Edge-Magic Total Labellings of Some Network Models

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Abstract—It has been known that edge-symmetric graphs can be used as models of some scale-free networks, such as hierarchical networks and self-similar networks, such as graph colorings can be used for distinguishing objects of communication and informa-tion networks. We study the edge-magic property of edge-symmetric graphs, and construct graphs having edge-magic total labelings from smaller graphs.

Keywords—network; tree; set-ordered labelings; edge-magic total labelling; edge-symmetric graphs

I. INTRODUCTION

Graph coloring theory is one of the most actively branch in graph theory. It involves in many fields, such as such as physics, chemistry, computer science, network theory, social science, etc. And graph labelings provide useful mathematical models for a wide range of applications, such as data security, cryptography (secret sharing schemes), astronomy, various coding theory problems, communication networks, mobile telecommunication systems, bioinforma-tics and X-ray crystallography. More detailed discussions on applications of graph labelings can be found in Bloom and Golomb's papers [11] and [12]. Many studies in graph labeling refer to Rosa’s research [13].

The mathematical model of scale-free networks in study of complex networks is closed to real networks. It has been known that edge-symmetric graphs can be used models of some scale-free networks, such as hierarchical networks and self-similar networks, etc. Li et al. [18] critically overviews the current understanding on scale-freeness and proposes its mathematical models. Yao et al. [19] present: The notation $N(t) = (p(u, k, t), G(t))$ denotes a dynamic network, where $p(u, k, t)$ is the probability such that the probability of a new node being adjacent to $k$ other nodes submits to $p(u, k, t)$; $G(t)$ is the connected topological structure (also, graph) of $N(t)$, $t \in [a, b]$; $G(a)$ is the initially connected graph of $N(t)$ at $t = a$. We say a node of $N(t)$ an all-time-hub node if it is not a leaf of any spanning tree $T(t)$ having maximal leaves in $G(t)$ for each time $t \in [a, b]$. By the method of analyzing spanning trees in scale-free networks, some problems are researched in [20]. As a result, finding a spanning tree with as many leaves as possible (MLATP is one of the classical NP-complete problems) is equal to finding a minimal connected dominating set in a connected network. It is very important to make network models for simulating real networks.

Sedlacek [14] published a paper about another kind of graph labeling, called the labeling magic. His definition was motivated by the magic square notion in number theory. A magic labeling is a function from the set of edges of a graph $G$ into the non-negative real numbers, so that the sums of the edge labels around any vertex in $G$ are all the same. Stewart [15] called magic labeling supermagic if the set of edge labels consisted of consecutive integers. Motivated by Sedlacek and Stewart’s research, many new related definitions have been proposed and new results have been found.

Conjecture 1. [16] Every tree admits an edge-magic total labelling.

Conjecture 2. [17] Every tree admits a super edge-magic total labelling.

An edge-symmetric graph $H$ is a connected graph having a non-empty proper subset $S \subseteq E(H)$ such that $H - S$ contains $m(\geq 2)$ components $H_1, H_2, \cdots, H_m$ with $H_i \cong H$. It has been known that edge-symmetric graphs can be used as models of researching scale-free networks, hierarchical networks and self-similar networks [2]. We use standard notation and terminology of graph theory here. The graphs mentioned are finite graphs without loops, multiple or directed edges. The shorthand notation $[m, n]$ stands for an integer set $\{m, m+1, \cdots, n\}$, where $n, m \geq 0$. A $(p, q)$-graph $G$ is one with $p$ vertices and $q$ edges.

Definition 1. [3] If a $(p, q)$-graph $G$ admits a mapping $f : V(T) \times [0, q]$ such that $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and the edge label set $\{f(uv) = |f(u) - f(v)| : u, v \in E(G)\} = \{1, q\}$, then $f$ is called a graceful labelling, also, $G$ is graceful.

We write $f(V(G)) = \{f(u) : u \in V(G)\}$ and $f(E(G)) = \{f(uv) : u, v \in E(G)\}$ hereafter.

