

A Variational Approach to the Stability of Periodic Peakons

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Abstract

The peakons are peaked traveling wave solutions of an integrable shallow water equation. We present a variational proof of their stability.

1 Introduction

The nonlinear partial differential equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (1.1)$$

models the unidirectional propagation of shallow water waves over a flat bottom, $u(x, t)$ representing the water's free surface in non-dimensional variables [1]. For a discussion of (1.1) in the context of water waves we refer to [13]. We consider periodic solutions of (1.1), i.e. $u : \mathbb{S} \times [0, T) \rightarrow \mathbb{R}$ where \mathbb{S} is the unit circle and $T > 0$ is the maximal existence time of the solution. The interest in periodic solutions is motivated by the observation that the majority of the waves propagating on a channel are approximately periodic. Equation (1.1) is a bi-Hamiltonian equation with infinitely many conservation laws [11]. The fact that (1.1) is a re-expression of the geodesic flow in the group of compressible diffeomorphisms of the circle [16], leads to a proof that it satisfies the Least Action Principle [6]. Moreover, for a large class of initial data, equation (1.1) is an infinite-dimensional completely integrable Hamiltonian system [7]. Associated to (1.1) there is a whole hierarchy of integrable equations [12]. Let us also point out that equation (1.1) models nonlinear waves in cylindrical axially symmetric hyperelastic rods, with $u(x, t)$ representing the radial stretch relative to a prestressed state [10].

Equation (1.1) has the periodic traveling solutions

$$u(x, t) = c\varphi(x - ct), \quad c \in \mathbb{R},$$

where $\varphi(x)$ is given for $x \in [0, 1]$ by

$$\varphi(x) = \frac{\cosh(1/2 - x)}{\sinh(1/2)},$$

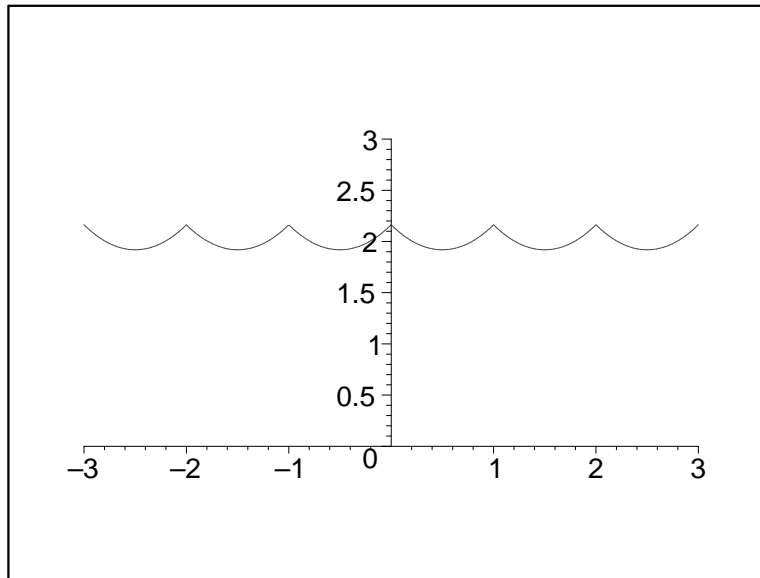


Figure 1. The peakon for $c = 1$.

and extends periodically to the real line. Because of their shape (they are smooth except for a peak at their crest, see Figure 1) these solutions are called (periodic) peakons. Note that the height of the peakon is proportional to its speed. Equation (1.1) can be rewritten in conservation form as

$$u_t + \frac{1}{2} \left(u^2 + \varphi * \left[u^2 + \frac{1}{2} u_x^2 \right] \right)_x = 0. \quad (1.2)$$

This is the exact meaning in which the peakons are solutions (see Section 3).

Equation (1.1) has the conservation laws

$$H_0[u] = \int_{\mathbb{S}} u dx, \quad H_1[u] = \frac{1}{2} \int_{\mathbb{S}} (u^2 + u_x^2) dx, \quad H_2[u] = \frac{1}{2} \int_{\mathbb{S}} (u^3 + uu_x^2) dx. \quad (1.3)$$

We consider the following variational problem:

$$\begin{aligned} & \text{Minimize } H_1[u] \text{ over the class of } u \in H^1(\mathbb{S}) \text{ such} \\ & \text{that } H_0[u] = H_0[c\varphi] \text{ and } H_2[u] = H_2[c\varphi]. \end{aligned} \quad (1.4)$$

Theorem 1. *The solutions of the variational problem (1.4) are exactly all translates $c\varphi(\cdot - \xi)$, $\xi \in \mathbb{R}$, of the peakons.*

For a wave profile $u \in H^1(\mathbb{S})$ the functional $H_1[u]$ represents kinetic energy (see the discussion in [6]). A standard principle in physics asserts that states of lowest energy are stable. Since Theorem 1 classifies the peakons as minima of constrained energy, this suggests that the peakons are stable. A small change in the shape of a peakon can yield another one with a different speed. The appropriate notion of stability is therefore that of orbital stability: a periodic wave with an initial profile close to a peakon remains close to some translate of it for all later times. That is, the shape of the wave remains

approximately the same for all times. For the peakons to be physically observable it is necessary that their shape remains approximately the same as time evolves. Therefore the stability of the peakons is of great interest. In [14] we proved that the peakons are orbitally stable.

