Fractal Structures of General Mandelbrot Sets and Julia Sets Generated From Complex Non-Analytic Iteration $F_m(z) = \bar{z}^m + c$

Yan Dejun,Wei Xiaodan, Zhang Hongpeng, Jiang Nan and Liu Xiangdong
Research Institute of Non-linear Information Technology
Dalian Nationalities University
Dalian 116600, P.R. China
yan-dejun@dlnu.edu.cn

***Abstract*—In this paper we use the same idea as the complex analytic dynamics to study general Mandelbrot sets and Julia sets generated from the complex non-analytic iteration . The definition of the general critical point is given, which is of vital importance to the complex non-analytic dynamics. The general Mandelbrot set is proved to be bounded, axial symmetry by real axis, and have (m+1)-fold rotational symmetry. The stability condition of periodic orbits and the boundary curve of stability region of one-cycle are given. And the general Mandelbrot sets are constructed by the escape-time method and the periodic scanning algorithm, which present a better understanding of the structure of the Mandelbrot sets. The filled-in Julia sets Km,c have m-fold structures. Similar to the complex analytic dynamics, the general Mandelbrot sets are kinds of mathematical dictionary or atlas that map out the behavior of the filled-in Julia sets for different values of c.**

Keywords-Complex Non-analytic Iteration; Critical Point; General Mandelbrot Set; Julia Set

I. INTRODUCTION

The work on fractals of the complex analytic functions is based on the works of Fatou and Julia. They investigated convergence properties of the families of functions $\{f^n(z)\}$ generated by rational complex functions $f(z)$, here $f^n(z) = f(f^{n-1}(z))$. The complex plane of the initial values of z is then separated into two disjoint subsets. In one subset, each value z has an orbit that escapes to infinity. For points in the other subset, their orbits are bounded. The latter subset is called the filled-in Julia set, and its boundary is called Julia set. Following the footsteps of Fatou and Julia, Mandelbrot found that the orbits generated from the critical points at which the derivative of the function is zero dominate the dynamical behavior. And he plotted the set of all values of parameter, that the orbit of the critical points is bounded firstly. Now we call the set the Mandelbrot set. Since the appearance of high-resolution images of the Mandelbrot set and Julia sets[1-4], the researches of complex dynamical systems have enjoyed a remarkable growth in popularity. The fractals generated from the complex polynomial family[5-9], $z \leftarrow z^{\alpha} + c$, and $z \leftarrow z^{\omega} + c$ have been studied[10-12]. For the definition of these fractal sets of the complex plane, the analyticity of the generating function was essential.

In this paper, we investigate the general Mandelbrot sets and Julia sets generated from the complex non-analytic iteration $F_m(z) = \bar{z}^m + c$.

II. REVIEW

The well-known Mandelbrot set is the parameter plane for iteration of the complex analytic iteration

$$f_c(z) = z^2 + c, \tag{1}$$

with $z, c \in C$, is considered. The Mandelbrot set M consists of those c values for which the orbit of 0, i.e., the sequence 0, $f_c(0)$, $f_c(f_c(0)) = f_c^2(0)$, $f_c^3(0)$, … is bounded. That is

$$M = \{c \in C : f_c^{\ n}(0) \text{ is bounded } \forall n \in N\}. \tag{2}$$

Here $f_c^{\ n}(z)$ denotes $f_c(f_c^{\ n-1}(z))$,and 0 is the critical point of the map f_c. A critical point of f_c is a point z_{cr} in the dynamical plane such that the derivative vanishes :

$$f_c'(z_{cr}) = 0. \tag{3}$$

The analyticity of the map is characterized by the fulfillment of the Cauchy-Riemann condition. Taking $f_c = u + iv$ and $z = x + iy$, this condition is written like

$$\frac{du}{dx} = \frac{dv}{dy}, \qquad \frac{du}{dy} = -\frac{dv}{dx}. \tag{4}$$

