On the Quantization of Yet Another Two Nonlinear Harmonic Oscillators

Francesco CALOGERO

Dipartimento di Fisica, Università di Roma "La Sapienza"
00185 Roma, Italy
E-mail: francesco.calogero@roma1.infn.it

Received August 03, 2003; Accepted September 08, 2003

Abstract

In two previous papers the quantization was discussed of three one-degree-of-freedom Hamiltonians featuring a constant c, the value of which does not influence at all the corresponding classical dynamics (which is characterized by *isochronous* solutions, *all* of them periodic with period 2π: "nonlinear harmonic oscillators"), but turned instead out to influence, possibly quite nontrivially, the results in the quantized case. The quantization of two analogous Hamiltonian systems is discussed in this paper. The outcome is analogous, if perhaps even more clear cut, than in the previous three cases; it also confirms that different quantized spectra may be obtained if quantization is performed *before* or *after* a (nonlinear) canonical transformation.

In two previous papers [1] [2] the (quantization of the) following three one-degree-of-freedom Hamiltonians was investigated:

$$\mathcal{H}_1(p,q) = \frac{1}{2}\left[\frac{p^2\,q^3}{c} + c\left(q + \frac{1}{q}\right)\right], \tag{1}$$

$$\mathcal{H}_2(p,q) = \frac{1}{2}\left[\frac{p^2\,\sin^2(q)\,\sin(2\,q)}{2\,c} + \frac{2\,c}{\sin(2\,q)}\right], \tag{2}$$

$$\mathcal{H}_3(p,q) = \frac{1}{2}\left[\frac{p^2\,\sin^2(q)\,\sin(2\,q)}{2\,c} + 2\,c\,\cot(2\,q)\right]. \tag{3}$$

These three Hamiltonians contain the arbitrary (positive) constant c, but the three corresponding Newtonian equations of motion that determine in the classical (unquantized) context the time-evolution of the coordinate $q(t)$ do not feature at all this constant, hence their *general* solutions $q(t)$ are as well independent of this constant, and moreover they are *all* periodic with period 2π (following Ref. [3] we call such systems, *all* solutions of which are *isochronous*, "nonlinear harmonic oscillators"). These three Hamiltonians were then quantized [1] [2], and, as might have been expected, for certain quantization prescriptions the equispaced energy spectrum

$$E_n = E_0 + n, \quad n = 0, 1, 2, ... \tag{4}$$

was obtained; but the evaluation of the ground-state energy E_0 turned out (to depend on the value of the constant c and possibly) to be complex (for sufficiently small values of c), corresponding then to a quantum Hamiltonian which lacked the property to be *self-adjoint* [1] [2]. In this paper we study two other Hamiltonian models closely related to the first of the three models mentioned above. The results we find in this case are analogous, if perhaps even more clear cut, than the previous findings [1] [2].

We consider the one-degree-of-freedom systems characterized by the following two Hamiltonians:

$$H^{(s)}(p, q) = \frac{1}{2}\left[\frac{p^2 q}{c} + c\left(q + \frac{s}{q}\right)\right], \quad s = \pm\,, \tag{5}$$

where c is an arbitrary *real* constant, which we hereafter assume (without significant loss of generality) to be *positive*,

$$c > 0. \tag{6}$$

The corresponding Hamiltonian equations of motion read

$$\dot{q} = \frac{q\,p}{c}, \quad \dot{p} = \frac{1}{2\,c}\left[p^2 + c^2\left(\frac{s}{q^2} - 1\right)\right], \quad s = \pm\,, \tag{7}$$

and they entail the Newtonian equations of motion

$$\ddot{q} = \frac{1}{2\,q}\left(s + \dot{q}^2 - q^2\right), \quad s = \pm. \tag{8}$$

Here and throughout superimposed dots denote of course time derivatives.

Note that the constant c does not appear at all in these Newtonian equations of motion.

The *general solution* of these Newtonian equations of motion reads

$$q(t) = q(0)\left[\cos\left(\frac{t}{2}\right)\right]^2 + \dot{q}(0)\,\sin(t) + \frac{s + [\dot{q}(0)]^2}{q(0)}\left[\sin\left(\frac{t}{2}\right)\right]^2, \quad s = \pm. \tag{9}$$

Obviously this solution $q(t) \equiv q(s; t)$ is independent of the constant c, nonsingular and clearly periodic in t with period 2π,

$$q(t + 2\pi) = q(t); \tag{10}$$

and it can be easily seen that

$$q_{\min} \leq q(t) \leq q_{\max} \tag{11a}$$

with

$$q_{\min} = \frac{s}{2\,q(0)}, \tag{11b}$$

$$q_{\rm max} = q_{\rm min} + \frac{[q(0)]^2 + [\dot{q}(0)]^2}{q(0)} = \frac{\frac{s}{2} + [q(0)]^2 + [\dot{q}(0)]^2}{q(0)} \ . \tag{11c}$$