Theorem 2. [14] *For every $\epsilon > 0$ there is a $\delta > 0$ such that if $u \in C([0, T]; H^1(\mathbb{S}))$ is a solution to (1.1) with*

$$\|u(\cdot, 0) - c\varphi\|_{H^1(\mathbb{S})} < \delta,$$

then

$$\|u(\cdot, t) - c\varphi(\cdot - \xi(t))\|_{H^1(\mathbb{S})} < \epsilon \quad \text{for } t \in (0, T),$$

where $\xi(t) \in \mathbb{R}$ is any point where the function $u(\cdot, t)$ attains its maximum. \square

In Theorem 2 a solution $u(x, t)$ of (1.1) on $[0, T]$ with $T > 0$ refers to a function $u \in C([0, T]; H^1(\mathbb{S}))$ such that (1.2) holds in distributional sense and the functionals $H_i[u]$, $i = 0, 1, 2$, defined in (1.3) are independent of $t \in [0, T]$.

In this paper we will show that the stability is a consequence of the fact that the peakons are solutions of the variational problem (1.4). It turns out that the stability derives purely from the three conservation laws in (1.3). No other structure of equation (1.1) is relevant. In fact, we have the following result.

Theorem 3. *For every $\epsilon > 0$ there is a $\delta > 0$ such that for $u \in H^1(\mathbb{S})$ we have*

$$\|u - c\varphi(\cdot - \xi)\|_{H^1(\mathbb{S})} < \epsilon \quad \text{whenever } |H_i[u] - H_i[c\varphi]| < \delta, \quad i = 0, 1, 2,$$

where $\xi \in \mathbb{R}$ is any point where u attains its maximum.

It is clear that Theorem 2 follows from Theorem 3. Indeed, if $u \in C([0, T]; H^1(\mathbb{S}))$ is a solution to (1.1) starting close to the peakon, then $H_i[u]$ is close to $H_i[\varphi]$, $i = 0, 1, 2$. Since the H_i 's remain constant with time, they stay close for $t \in [0, T]$. We deduce from Theorem 3 that $u(\cdot, t)$ stays close to some translate of the peakon $c\varphi$ for $t \in [0, T]$.

In Section 2 we prove Theorem 1 and Theorem 3. We conclude the paper with Section 3 where we discuss various results on the existence of solutions to (1.1).

2 Proofs

For simplicity we henceforth take $c = 1$. We will identify \mathbb{S} with $[0, 1)$ and view functions u on \mathbb{S} as periodic functions on the real line with period one. For an integer $n \geq 1$, we let $H^n(\mathbb{S})$ be the Sobolev space of all square integrable functions $f \in L^2(\mathbb{S})$ with distributional derivatives $\partial_x^i f \in L^2(\mathbb{S})$ for $i = 1, \dots, n$. These Hilbert spaces are endowed with the inner products

$$\langle f, g \rangle_{H^n(\mathbb{S})} = \sum_{i=0}^n \int_{\mathbb{S}} (\partial_x^i f)(x) (\partial_x^i g)(x) dx.$$

For $u \in H^1(\mathbb{S})$ we write $M_u = \max_{x \in \mathbb{S}} \{u(x)\}$ and $m_u = \min_{x \in \mathbb{S}} \{u(x)\}$.

Note that φ is continuous on \mathbb{S} with peak at $x = 0$,

$$M_\varphi = \varphi(0) = \frac{\cosh(1/2)}{\sinh(1/2)}, \quad m_\varphi = \varphi(1/2) = \frac{1}{\sinh(1/2)},$$

and

$$H_0[\varphi] = \int_0^1 \frac{\cosh(1/2 - x)}{\sinh(1/2)} dx = 2.$$

Moreover, φ is smooth on $(0, 1)$, $\varphi_x(x) \rightarrow 1$ as $x \uparrow 1$, and $\varphi_x(x) \rightarrow -1$ as $x \downarrow 0$. This gives, as $\varphi_{xx} = \varphi$ on $(0, 1)$, that the integration by parts formula $\int \varphi_{xx} f dx = -\int \varphi_x f_x dx$, $f \in H^1(\mathbb{S})$, holds with $\varphi_{xx} = \varphi - 2\delta$. Here δ denotes the Dirac delta distribution and, for simplicity, we abuse notation by writing integrals instead of the $H^{-1}(\mathbb{S})/H^1(\mathbb{S})$ duality pairing. We obtain

$$\begin{aligned} H_1[\varphi] &= \frac{1}{2} \int_{\mathbb{S}} (\varphi^2 + \varphi_x^2) dx = \frac{1}{2} \int_{\mathbb{S}} (\varphi^2 - \varphi \varphi_{xx}) dx \\ &= \frac{1}{2} \int_{\mathbb{S}} (\varphi^2 - \varphi(\varphi - 2\delta)) dx = \varphi(0) = M_\varphi. \end{aligned}$$

Using the identity $\cosh 3x = \cosh^3 x + 3 \cosh x \sinh^2 x$ we also compute

$$\begin{aligned} H_2[\varphi] &= \frac{1}{2} \int_{\mathbb{S}} \varphi(\varphi^2 + \varphi_x^2) dx = \frac{1}{2 \sinh^3(1/2)} \int_{-1/2}^{1/2} (\cosh^3 x + \cosh x \sinh^2 x) dx \\ &= \frac{1}{2 \sinh^3(1/2)} \int_{-1/2}^{1/2} (\cosh 3x - 2 \cosh x \sinh^2 x) dx \\ &= \frac{\sinh 3x - 2 \sinh^3 x}{6 \sinh^3(1/2)} \Big|_{-1/2}^{1/2} = \frac{\sinh(3/2) - 2 \sinh^3(1/2)}{3 \sinh^3(1/2)}. \end{aligned}$$

Employing the identity $\sinh 3x = 4 \sinh^3 x + 3 \sinh x$ we can rewrite this as

$$H_2[\varphi] = \frac{2}{3} + \frac{1}{\sinh^2(1/2)}. \quad (2.1)$$

We need the following lemmas.