The filled-in Julia set K_c is defined as the set of all points of dynamical plane that have bounded orbit with respect to f_c

$$K_c = \{z \in C : f_c^{\ n}(z) \text{ is bounded } \forall n \in N\}. \tag{5}$$

The reason for singling out the orbit of 0 in (2) is the following important fact from complex analytic dynamics: if f_c possesses an attracting periodic orbit, then the orbit of 0, the critical point, must converge to that periodic orbit. Recall that a periodic orbit also called a cycle is an orbit z_0, $f_c(z_0)$, …, $f_c^{\ p}(z_0) = z_0$ that returns to itself after p iterations. A point z_0 in the complex plane is called a periodic point of f_c with period p if p is the smallest integer satisfies $f_c^{\ p}(z_0) = z_0$. Such a periodic orbit is called attracting if

$$|(f_c^p)'(z_0)|<1. \quad (6)$$

For $z\in K_c$, $f_c^n(z)$ is bounded, implies that the orbit z, $f_c(z)$, $f_c^2(z)$, $f_c^3(z)$, … is a periodic orbit or falls eventually into an attracting periodic orbit determined by the fate of the orbit of the critical point 0.

In a word, the analyticity of generating functions is essential and critical points play an important role in the study of classical Mandelbrot set and Julia set.

III. GENERAL MANDELBROT SETS

In this Sec., we will use the same idea as the classical analytical methods to study the structures of the general Mandelbrot sets generated from complex non-analytic iteration $F_m(z)=\bar{z}^m+c$.

A. The General Critical Point

The complex iteration $F_m(z)=\bar{z}^m+c$ ($m \ge 2$) is non-analytic since it does not satisfy the Cauchy-Riemann condition (4). Let z_0 be a initial point, and may be written as $z_0=F_m{}^0(z_0)$ and $z_n=F_m{}^n(z_0)$. Then the iteration $F_m(z)$ is equivalent to iteration

$$z_{n+1}=\bar{z}_n^m+c. \quad (7)$$

Denote $z_n=x_n+y_n i$ and $c=a+bi$, the iteration is equivalent to the two-dimension iteration

$$T:\begin{cases} x_{n+1}=R(x_n,y_n)+a \\ y_{n+1}=I(x_n,y_n)+b \end{cases}, \quad (8)$$

where $R(x_n, y_n) = x_n^m - C_m^2x_n^{m-2}y_n^2 + C_m^4x_n^{m-4}y_n^4 + \dots$, $I(x_n, y_n) = - C_m^1x_n^{m-1}y_n + C_m^3x_n^{m-3}y_n^3 - C_m^5x_n^{m-5}y_n^5 + \dots$.

Theorem 1. For complex iteration $z_{n+1}=\bar{z}_n^m+c$, with an integer m, $m\ge 2$, the Jacobi matrix of the corresponding two-dimensional discrete map T is symmetric, and its proper polynomial is

$$\lambda^2-m^2(x_n^2+y_n^2)^{m-1}. \quad (9)$$

Proof. Mathematical induction can be used to prove this theorem.

When $m = 2$, Eq. (8) may be written as

$$T:\begin{cases} x_{n+1}=x_n^2-y_n^2+a \\ y_{n+1}=-2x_ny_n+b \end{cases}. \quad (10)$$

Its Jacobi matrix is

$$J_2=\begin{pmatrix} 2x_n & -2y_n \\ -2y_n & -2x_n \end{pmatrix}, \quad (11)$$

which is symmetric, and its proper polynomial is

$$\begin{vmatrix} 2x_n-\lambda & -2y_n \\ -2y_n & -2x_n-\lambda \end{vmatrix}=\lambda^2-2^2(x_n^2+y_n^2). \quad (12)$$