The corresponding expression of the canonically conjugate coordinate $p(t) \equiv p(s;t)$ reads

$$p(t) = c \, \frac{s + [\dot{q}(0)]^2 - [q(0)]^2 + 2\, q(0)\, \dot{q}(0) \, \cos(t)}{2\, q(0)\, q(t)}; \tag{12}$$

note that it is as well independent of the value of the constant c, except for a trivial overall rescaling; and it is of course as well periodic with period 2π,

$$p(t + 2\pi) = p(t). \tag{13}$$

Let us conclude this treatment of the classical dynamics entailed by the two one-degree-of-freedom Hamiltonian models (5) by noting a difference between the two cases with $s = +$ and $s = -$, which is sufficiently important (see especially below) to justify our consideration of these two Hamiltonian models as two different cases. Let us hereafter suppose, for definiteness (and without significant loss of generality), that $q(0)$ is positive, $q(0) > 0$. It is then clear that, in the $s = +$ case, $q(t)$ is as well positive for all time, $q(t) > 0$, see (11) (in particular (11b)); and this entails that $p(t)$ is *nonsingular*, see (12). In the $s = -$ case $q_{\rm min}$ is instead negative, see (11b), while $q_{\rm max}$ might be positive or negative, depending on the initial conditions, in particular on the magnitude of both $q(0)$ and $\dot{q}(0)$ (see (11c)). There is therefore in this second, $s = -$, case the possibility that, at some times $t = t_0 \mod(2\pi)$, the coordinate q vanish, $q(t_0 + 2n\pi) = 0$, and therefore that at those times $p(t)$ diverge, see (12).

The periodic, indeed *isochronous*, character of the time evolution of the classical canonical variables, see (10) and (13), suggests that the corresponding quantized system – at least for some appropriate quantization prescription (see below) – feature the discrete equispaced spectrum (4).

To quantize the two Hamiltonian systems (5) we rewrite first of all these Hamiltonians as follows:

$$H^{(s)}(p,q) = \frac{1}{2} \left[\lambda \, \frac{p\,q\,p}{c} + \frac{(1-\lambda)\, \left(q\,p^2 + p^2\,q\right)}{2\,c} + c \left(q + \frac{s}{q} \right) \right], \quad s = \pm \ . \tag{14}$$

Clearly in the classical case, namely when p and q commute, these Hamiltonians, (14), are independent of the value of the constant λ and coincide with the Hamiltonian (5); and, as it happens, this is as well true in the quantum case, when we perform the replacement

$$q \Rightarrow x, \quad p \Rightarrow -i \, \frac{d}{dx} \tag{15}$$

(corresponding to the fact that p and q are noncommuting operators, indeed the commutator $[q,p]$ equals now the imaginary unit, $[q,p] = i$), and we must then solve the Schrödinger equation

$$H^{(s)}(-i \, \frac{d}{dx}, x) \ \psi_n(x) = E_n \, \psi_n(x), \quad s = \pm \ . \tag{16}$$

Indeed via (5) and (15) this Schrödinger equation (turns out to be independent of λ and) reads

$$-x\,\psi_n'' - \psi_n' + c^2\left(x + \frac{s}{x}\right)\psi_n = 2\,c\,E_n\,\psi_n, \quad s = \pm\ . \tag{17}$$

Here and throughout primes denote of course differentiations with respect to the argument of the functions they are appended to, and we are omitting for notational simplicity to indicate explicitly the dependence of the eigenfunction $\psi_n(x) \equiv \psi_n(s;x)$ and the eigenvalue $E_n \equiv E_n(s)$ on the parameter $s = \pm$ that distinguishes the two models under consideration.

Note that the same outcome would be produced by the "Weyl" quantization prescription (see, for instance, the Appendix of [1], and the references cited there) which entails

$$\left[F(q)\,p^2\right]_{\mathrm{Weyl}} = -\left[F(x)\,\frac{d^2}{dx^2} + F'(x)\,\frac{d}{dx} + \frac{1}{4}\,F''(x)\right], \tag{18}$$

hence it yields, directly from (5), again the Schrödinger equation (17).

To solve the Schrödinger equation (17) we set

$$\psi_n(x) = \exp\left(-c\,x\right)\,\sum_{m=0}^{n} a_{n,m}\,x^{m+\alpha}. \tag{19}$$

We then get for the coefficients $a_{n,m}$ the recursion relation

$$a_{n,m+1} = -2\,c\,a_{n,m}\,\frac{E_n - (m + \alpha + \frac{1}{2})}{(m + \alpha + 1)^2 - s\,c^2}, \quad s = \pm, \tag{20}$$

with the following two conditions:

$$\alpha^2 = s\,c^2\,, \quad s = \pm, \tag{21}$$

$$E_n = n + \alpha + \frac{1}{2}, \quad n = 0, 1, 2, \dots\ . \tag{22}$$

The first of these two conditions, (21), is consistent with the fact that the sum at the right-hand side of (19) starts at $m = 0$, and the second condition, (22), is consistent with the fact that the sum in the right-hand side of (19) ends at $m = n$.