Lemma 1. [14] For every $u \in H^1(\mathbb{S})$ and $\xi \in \mathbb{R}$,

$$H_1[u] - H_1[\varphi] = \frac{1}{2} \|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 + 2u(\xi) - 2M_\varphi,$$

where $M_\varphi = \max_{x \in \mathbb{S}} \{\varphi(x)\}$. □

Lemma 2. [14] We have

$$\max_{x \in \mathbb{S}} |f(x)| \leq \sqrt{\frac{\cosh(1/2)}{2 \sinh(1/2)}} \|f\|_{H^1(\mathbb{S})}, \quad f \in H^1(\mathbb{S}). \quad (2.2)$$

Moreover, $\sqrt{\frac{\cosh(1/2)}{2 \sinh(1/2)}}$ is the best constant and equality holds in (2) if and only if $f = c\varphi(\cdot - \xi)$ for some $c, \xi \in \mathbb{R}$, i.e. if and only if f has the shape of a peakon. □

Let $\Gamma = \{(M, m) \in \mathbb{R}^2 : M \geq m > 0\}$. For any positive $u \in H^1(\mathbb{S})$ we define $F_u : \Gamma \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_u(M, m) &= M \left[H_1[u] - \frac{1}{2} m^2 - M \sqrt{M^2 - m^2} + m^2 \ln \left(\frac{M + \sqrt{M^2 - m^2}}{m} \right) \right] \\ &\quad + \frac{1}{2} m^2 H_0[u] + \frac{2}{3} (M^2 - m^2)^{3/2} - H_2[u]. \end{aligned} \quad (2.3)$$

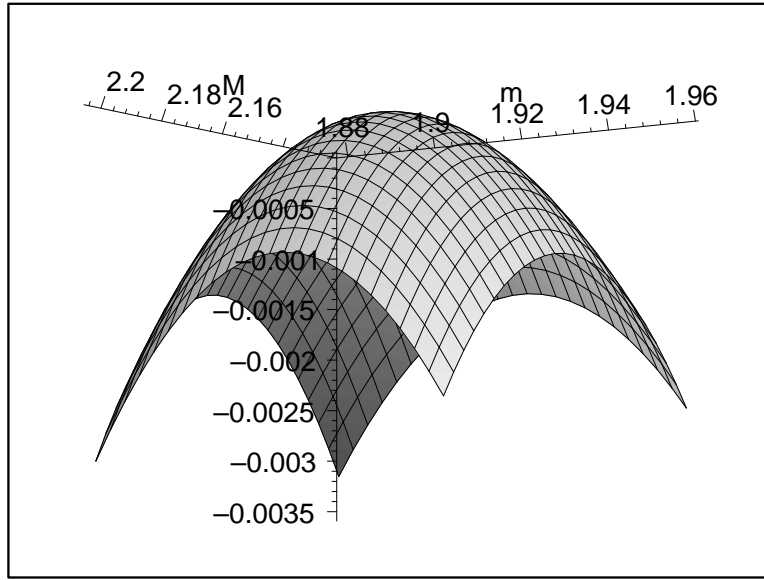


Figure 2. The function $F_\varphi(M, m)$ near the point (M_φ, m_φ) .

Lemma 3. [14] For any positive $u \in H^1(\mathbb{S})$ we have

$$F_u(M_u, m_u) \geq 0,$$

where $M_u = \max_{x \in \mathbb{S}}\{u(x)\}$ and $m_u = \min_{x \in \mathbb{S}}\{u(x)\}$. \square

The next lemmas highlight some properties of the function $F_\varphi(M, m)$ associated to the peakon. The graph of $F_\varphi(M, m)$ is shown in Figure 2 and Figure 4.

Lemma 4. [14] For the peakon φ we have

$$\begin{aligned} F_\varphi(M_\varphi, m_\varphi) &= 0, \\ \frac{\partial F_\varphi}{\partial M}(M_\varphi, m_\varphi) &= 0, \quad \frac{\partial F_\varphi}{\partial m}(M_\varphi, m_\varphi) = 0, \\ \frac{\partial^2 F_\varphi}{\partial M^2}(M_\varphi, m_\varphi) &= -2, \quad \frac{\partial^2 F_\varphi}{\partial M \partial m}(M_\varphi, m_\varphi) = 0, \quad \frac{\partial^2 F_\varphi}{\partial m^2}(M_\varphi, m_\varphi) = -2. \end{aligned}$$

\square

Lemma 5. The function $F_\varphi(M, m)$ has no other critical points (i.e. points where $\frac{\partial F_\varphi}{\partial M} = \frac{\partial F_\varphi}{\partial m} = 0$) in Γ except (M_φ, m_φ) .