Definition 2. ([7],[9]) A bipartite graph $G$ admits a graceful labelling $f$. If $\max \{f(x) : x \in X\} < \min \{f(y) : y \in Y\}$, then $(X,Y)$ is the bipartition of $V(G)$, then $f$ is called a set-ordered graceful labelling, and this case is denoted as $f(X) < f(Y)$.

Definition 3. ([3],[5]) Let $G$ be a $(p, q)$-graph. If there exists a constant $\lambda$ and a bijection $f : V(G) \cup E(G) \rightarrow [1, p+q]$ such that $f(u)+f(v)+f(uv)=\lambda$ for every edge $u, v \in E$, then we say $f$ is an edge-magic total labelling of $G$, and $\lambda$ a magic constant. Furthermore, if $G$ is a bipartite graph with bipartition $(X,Y)$, and $f$ holds $f(V(G))=[1, p]$ and $\max \{f(x)$. 


that every $H_k$ admits super set-ordered edge-magic total labellings for $k \in [1, p]$. Then the uniformly edge-symmetric tree $(T; H_1, H_2, \cdots, H_p)$ admits super set-ordered edge-magic total labellings.

**Corollary 4.** Let $T$ be a set-ordered graceful tree on $p$ vertices, and its bipartition $(X, Y)$ hold $|X|=|Y|=1$. Suppose that every $H_i$ admits set-ordered graceful labellings for $k \in [1, p]$. Then the uniformly edge-symmetric tree $(T; H_1, H_2, \cdots, H_p)$ admits set-ordered graceful labellings.

**Corollary 5.** Let $T$ be a set-ordered graceful tree on $p$ vertices, and its bipartition $(X, Y)$ hold $|X|=|Y|=1$. Suppose that every $H_i$ admits set-ordered edge-magic total labellings for $k \in [1, p]$. Then the uniformly edge-symmetric tree $(T; H_1, H_2, \cdots, H_p)$ admits set-ordered edge-magic total labellings.

**Corollary 6.** Let $T$ be a super set-ordered edge-magic total tree on $p$ vertices, and its bipartition $(X, Y)$ hold $|X|=|Y|=1$. Suppose that every $H_i$ admits super set-ordered edge-magic total labellings for $k \in [1, p]$. Then the uniformly edge-symmetric tree $(T; H_1, H_2, \cdots, H_p)$ admits super set-ordered edge-magic total labellings.

**Proof of Theorem 1.** Use the notations of Definition 4 for $G=(T; H_1, H_2, \cdots, H_p)$. Let $(X, Y)$ be the bipartition of a set-ordered graceful tree $T$, where $|X|=|Y|=1$, $X=\{w_1, w_2, \cdots, w_r\}$ and $Y=\{w_{r+1}, w_{r+2}, \cdots, w_{r+p}\}$. By this theorem's hypothesis, $T$ has a set-ordered graceful labelling $f$ such that $f(w_i)=i-1$, $i \in [1, p]$, since $f(X)<f(Y)$.

Notice that $G$ is a uniformly edge-symmetric tree. By Definition 4 we can prove the following properties of $G$.

**P1.** Every tree $H_i$ has the same number $|H_i|=n$ from $s=|X_i|=|X|$ and $r=|Y_i|=|Y|$ for $i \neq k$.
P2. Every tree $H_k$ admits a super set-ordered edge-magic total labelling $g_k$ such that $g(u_k) + g(v_k) + g(u_{vk}) + g(v_{vk}) = 0$ (because $g$ is every $H_k$ admits super set-ordered edge-magic total labellings for $k \in [1, p]$).

P3. For every edge $u_{vk,j}$ of each tree $H_k$ holds $g(u_{vk}) + g(u_{vk}) + g(v_{vk}) = 0$ for edges $w_{vk,j} \in E(T)$.

P4. Two trees $H_1$ and $H_2$ both have the same label sets $g_1(V(H_1)) = [1, n]$, $g_1(E(H_1)) = [n+1, 2n-1]$ for $k \in [1, p]$.

