and up to the center of mass motion, which is of course uniform), entailing that again *all* solutions of the corresponding N-body model (1.1) or (1.3) are in this case (with (1.10) and (1.15)) completely periodic indeed *isochronous* with period T, see (1.2). Note that this model is the N-body generalization of the 3-body case (ii), see (1.9b), to which it reduces for $N = 3$. And let us emphasize that this N-body model in the plane provides another remarkable example of *nonlinear harmonic oscillators*. [11]

The solution of these two problems is tersely outlined in the following two sections, using a rather straightforwarded approach that originated from the previous treatments given in Refs. [4], [5] and [6]. A more detailed treatment of the second model (to the extent of obtaining the explicit solution of the initial-value problem, as exhibited above for the first model) is however forsaken, because, after the results reported in this paper had been obtained, we found out that the second of these two N-body problems had already been fully solved (in fact, in more than one manner, although not quite by the same technique as described below) more than a decade ago by Mario Bruschi and Orlando Ragnisco [12]; a fact of which we were unfortunately unaware until very recently, and in particular when writing Refs. [5] and [3], although we had identified in the latter (see Section 4.4.6 of this book) the solvable character of these models, on the basis of the fact that they can be obtained as appropriate limiting cases of the "relativistic Toda" N-body problem previously introduced and solved by Simon Ruijsenaars [13].

2 Solution of the first model

Let us start by treating the equations of motion (1.11). Let us recall [2] that, associated with these equations of motion, there exists the constant of motion

$$K = \frac{\prod_{n=1}^{N} \zeta_n'}{\prod_{n=1}^{N-1} (\zeta_n - \zeta_{n+1})}. \tag{2.1}$$

The verification that K is indeed a constant of motion is plain (the diligent reader will easily check it by differentiating logarithmically K, see (2.1), and then using the equations of motion (1.11)).

We now focus on the auxiliary variables $\varphi_n^{(m)}(\tau)$, see (1.14), and note first of all that

$$\varphi_1^{(N)}(\tau) = K \tag{2.2a}$$

hence this quantity is actually time-independent,

$$\varphi_1^{(N)}(\tau) = \varphi_1^{(N)}(0). \tag{2.2b}$$

Next we note that there hold, as a consequence of the equations of motion (1.11), the following remarkably simple (linear!) relations:

$$\varphi_1^{(m)\,\prime} = -\varphi_1^{(m+1)}, \quad m = 2, ..., N-1, \tag{2.3a}$$

$$\varphi_n^{(m)\,\prime} = \varphi_{n-1}^{(m+1)} - \varphi_n^{(p+1)}, \quad n = 2, ..., N - m; \quad m = 2, ..., N-2; \tag{2.3b}$$

$$\varphi_{N+1-m}^{(m)}\,{}' = \varphi_{N-m}^{(m+1)}, \quad m = 2, ..., N-1. \tag{2.3c}$$

It is then easily seen that the first of these ODEs, (2.3a), together with (2.2b), entail the formula

$$\varphi_1^{(m)}(\tau) = \sum_{\ell=0}^{N-m} \frac{\tau^\ell}{\ell!} (-)^\ell \varphi_1^{(m+\ell)}(0), \quad m = N, N-1, ..., 2. \tag{2.4}$$

Likewise one gets, from (2.3c) together with (2.2b),

$$\varphi_{N+1-m}^{(m)}(\tau) = \sum_{\ell=0}^{N-m} \frac{\tau^\ell}{\ell!} \varphi_{N+1-m-\ell}^{(m+\ell)}(0), \quad m = N, N-1, ..., 2. \tag{2.5}$$

The integration of the ODEs (2.3c) can then be performed, and one thus obtains the explicit formula

$$\begin{aligned} \varphi_n^{(m)}(\tau) &= \sum_{\ell=0}^{N-m} \frac{\tau^\ell}{\ell!} \sum_{s=\max(0,\, n+m+\ell-N-1)}^{\min(\ell,\, n-1)} (-)^{\ell-s} \binom{\ell}{s} \varphi_{n-s}^{(m+\ell)}(0), \\ m &= 2, ..., N; \quad n = 1, ..., N+1-m. \end{aligned} \tag{2.6}$$

The reader who is skeptic about the validity of this formula can check that this expression of the auxiliary quantities $\varphi_n^{(m)}(\tau)$ satisifes these ODEs, (2.3c), and that it reproduces (2.4) respectively (2.5) for $n = 1$ respectively for $n = N + 1 - m$.