Proof. Differentiation of F_u , as defined in (2.3), gives

$$\begin{aligned} \frac{\partial F_u}{\partial M} &= \left[H_1[u] - \frac{1}{2}m^2 - M\sqrt{M^2 - m^2} + m^2 \ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) \right] \\ &+ M \left[-\sqrt{M^2 - m^2} - \frac{M^2}{\sqrt{M^2 - m^2}} + \frac{m^2}{M + \sqrt{M^2 - m^2}} \left(1 + \frac{M}{\sqrt{M^2 - m^2}}\right) \right] \\ &+ 2M\sqrt{M^2 - m^2} = H_1[u] - \frac{1}{2}m^2 + m^2 \ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - M\sqrt{M^2 - m^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F_u}{\partial m} = & M \left[-m + \frac{Mm}{\sqrt{M^2 - m^2}} + 2m \ln \left(\frac{M + \sqrt{M^2 - m^2}}{m} \right) \right. \\ & \left. - \frac{m^3}{(M + \sqrt{M^2 - m^2})\sqrt{M^2 - m^2}} \right] - Mm + mH_0[u] - 2m\sqrt{M^2 - m^2}. \end{aligned}$$

In the above expression for $\frac{\partial F_u}{\partial m}$ the first, second and fourth term within the bracket cancel to give

$$\frac{\partial F_u}{\partial m} = 2Mm \ln \left(\frac{M + \sqrt{M^2 - m^2}}{m} \right) - Mm + mH_0[u] - 2m\sqrt{M^2 - m^2}.$$

We introduce new variables $x, y > 0$ by

$$M = \frac{\cosh x}{\sinh y}, \quad m = \frac{1}{\sinh y}.$$

We get

$$\sqrt{M^2 - m^2} = \frac{\sinh x}{\sinh y} \quad \text{and} \quad \ln \left(\frac{M + \sqrt{M^2 - m^2}}{m} \right) = \ln(\cosh x + \sinh x) = x.$$

Hence

$$\frac{\partial F_u}{\partial M} = H_1[u] - \frac{1}{2\sinh^2 y} + \frac{x}{\sinh^2 y} - \frac{\cosh x \sinh x}{\sinh^2 y},$$

and

$$\frac{\partial F_u}{\partial m} = \frac{2x \cosh x}{\sinh^2 y} - \frac{\cosh x}{\sinh^2 y} + \frac{H_0[u]}{\sinh y} - \frac{2 \sinh x}{\sinh^2 y}.$$

Now take $F_u = F_\varphi$. Recalling that $H_0[\varphi] = 2$, we obtain

$$\frac{\partial F_\varphi}{\partial M} = H_1[\varphi] + \frac{1}{\sinh^2 y} \left(x - \frac{1}{2} \right) - \frac{\cosh x \sinh x}{\sinh^2 y},$$

and

$$\frac{\partial F_\varphi}{\partial m} = \frac{2 \cosh x}{\sinh^2 y} \left(x - \frac{1}{2} \right) + \frac{2}{\sinh^2 y} (\sinh y - \sinh x).$$

Therefore any critical point of $F_\varphi(M, m)$ in Γ corresponds to a solution $x, y > 0$ of

$$\begin{cases} \sinh^2 y H_1[\varphi] + \left(x - \frac{1}{2} \right) - \cosh x \sinh x = 0, \\ \left(x - \frac{1}{2} \right) \cosh x + (\sinh y - \sinh x) = 0. \end{cases}$$

Solving the second equation for $\sinh y$, we obtain the equivalent system of equations

$$g_1(x) = 0, \quad g_2(x) = \sinh y, \quad x > 0, \quad y > 0, \quad (2.4)$$

where

$$g_1(x) = g_2^2(x) H_1[\varphi] + \left(x - \frac{1}{2} \right) - \cosh x \sinh x, \quad g_2(x) = \sinh x - \left(x - \frac{1}{2} \right) \cosh x.$$

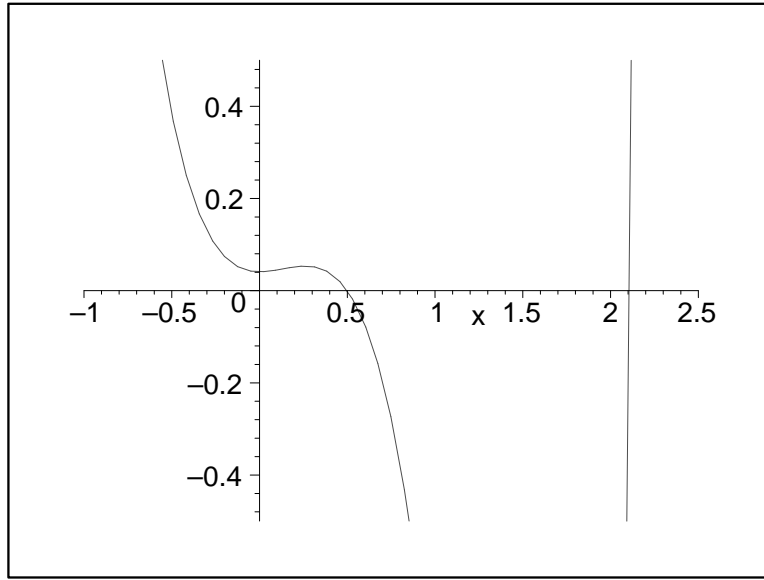


Figure 3. The graph of the function $g_1(x)$. The zero at $\frac{1}{2}$ corresponds to the critical point at (M_φ, m_φ) . The zero close to $x = 2.1$ does not correspond to a point in Γ .