At this point we must consider separately the two cases with $s = +$ and $s = -$.

In the $s = +$ case, the condition (21), together with (6), entails

$$\alpha = c > 0\ . \tag{23}$$

Note that this implies that the eigenfunction $\psi_n(x)$, see (19), vanishes at $x = 0$, $\psi_n(0) = 0$. This is consistent with the fact that in this case the classical motion is confined to the positive real axis, $0 < q(t) < \infty$, as discussed above. We can therefore limit consideration of the eigenfunction $\psi_n(x)$ to the interval $0 \le x < \infty$, with the boundary conditions

$$\psi_n(0) = 0, \quad \lim_{x\to\infty}\left[x^p\,\psi_n(x)\right] = 0, \tag{24}$$

where p is an arbitrary (of course finite) number; and it is then clear that in the functional space characterized by such restrictions (as well as by the requirement that $\psi_n(x)$ be nonsingular for $0 < x < \infty$) the quantum Hamiltonian operator $H^{(+)}(p,q)$ (see (14)) is *self-adjoint*. And indeed in this case, at least for the (most natural) quantization prescription as given above, which in this case also happens to coincide with the "Weyl" rule [1], we get (see (22) with (23)) the equispaced spectrum (4) with the following simple identification of the ground-state energy,

$$E_0 = \frac{1}{2} + c\ . \tag{25}$$

It is interesting to compare these findings with the results obtained [1] by quantizing the Hamiltonian $\mathcal{H}_1(p,q)$, see (1), which is related to the Hamiltonian $H^{(+)}(p,q)$ treated here, see (5), by the simple canonical transformation

$$q \Rightarrow \frac{1}{q}, \quad p \Rightarrow -p\,q^2\ . \tag{26}$$

For the quantized Hamiltonian $\mathcal{H}_1(p,q)$, see (1), the equispaced spectrum (4) was also obtained [1], but with the following expression of the ground-state energy:

$$E_0 = \frac{1}{2} + \sqrt{c^2 - \rho}\,, \tag{27}$$

where ρ is a constant which depends on the choice of the quantization prescription [1] and takes the value $\rho = \frac{1}{2}$ if the "Weyl" rule is adopted. Clearly the condition $c^2 \geq \rho$ is necessary in order that this expression, (27), of the ground-state energy E_0 be *real*, and indeed the same condition is required [1] in order that the quantized Hamiltonian $\mathcal{H}_1(p,q)$, see (1), be *self-adjoint*. No analogous restriction emerged in the (quite natural) quantization of the Hamiltonian $H^{(+)}(p,q)$, as described herein.

This is of course one more illustration of the well-known fact [4] that "canonical transformations and quantization" need not commute, namely that a different outcome (for instance a different ground-state energy, or even a quantum Hamiltonian operator that does, or does not, possess the essential property to be *self-adjoint*) may be obtained by quantizing a system *before* or *after* performing a canonical transformation.

Let us finally proceed and consider the other case, $s = -$.

Then from the condition (21) we infer that α is imaginary, hence we see from (22) that the energy spectrum is no longer real. This is of course an indication that the quantum Hamiltonian operator $H^{(-)}(-i\,\frac{d}{dx},\,x)$, see (14), is *not* self-adjoint in the functional space which contains the eigenfunction $\psi_n(x)$, which indeed in this case does not vanish at $x = 0$, where the Hamiltonian operator is singular. We therefore must conclude that in this $s = -$ case quantization runs into difficulties (independently of the value of c), in spite of the fact that, as we saw above, the classical solution $q(t)$ of the Newtonian equation of motion yielded by this Hamiltonian is nonsingular and periodic with period 2π (but let us recall that in this $s = -$ case, in contrast to the $s = +$ case discussed above, the other canonical variable, $p(t)$, might become singular at certain times; while of course the Hamiltonian remains itself constant throughout the classical motion).

The analogies and differences of the findings for the models considered herein, and for the three models previously considered [1] [2], move us to conclude by reiterating [2] our hunch that these results are probably pedagogically useful: perhaps they should be included in the teaching of elementary quantum mechanics courses.

Acknowledgements

We are grateful to Sandro Graffi, with whom we began these researches, for many illuminating conversations, and we regret that the pressure of other duties did not allow him to participate in the development of the findings reported in this paper to an extent considered by him sufficient to justify co-authorship.

References

[1] Calogero F and Graffi S, On the quantization of a nonlinear Hamiltonian oscillator, *Phys. Lett. A* **313** (2003), 356–362.

[2] Calogero F, On the quantization of two other nonlinear harmonic oscillators, *Phys. Lett. A* **319** (2003), 240-245.

[3] Calogero F and Inozemtsev V I, Nonlinear harmonic oscillators, *J. Phys. A: Math. Gen.* **35** (2002), 10365–10375.

[4] van Hove L, Sur certaines repré sentations unitaires d'un groupe infini de transformations, *Mem. Acad. Royale de Belgique, Classe de Sciences* **26** (6) (1951), 1–102.