It is on the other hand plain, see (1.14), that the equations of motion (1.11) can be rewritten as follows:

$$\zeta_1''(\tau) = -\,\varphi_1^{(2)}(\tau), \tag{2.7a}$$

$$\zeta_n''(\tau) = \varphi_{n-1}^{(2)}(\tau) - \varphi_n^{(2)}(\tau), \quad n = 2, ..., N-1\,, \tag{2.7b}$$

$$\zeta_N''(\tau) = \varphi_{N-1}^{(2)}(\tau), \tag{2.7c}$$

and it is then clear, using the explicit expression of $\varphi_n^{(2)}(\tau)$ (see (2.6) with $m = 2$), that these ODEs can now be easily integrated to yield the explicit expressions (1.12), which are therefore now proven. Note that the two internal sums in the right-hand side of (1.12b) could be combined into a single one, but the resulting expression is generally not simpler than the one we wrote; although it might be simpler for specific values of N and n.

3 Solution of the second model

Let us now consider the second model, which is characterized by the following equations of motion:

$$\zeta_n'' = -\,\zeta_n' \left(\frac{\zeta_{n-1}'}{\zeta_n - \zeta_{n-1}} + \frac{\zeta_{n+1}'}{\zeta_n - \zeta_{n+1}} \right), \quad n = 1, ..., N \quad \mathrm{mod}(N). \tag{3.8}$$

Note that here we conveniently assume the index n to be defined mod(N), and we shall hereafter maintain this convention, which for instance entails $\zeta_{N+1} = \zeta_1$.

Let us recall [2] that, associated with these equations of motion, (3.8), there exists the constant of motion

$$\tilde{K} = \prod_{n=1}^{N} \left(\frac{\zeta_n'}{\zeta_n - \zeta_{n+1}} \right) . \tag{3.9}$$

The verification that $\tilde{K}$ is indeed a constant of motion is plain (the diligent reader will easily check it by differentiating logarithmically $\tilde{K}$, see (3.9), and then using the equations of motion (3.8)).

We now focus again on the auxiliary variables $\varphi_n^{(m)}(\tau)$, see (1.14), and we note that there now holds the relation (see (3.9))

$$\varphi_n^{(N)} = \tilde{K} \, (\zeta_{n-1} - \zeta_n) \, , \quad n = 1, ..., N \quad \mathrm{mod}(N), \tag{3.10}$$

as well as (see (3.8))

$$\zeta_n'' = \varphi_{n-1}^{(2)} - \varphi_n^{(2)}, \quad n = 1, ..., N \quad \mathrm{mod}(N). \tag{3.11}$$

On the other hand logarithmic differentiation of the definition (1.14) of $\varphi_n^{(p)}(\tau)$ easily yields, via (3.8), the set of differential relations

$$\varphi_n^{(m)\,\prime} = \varphi_{n-1}^{(m+1)} - \varphi_n^{(m+1)}, \quad n = 1, ..., N \quad \mathrm{mod}(N). \tag{3.12}$$

The alert reader will note the similarity, as well as the difference, among these equations, (3.10), (3.11) respectively (3.12), and the analogous ones for the previous models, namely (2.2a), (2.7) respectively (2.3).

We now τ-differentiate (3.11) and, via (3.12), we immediately get

$$\zeta_n''' = \varphi_{n-2}^{(3)} - 2\, \varphi_{n-1}^{(3)} + \varphi_n^{(3)}, \quad n = 1, ..., N \quad \mathrm{mod}(N), \tag{3.13}$$

and this relation (perhaps after one more differentiation) suggests to conjecture the following formula:

$$\left(\frac{d}{d\tau} \right)^p \zeta_n = K \sum_{m=0}^{p-1} (-)^m \begin{pmatrix} p-1 \\ m \end{pmatrix} \varphi_{n-m}^{(p)} \, , \quad p = 2, ..., N, \quad n = 1, ..., N \quad \mathrm{mod}(N). \tag{3.14}$$

For $p = 2$ this formula, (3.14), is clearly consistent with (3.11) (and for $p = 3$ it is as well consistent with (3.13)), hence to prove its validity for all integer values of $p > 2$ it is sufficient to show that its validity at p entails its validity at $p+1$. Via (3.12), this is quite easy. Hence this formula, (3.14), is now proven.

But, for $p = N$, this formula entails (via (3.10))

$$\left(\frac{d}{d\tau} \right)^N \zeta_n = K \sum_{m=0}^{N-1} (-)^m \begin{pmatrix} N-1 \\ m \end{pmatrix} (\zeta_{n-m-1} - \zeta_{n-m}) \, , \quad n = 1, ..., N \quad \mathrm{mod}(N),$$

(3.15a)

and it is also easy to see that this last formula can be rewritten in the following simpler form:

$$\left(\frac{d}{d\tau}\right)^N \zeta_n = -K \sum_{m=0}^{N} (-)^m \binom{N}{m} \zeta_{n-m} , \quad n = 1, ..., N \quad \text{mod}(N). \tag{3.15b}$$