We compute

$$g_2'(x) = -\left(x - \frac{1}{2}\right) \sinh x,$$

and so

$$g_1'(x) = -2g_2(x) \left(x - \frac{1}{2}\right) \sinh(x) H_1[\varphi] - 2 \sinh^2 x.$$

To show that there is no solution of (2.4) for $x > \frac{1}{2}$, note first that $g_2(1/2) > 0$. Since $g_2'(x) < 0$ for $x > \frac{1}{2}$ and $\lim_{x \rightarrow \infty} g_2(x) = -\infty$, there is an $x_0 > \frac{1}{2}$ such that $g_2(x) > 0$ for $\frac{1}{2} < x < x_0$ and $g_2(x) \leq 0$ for $x \geq x_0$. The condition $y > 0$ rules out all solutions with $x \geq x_0$. Moreover, on $\frac{1}{2} < x < x_0$ we obtain $g_1'(x) < 0$. Hence, as $g_1(\frac{1}{2}) = 0$, there can be no solutions of (2.4) with $x > \frac{1}{2}$.

To prove that $x < \frac{1}{2}$ is equally impossible, recall that $H_1[\varphi] = \cosh(1/2)/\sinh(1/2)$. We get $g_1(\frac{1}{2}) = 0$ and $g_1'(\frac{1}{2}) < 0$. Since $g_2(0) = \frac{1}{2}$ and $H_1[\varphi] > 2$, we have $g_1(0) > 0$. Consequently, if we can show that $g_1'(x)$ only has one zero on the open interval $(0, 1/2)$, then $g_1(x) > 0$ on $(0, 1/2)$. But $g_1'(x) = 0$ is equivalent to $w(x) = 0$ where

$$w(x) = \left(x - \frac{1}{2}\right) g_2(x) H_1[\varphi] + \sinh x.$$

Observe that $w(0) < 0$ and $w(\frac{1}{2}) > 0$. Also,

$$w'(x) = \left[\left(1 - \left(x - \frac{1}{2}\right)^2\right) \sinh x - \left(x - \frac{1}{2}\right) \cosh x \right] H_1[\varphi] + \cosh x,$$

so that $w'(x) > 0$ on $(0, 1/2)$. This proves that $g_1'(x)$ only has one zero in $(0, 1/2)$, and hence that there are no solutions of (2.4) with $x < \frac{1}{2}$.

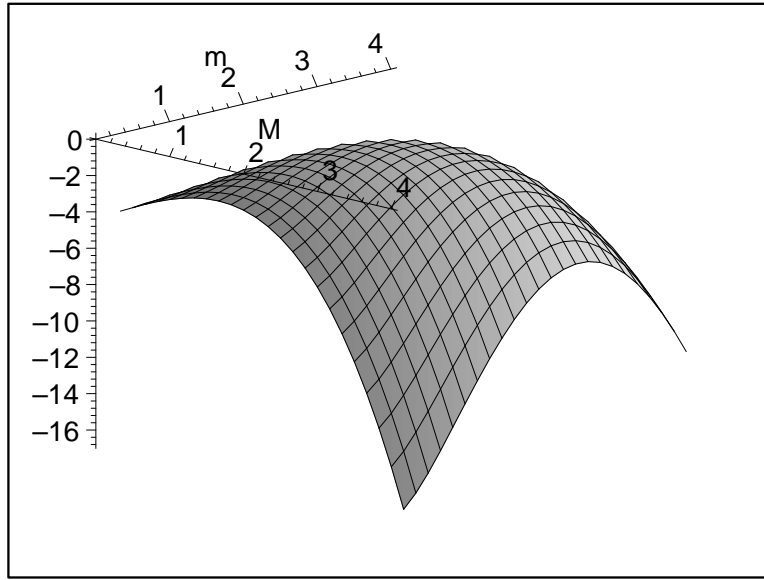


Figure 4. The function $F_\varphi(M, m)$ only gets close to zero at (M_φ, m_φ) .

$$(M_\varphi = \frac{\cosh(1/2)}{\sinh(1/2)} \approx 2.16, m_\varphi = \frac{1}{\sinh(1/2)} \approx 1.92.)$$

We proved that all solutions of (2.4) must have $x = \frac{1}{2}$. But $x = \frac{1}{2}$ implies $y = \frac{1}{2}$, so that the corresponding critical point is exactly

$$\left(\frac{\cosh(1/2)}{\sinh(1/2)}, \frac{1}{\sinh(1/2)} \right) = (M_\varphi, m_\varphi).$$

This finishes the proof of the lemma. \square

Lemma 6. *We have $F_\varphi(M, m) < 0$ for $(M, m) \in \Gamma$, $(M, m) \neq (M_\varphi, m_\varphi)$. Moreover, F_φ stays bounded away from zero near the boundary of Γ and there is a constant $a > 0$ such that*

$$F_\varphi(M, m) < -a\|(M, m)\|_{\mathbb{R}^2}^3 \quad \text{for all large } (M, m) \in \Gamma. \quad (2.5)$$