We can take a tree $H_1$ as a representative of $H_1, H_2, \ldots, H_p$, where $V(H_1) = X \cup Y$ with $X = \{u; i \in [1, s] \}$ and $Y = \{v; j \in [1, t] \}$. Furthermore, $H_1$ admits a super set-ordered edge-magic total labelling $g_1$ such that $g_1(u_{u}) + g_1(u_{v}) + g_1(v_{u}) + g_1(v_{v}) = 0$ for $i \in [1, s], j \in [1, t]$. According to the above deduction, we have $g_1(u_{u}) + g_1(u_{v}) + g_1(v_{u}) + g_1(v_{v}) = 0$ for $i \in [1, s], j \in [1, t]$.

We use the labelling of $T$ to define another labelling $f'$ of $T$ as:

$$f'(w) = f(w), \quad t \in [1, p].$$

By the parity of $p$, we define a labelling $g$ of the uniformly edge-symmetric tree $G$ in the following.

**Case 1.** If $p = 2\beta + 1$, then $|X| - |Y| \leq 1$.

Let $\lambda = (5\beta + 2)n + g_1(v_0)$. The labelling $g$ of $G$ can be defined as:

(i) When $k \in [1, \beta + 1]$, set $g(u_{u}) = m(k-1) + g_1(u_{u}) - g_1(v_{u})$, $g(u_{v}) = m(k-1) + g_1(u_{v}) - g_1(v_{v})$, and for every edge $u_{u} \in E(G)$, let $g(u_{u}) = g_1(u_{u}) + 2n(2\beta - 1) - k$, $k \in [1, \beta + 1]$.

(ii) When $k \in [\beta + 2, 2\beta + 1]$, set $g(v_{u}) = m(2\beta - 2) - g_1(v_{u}) + g_1(v_{v})$, $g(v_{v}) = m(2\beta - 2) - g_1(v_{v}) + g_1(u_{v})$, $g(u_{u}) = g_1(u_{u}) + 2n(2\beta - 1) - k$, $k \in [1, \beta + 1]$.

The above shows that for $k = 2\beta + 1$, $H_1$ and $H_2$ form a matchable pair. Next, our aim is to prove that $g$ is a super set-ordered edge-magic total labelling of $G$. For $k \in [1, \beta + 1]$, $i \in [1, s]$ and $j \in [1, t]$, each edge $u_{u} \in E(H_1)$ holds $g(u_{u}) + g(u_{v}) + g(v_{u}) + g(v_{v}) = \lambda$. For $k = [\beta + 2, 2\beta + 1]$, $i \in [1, s]$ and $j \in [1, t]$, every edge $u_{u} \in E(H_1)$ holds $g(u_{u}) + g(u_{v}) + g(v_{u}) + g(v_{v}) = 0$.

For $k \in [1, 2\beta + 1]$ and $i \in [1, s]$, if we identify the vertex $u_{u} \in X \subset V(H_1)$ with the vertex $w_{vk,j} \in V(T)$ and then substitute $f'(w)$ by $g(u_{u})$. Finally, set $g(w_{vk}) = f'(w)$ for edges $w_{vk,j} \in E(T)$.

For $k = [1, 2\beta + 1]$ and $j \in [1, t]$, if we identify the vertex $v_{v} \in Y \subset V(H_1)$ with the vertex $w_{vk,j} \in V(T)$ and then substitute $f'(w)$ by $g(v_{v})$, then define $g(w_{vk}) = f'(w)$ for edges $w_{vk,j} \in E(T)$. Here, the labelling $g$ of $G$ is defined well.

We verify that $f'(w) + f'(w) + f'(w) = 0$ for every edge $w_{vk,j} \in E(T)$.

(1.1) For a fixed $i \in [1, s]$, we identify the vertex $u_{u} \in X \subset V(H_1)$ with the vertex $w_{vk,j} \in V(T)$ into one vertex.

For $i \in [1, \beta + 1], j \in [\beta + 2, 2\beta + 1]$ and $i \in [1, s]$, every edge $w_{vk,j} \in E(T) \subset E(G)$ satisfies $f'(w) + f'(w) + f'(w) = 0$.

(1.2) For a fixed $j \in [1, t]$, we identify the vertex $v_{v} \in Y \subset V(H_1)$ with the vertex $w_{vk,j} \in V(T)$ into one vertex.