The remarkable fact that this is now a system of N *linear* ODEs (of N-th order) should be emphasized: the equations of motion (3.8) have been completely linearized! This of course entails that *all* the solutions $\zeta_n(\tau)$ of this system, hence as well *all* the solutions $\zeta_n(\tau)$ of the original equations of motion (3.8) are *entire*, indeed, up to degeneracies, just *exponentials*. In fact, the general solution of these system of linear ODEs, (3.15), can be obtained in fairly explicit form, by focussing firstly on the special solutions

$$\zeta_n(\tau) = u_n^{(j)}(\lambda) \exp\left[\exp\left(\frac{2\,\pi\,i\,j}{N}\right)(-K\,\lambda)^{\frac{1}{N}}\,\tau\right], \quad n = 1, ..., N. \tag{3.16}$$

Here and below a definite determination must be taken of the N-th root $(-K\,\lambda)^{\frac{1}{N}}$; different determinations are then explicitly taken care of by the factor $\exp\left(\frac{2\pi i j}{N}\right)$ multiplying this root. In this formula λ is of course the eigenvalue of the set of linear algebraic equations

$$\lambda\, u_n(\lambda) + \sum_{m=0}^{N} (-)^m \binom{N}{m} u_{n-m}(\lambda) = 0 , \quad n = 1, ..., N \quad \text{mod}(N), \tag{3.17}$$

and the constants $u_n(\lambda)$ are the N components of the corresponding eigenvectors. These N eigenvalues λ_k can be easily computed (using the known formula for the eigenvalues of a circulant matrix [14], as well as the basic properties of the binomial coefficients),

$$\lambda_k = (-)^k \left[2\,i\,\sin\left(\frac{k\,\pi}{N}\right)\right]^N, \quad k = 1, ..., N, \tag{3.18}$$

and the corresponding eigenvectors can as well be obtained,

$$u_n(\lambda_k) = \exp\left(\frac{-2\,\pi\,i\,k\,n}{N}\right), \quad k = 1, ..., N, \quad n = 1, ..., N. \tag{3.19}$$

The eigenvalue $\lambda_N = 0$, and the corresponding eigenvector $u_n(0) = 1$, $n = 1, ..., N$, correspond of course to the center-of-mass motion. Also note that, for even N, all these eigenvalues λ_k are *real*, that the largest of them is $\lambda_{\frac{N}{2}} = 2^N$, and that there are $\frac{N}{2} - 1$ others, each of them twice degenerate. For odd N the $N-1$ nonvanishing eigenvalues λ_k are instead all *imaginary*, and all different.

We thus conclude that the general solution of the set of linear ODEs (3.15) reads

$$\zeta_n(\tau) = \sum_{j,k=1}^{N} c_{jk}\, u_n^{(j)}(\lambda_k) \exp\left[\exp\left(\frac{2\,\pi\,i\,j}{N}\right)(-K\,\lambda_k)^{\frac{1}{N}}\,\tau\right], \quad n = 1, ..., N, \tag{3.20}$$

with the eigenvalues λ_k given by (3.18), the eigenvectors $u_n^{(j)}(\lambda_k)$ given by (3.19), and the N^2 constants c_{jk} arbitrary. Of course, the usual modification (namely, multiplication by a polynomial of degree $M-1$ in τ) must be introduced in case of degeneracies of order M, namely whenever M of the quantities $\exp\left(\frac{2\,\pi\,i\,j}{N}\right)\,(-K\,\lambda_k)^{\frac{1}{N}}$ with different values of j and k coincide.

This *general* solution of the ODEs (3.15) features N^2 arbitrary constants c_{jk}. It includes of course the *general* solution of the equations of motion (3.8), since the ODEs (3.15) are entailed by the equations of motion (3.8). In fact, by taking into account the obvious fact that the equations of motion (3.8) entail that the center-of-mass

$$\bar{\zeta}(\tau)=\frac{1}{N}\sum_{n=1}^{N}\zeta_n(\tau), \tag{3.21}$$

moves uniformly,

$$\bar{\zeta}(\tau)=\bar{\zeta}(0)+\bar{\zeta}'(0)\,\tau, \tag{3.22}$$

and the vanishing of the eigenvalue λ_N, the formula that includes the *general* solution of the equations of motion (3.8) can be written as follows:

$$\zeta_n(\tau)=\bar{\zeta}(\tau)+\sum_{k=1}^{N-1}\sum_{j=1}^{N}c_{jk}\,u_n^{(j)}(\lambda_k)\,\exp\left[\exp\left(\frac{2\,\pi\,i\,j}{N}\right)\,(-K\,\lambda_k)^{\frac{1}{N}}\,\tau\right],\quad n=1,...,N. \tag{3.23}$$

This formula still features $N(N-1)$ constants c_{jk}, while of course the *general* solution of the equations of motion (3.8) features only $2N$ arbitrary constants. For the reasons indicated at the end of the previous section we do not elaborate here on the additional restrictions that must be imposed on the $N(N-1)$ constants c_{jk} in order that (3.23) satisfy (3.8) (for a detailed discussion in the $N=3$ case see Ref. [5]).

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