Proof. We first establish the behavior of F_φ near the boundary of Γ . It is easy to see that on the boundary $\{m = M\}$ of Γ we have $F_\varphi(M, m) = h(M)$, where

$$h(M) = -\frac{1}{2}M^3 + \frac{1}{2}H_0[\varphi]M^2 + H_1[\varphi]M - H_2[\varphi].$$

We will show that $h(M) < 0$ for $M > 0$. Assume the contrary. Then, since $h(0) < 0$ and $h(M) \rightarrow -\infty$ as $M \rightarrow \infty$, there is a point $M_0 > 0$ such that $\max_{M \geq 0} h(M) = h(M_0) \geq 0$ and $h'(M_0) = 0$. We have, recalling that $H_0[\varphi] = 2$,

$$h'(M) = -\frac{3}{2}M^2 + 2M + H_1[\varphi],$$

with zeros at

$$M = \frac{2}{3} \pm \frac{1}{3} \sqrt{4 + 6H_1[\varphi]}.$$

We infer that

$$M_0 = \frac{2}{3} + \frac{1}{3}\sqrt{4 + 6H_1[\varphi]}.$$

As a computation gives $h(M_0) < 0$, we conclude that $h(M) < 0$ for $M > 0$. Hence $F_\varphi(M, m)$ stays away from zero near the boundary $\{M = m\}$ of Γ .

Similarly, for $M > 0$ we have

$$F_\varphi(M, m) \rightarrow h_2(M) \quad \text{as } m \downarrow 0 \quad \text{where} \quad h_2(M) = -\frac{1}{3}M^3 + H_1[\varphi]M - H_2[\varphi].$$

Just as above we prove that $h_2(M) < 0$ for $M > 0$, so that $F_\varphi(M, m)$ is bounded away from zero also along $\{m = 0\}$.

To determine the behavior of $F_\varphi(M, m)$ as $(M, m) \rightarrow \infty$, note that as $m \rightarrow \infty$ with $M = \alpha m$, $\alpha \geq 1$, we have

$$F_\varphi(M, m) = \left(\alpha \ln(\alpha + \sqrt{\alpha^2 - 1}) - \frac{\alpha}{2} - \alpha^2 \sqrt{\alpha^2 - 1} + \frac{2}{3}(\alpha^2 - 1)^{3/2} \right) m^3 + O(m^2).$$

To establish (2.5) it is clearly enough to show that

$$\alpha \log(\alpha + \sqrt{\alpha^2 - 1}) - \frac{\alpha}{2} - \alpha^2 \sqrt{\alpha^2 - 1} + \frac{2}{3}(\alpha^2 - 1)^{3/2} \leq -\frac{1}{2}, \quad \alpha \geq 1. \quad (2.6)$$

Performing the change of variables $\cosh \beta = \alpha$, (2.6) is seen to be equivalent to

$$h_3(\beta) = \left(\beta - \frac{1}{2} \right) \cosh \beta - \cosh^2 \beta \sinh \beta + \frac{2}{3} \sinh^3 \beta \leq -\frac{1}{2}, \quad \beta \geq 0.$$

Differentiation gives

$$h'_3(\beta) = \left(\left(\beta - \frac{1}{2} \right) - \cosh \beta \sinh \beta \right) \sinh \beta.$$

Note that $h'_3(0) = 0$. For $\beta > 0$, we have $h'_3(\beta) = 0$ if and only if $\beta - \frac{1}{2} = \cosh \beta \sinh \beta$. We deduce that $h'_3(\beta) \neq 0$ for $\beta > 0$ and since $\lim_{\beta \rightarrow \infty} h'_3(\beta) = -\infty$, this means that $h'_3(\beta) < 0$ for $\beta > 0$. Hence $\max_{\beta \geq 0} h_3(\beta) = h_3(0) = -1/2$. This proves (2.6) and also (2.5).

It remains to prove that $F_\varphi(M, m) < 0$ for $(M, m) \in \Gamma$, $(M, m) \neq (M_\varphi, m_\varphi)$. In view of the behavior near the boundary, it is enough to show that the only critical point of F_φ in Γ is (M_φ, m_φ) . But that is the statement of Lemma 5. \square

Lemma 7. *To any $u \in H^1(\mathbb{S})$ associate the polynomial*

$$P_u(M) = M_\varphi \left(\frac{M^2}{2} - H_0[u]M + H_1[u] \right).$$

Then

$$(M_u - m_u)^2 \leq P_u(M_u).$$

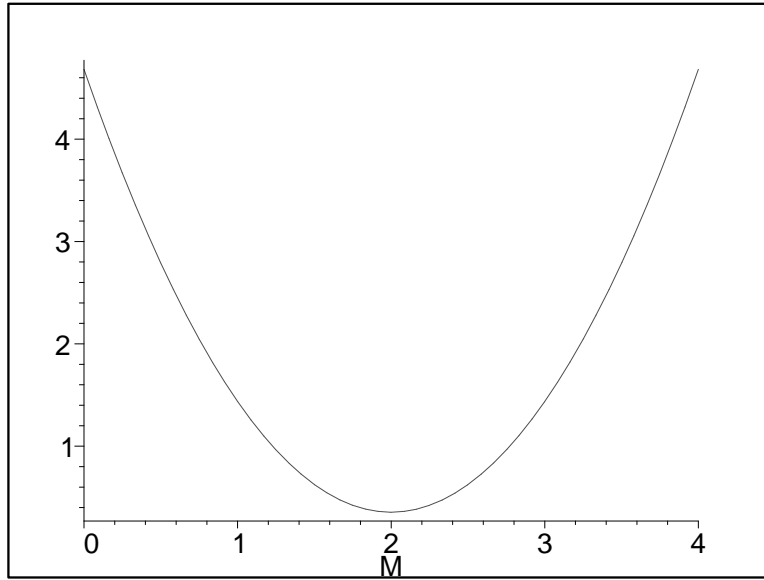


Figure 5. The graph of the polynomial $P_\varphi(M)$. As $M_\varphi \approx 2.16$, we clearly have $P_\varphi(M, m) < 1$ in the interval $(2, M_\varphi)$.