For $i \in [1, \beta + 1], j \in [\beta + 2, 2\beta + 1]$ and $i \in [1, s]$, every edge $w_{vk,j} \in E(T) \subset E(G)$ satisfies $f'(w) + f'(w) + f'(w) = 0$.
$g(u_1)$, finally, set $g(wv) = f^*(wv)$ for edges $wv \in E(T)$. On the other hand, for $k \in \{1, 2\}$ and $j \in \{1, 2\}$, if we identify the vertex $v_k \in V(T)$ with the vertex $w_j \in V(T)$ into one, we replace $f^*(wv)$ by $g(v_k)$, and then define $g(wv) = f^*(wv)$ for edges $wv \in E(T)$.

Here, the labelling $h$ of $G$ is defined well. We verify that $f^*(wv) + f^*(wv) = (5 \beta + 1)\alpha$ for every edge $wv \in E(T) \subseteq E(G)$. 

(2.1) For a fixed $i_1 \in \{1, 2\}$ and $k \in \{1, 2\}$, we identify the vertex $v_{i_1,k} \in V(T)$ with the vertex $w_k \in V(T)$ into one vertex. Then, for $i \in \{1, 2\}$, $j \in \{1, \beta, 1, 2\}$ and $l \in \{1, 2\}$, every edge $wv \in E(T) \subseteq E(G)$ holds $f^*(wv) + f^*(wv) + f^*(wv) = f^*(wv) + h(u_{i_1}) + h(w_{i_1}) + h(u_{i_1}) + h(w_{i_1}) + h(w_{i_1}) + h(w_{i_1}) + n(f(wv) - f(wv)) + 2n\beta h(u_{i_1}) + n(i-1) + 1 + g(w_{i_1}) + g(w_{i_1}) + g(w_{i_1}) + n(f(wv) - f(wv)) + 2n\beta n(3\beta + 1 - j) - g(w_{i_1}) + g(w_{i_1}) + n(i-1) + 2\beta + 5\beta + 1 + n(i-1) + 5n\beta + 1 = \alpha$.

(2.2) For a fixed $j \in \{1, 2\}$ and $k \in \{1, 2\}$, we identify the vertex $v_{i_1,k} \in V(T)$ with the vertex $v_{i_1,k} \in V(T)$ into one vertex. Then, for $i \in \{1, 2\}$, $j \in \{1, \beta, 1, 2\}$ and $l \in \{1, 2\}$, each edge $wv \in E(T) \subseteq E(G)$ holds $f^*(wv) + f^*(wv) + f^*(wv) = f^*(wv) + h(u_{i_1}) + h(w_{i_1}) + h(u_{i_1}) + h(w_{i_1}) + h(w_{i_1}) + h(w_{i_1}) + n(f(wv) - f(wv)) + 2n\beta h(u_{i_1}) + n(i-1) + 1 + g(u_{i_1}) + g(w_{i_1}) + n(f(wv) - f(wv)) + 2n\beta n(2\beta + 1 - j) - g(w_{i_1}) + g(u_{i_1}) + n(i-1) + 1 + n(f(wv) - f(wv)) + 2n\beta n(2\beta + 1 - j) - n(i-1) + 2\beta + 5\beta + 1 + n(i-1) + 5n\beta + 1 = \alpha$.

The above facts enable us to conclude that the bipartition $(X^*, Y^*)$ of $G$ holds $h(X^*) = h(Y^*)$, where $h(X^*) = \{h(u_{i_1}) : k \in \{1, 2\}, i \in \{1, 2\}\} \cup \{h(v_{i_1,k}) : k \in \{1, 2\}, j \in \{1, 2\}\}$, $h(Y^*) = \{h(u_{i_1}) : k \in \{1, 2\}, i \in \{1, 2\}\} \cup \{h(v_{i_1,k}) : k \in \{1, 2\}, j \in \{1, 2\}\}$.

Furthermore, $h(E(G)) = 2n\beta + 1, 4n\beta - 1$. Hence, $g$ is a super set-ordered edge-magic total labelling of $G$ for even $p$, as desired. The theorem follows the proof of Case 1 and Case 2.

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