Proof. Applying Lemma 2 to $M_u - u$ we get

$$\max_{x \in \mathbb{S}} |M_u - u(x)| \leq \sqrt{\frac{M_\varphi}{2}} \|M_u - u\|_{H^1(\mathbb{S})}. \quad (2.7)$$

Observe that $\max_{x \in \mathbb{S}} |M_u - u(x)| = M_u - m_u$. Squaring (2.7) yields

$$(M_u - m_u)^2 \leq \frac{M_\varphi}{2} \|M_u - u\|_{H^1(\mathbb{S})}^2.$$

Noting that

$$\frac{1}{2} \|M_u - u\|_{H^1(\mathbb{S})}^2 = \frac{1}{2} \int_{\mathbb{S}} \left((M_u - u)^2 + u_x^2 \right) dx = \frac{M_u^2}{2} - H_0[u]M_u + H_1[u],$$

the lemma follows. \square

Proof of Theorem 1. Take $u \in H^1(\mathbb{S})$ with $H_0[u] = H_0[\varphi]$ and $H_2[u] = H_2[\varphi]$. Suppose $H_1[u] < H_1[\varphi]$. Let us first show that u is a strictly positive function. Since $\int_{\mathbb{S}} u dx = H_0[u] = H_0[\varphi] = 2$, we have $M_u \geq 2$. From Lemma 1 we obtain

$$\frac{1}{2} \|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 + 2u(\xi) - 2M_\varphi = H_1[u] - H_1[\varphi], \quad \xi \in \mathbb{R}.$$

As $H_1[u] - H_1[\varphi] < 0$ we deduce that $u(\xi) - M_\varphi < 0$ for $\xi \in \mathbb{R}$. Hence $M_u < M_\varphi$. By Lemma 7 we have

$$(M_u - m_u)^2 \leq P_u(M_u) \quad \text{where} \quad P_u(M) = M_\varphi \left(\frac{M^2}{2} - H_0[u]M + H_1[u] \right).$$

Since $2 \leq M_u < M_\varphi$ and $P_u(M)$ is less than, say 1, on the interval $[2, M_\varphi)$, we get $(M_u - m_u)^2 \leq 1$. This gives $m_u \geq 1$ so that u is indeed strictly positive.

Now since $H_0[u] = H_0[\varphi]$, $H_2[u] = H_2[\varphi]$ and $H_1[u] < H_1[\varphi]$, it follows from the definition (2.3) of F_u that $F_u(M, m) < F_\varphi(M, m)$ in Γ . We infer from Lemma 6 that $F_u(M, m) < 0$, $(M, m) \in \Gamma$. Since $(M_u, m_u) \in \Gamma$, this contradicts the fact that $F_u(M_u, m_u) \geq 0$ in view of Lemma 3. We conclude that any $u \in H^1(\mathbb{S})$ with $H_0[u] = H_0[\varphi]$ and $H_2[u] = H_2[\varphi]$ has $H_1[u] \geq H_1[\varphi]$.

It remains to show that any $u \in H^1(\mathbb{S})$ with $H_i[u] = H_i[\varphi]$, $i = 0, 1, 2$, is equal to a translate of the peakon. For any such u we have $F_u = F_\varphi$. Therefore, from Lemma 6 and Lemma 4 we deduce that $F_u(M, m) \geq 0$ only at the point $(M, m) = (M_\varphi, m_\varphi)$. As we have $F_u(M_u, m_u) \geq 0$ by Lemma 3, this implies $(M_u, m_u) = (M_\varphi, m_\varphi)$. Lemma 1 says that

$$\frac{1}{2} \|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 = 2M_\varphi - 2u(\xi) + H_1[u] - H_1[\varphi], \quad \xi \in \mathbb{R}.$$

Taking $\xi \in \mathbb{R}$ such that $u(\xi) = M_u = M_\varphi$, the right hand side is zero. Thus $u = \varphi(\cdot - \xi)$ for any $\xi \in \mathbb{R}$ such that $u(\xi) = M_u$. This completes the proof of Theorem 1. \square

Proof of Theorem 3. Take $u \in H^1(\mathbb{S})$ with $H_i[u] = H_i[\varphi] + \delta_i$ for $i = 0, 1, 2$. Let us first show that u is strictly positive whenever the δ_i 's are small. Since $\int_{\mathbb{S}} u dx = H_0[u] = H_0[\varphi] + \delta_0 = 2 + \delta_0$, we have $M_u \geq 2 + \delta_0$. From Lemma 1 we obtain

$$\frac{1}{2} \|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 + 2u(\xi) - 2M_\varphi = H_1[u] - H_1[\varphi] = \delta_1, \quad \xi \in \mathbb{R}.$$

We deduce that $u(\xi) - M_\varphi \leq \frac{\delta_1}{2}$ for $\xi \in \mathbb{R}$. Hence $M_u \leq M_\varphi + \frac{\delta_1}{2}$. Let $P_u(M)$ be the polynomial defined in Lemma 7. We get

$$P_u(M) = P_\varphi(M) - \delta_0 M_\varphi M + M_\varphi \delta_1,$$

so that P_u is a small perturbation of $P_\varphi(M)$. Taking the δ_i 's small, we can get $P_u(M)$ less than, say 1, on the interval $[2 + \delta_0, M_\varphi + \frac{\delta_1}{2}]$. As we showed that $M_u \in [2 + \delta_0, M_\varphi + \frac{\delta_1}{2}]$, Lemma 7 gives $(M_u - m_u)^2 \leq P_u(M_u) < 1$. With δ_0 small, this implies $m_u > 0$. Therefore we can restrict our attention to strictly positive $u \in H^1(\mathbb{S})$.

Take a positive $u \in H^1(\mathbb{S})$ with $H_i[u] = H_i[\varphi] + \delta_i$ for $i = 0, 1, 2$. Let $\epsilon > 0$ be given. Employing Lemma 1 we see that if

$$|M_\varphi - M_u| < \frac{\epsilon^2}{6} \quad \text{and} \quad |H_1[u] - H_1[\varphi]| < \frac{\epsilon^2}{6}, \quad (2.8)$$

then

$$\|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 = 4(M_\varphi - M_u) + 2(H_1[u] - H_1[\varphi]) < \epsilon^2, \quad \xi \in \mathbb{R},$$

where $\xi \in \mathbb{R}$ is any point with $u(\xi) = M_u$. We have

$$F_u(M, m) = F_\varphi(M, m) + M\delta_1 + \frac{1}{2}m^2\delta_0 - \delta_2,$$

so that F_u is a small perturbation of F_φ . On any bounded subset of Γ we can make the perturbation arbitrarily small by choosing the δ_i 's small. Moreover, Lemma 6 says that there is an $a > 0$ such that $F_\varphi(M, m) < -a\|(M, m)\|_{\mathbb{R}^2}^3$ for all large $(M, m) \in \Gamma$.

Consequently, as the perturbation is $O(\|(M, m)\|_{\mathbb{R}^2}^2)$, $F_u(M, m)$ is clearly negative for large $(M, m) \in \Gamma$. Therefore in view of Lemma 4 and Lemma 6 we can, by taking the δ_i 's small, make the set of $(M, m) \in \Gamma$ where $F_u(M, m) \geq 0$, an arbitrarily small neighborhood around (M_φ, m_φ) . We conclude that there is a $\delta > 0$ such that

$$|M_\varphi - M_u| < \frac{\epsilon^2}{6} \quad \text{whenever} \quad |H_i[u] - H_i[\varphi]| < \delta, \quad i = 0, 1, 2.$$

Shrinking δ if necessary so that $\delta < \frac{\epsilon^2}{6}$, we see that (2.8) holds. Hence, if $\xi \in \mathbb{R}$ is any point where u attains its maximum, we have $\|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})} < \epsilon$ whenever $|H_i[u] - H_i[\varphi]| < \delta$, $i = 0, 1, 2$. This proves Theorem 3. \square

3 Comments

The only way that a classical solution to (1.1) blows up in finite time is if the wave breaks: the solution remains bounded while its slope becomes unbounded in finite time [5]. Certain classical solutions of (1.1) exist for all time while others break [2, 3, 4]. We would like to emphasize that a shallow water equation which exhibits both wave breaking as well as peaked waves of permanent form was long time sought after [19].

We now review the issue of well-posedness. For $u_0 \in H^3(\mathbb{S})$ there exists a maximal time $T = T(u_0) > 0$ such that (1.1) has a unique solution $u \in C([0, T]; H^3(\mathbb{S})) \cap C^1([0, T]; H^2(\mathbb{S}))$ with H_0, H_1, H_2 conserved. For $u_0 \in H^r(\mathbb{S})$ with $r > 3/2$, equation (1.1) has a unique strong solution $u \in C([0, T]; H^r(\mathbb{S}))$ for some $T > 0$, with H_0, H_1, H_2 conserved [15, 18]. Since the peakons do not belong to the spaces $H^r(\mathbb{S})$ with $r > 3/2$, they have to be regarded as weak solutions to (1.1). It is known [8] that if $u_0 \in H^1(\mathbb{S})$ is such that $(1 - \partial_x^2)u_0$ is a positive Radon measure with bounded total variation (e.g. $u_0 = \varphi$), then (1.1) has a unique solution $u \in C([0, \infty); H^1(\mathbb{S})) \cap C^1([0, \infty); L^2(\mathbb{S}))$ and H_0, H_1, H_2 are conserved functionals. Note that if $(1 - \partial_x^2)u_0 \in C^1(\mathbb{S})$ changes sign, then the solution to (1.1) will develop into a breaking wave [5, 17]. Since $(1 - \partial_x^2)\varphi = 2\delta$, we infer that close to a peakon there exist profiles that develop into breaking waves as well as profiles that lead to globally existing waves. Our results apply in both cases up to breaking time.

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