Assume the theorem holds for m, that is,

$$J_m=\begin{pmatrix} \frac{\partial R}{\partial x_n} & \frac{\partial R}{\partial y_n} \\ \frac{\partial I}{\partial x_n} & \frac{\partial I}{\partial y_n} \end{pmatrix} \quad (13)$$

satisfies

$$\frac{\partial R}{\partial x_n}=-\frac{\partial I}{\partial y_n}, \frac{\partial R}{\partial y_n}=\frac{\partial I}{\partial x_n}, \quad (14)$$

and

$$\begin{vmatrix} \frac{\partial R}{\partial x_n}-\lambda & \frac{\partial R}{\partial y_n} \\ \frac{\partial I}{\partial x_n} & \frac{\partial I}{\partial y_n}-\lambda \end{vmatrix}=\lambda^2-m^2(x_n^2+y_n^2)^{m-1}. \quad (15)$$

Using the induction hypothesis, for $m+1$,

$$z_{n+1}=\bar{z}_n^{m+1}+c=\bar{z}_n^m\bar{z}_n+c$$
$$=(R(x_n,y_n)+I(x_n,y_n)i)(x_n-y_ni)+a+bi.$$

Then, Eq. (8) may be written as

$$T:\begin{cases} x_{n+1}=x_nR(x_n,y_n)+y_nI(x_n,y_n)+a \\ y_{n+1}=x_nI(x_n,y_n)-y_nR(x_n,y_n)+b \end{cases}. \quad (16)$$

Its Jacobi matrix is

$$J_{m+1}=\begin{pmatrix} R(x_n,y_n)+x_n\frac{\partial R}{\partial x_n}+y_n\frac{\partial I}{\partial x_n} & I(x_n,y_n)+x_n\frac{\partial R}{\partial y_n}+y_n\frac{\partial I}{\partial y_n} \\ I(x_n,y_n)+x_n\frac{\partial I}{\partial x_n}-y_n\frac{\partial R}{\partial x_n} & -R(x_n,y_n)+x_n\frac{\partial I}{\partial y_n}-y_n\frac{\partial R}{\partial y_n} \end{pmatrix}.$$

With the help of Eq. (14), we obtain that the J_{m+1} is symmetric and its proper polynomial is

$$|J_{m+1}-\lambda E|=\lambda^2-x_n\frac{\partial(R^2+I^2)}{\partial x_n}-y_n\frac{\partial(R^2+I^2)}{\partial y_n}$$
$$-(R^2(x_n,y_n)+I^2(x_n,y_n))$$
$$+(x_n^2+y_n^2)\left(\frac{\partial R}{\partial x_n}\frac{\partial I}{\partial y_n}-\frac{\partial R}{\partial y_n}\frac{\partial I}{\partial x_n}\right). \quad (17)$$

From Eqs. (14) and (15), we have

$$\frac{\partial R}{\partial x_n}\frac{\partial I}{\partial y_n}-\frac{\partial R}{\partial y_n}\frac{\partial I}{\partial x_n}=-m^2(x_n^2+y_n^2)^{m-1} \quad (18)$$

and

$$R^2+I^2=(x_n^2+y_n^2)^m. \quad (19)$$

Then we obtain

$$|J_{m+1}-\lambda E|=\lambda^2-(m+1)^2(x_n^2+y_n^2)^m. \qquad \square$$

For the iteration T, the iterative property of its fixed point or periodic point w is determined by two eigenvalues λ_1 and λ_2 of its Jacobi matrix at w. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then w is attracting. If $|\lambda_1| >1$ and $|\lambda_2| > 1$, then w is repelling. Here two eigenvalues λ_1 and λ_2 play the same important roles just like the derivative of analytic families.

Definition 1. For complex non-analytic iteration $F_m(z)=\bar{z}^m+c$ (m is an integer, $m \ge 2$), we call the point $z = 0$, which the eigenvalues of the Jacobi matrix of the two-dimension iteration T equal to zero, the general critical point of F_m.

B. The Definition of General Mandelbrot Set

Definition 2. Let $F_m(z) = \bar{z}^m + c$, where $z, c \in C$. The general Mandelbrot set of $F_m(z)$ is defined as follows:

$$M_m = \{c \in C : F_m{}^n(0) \text{ is bounded } \forall n \in N\}. \quad (20)$$

Here $F_m{}^n(z)$ denotes $F_m(F_m{}^{n-1}(z))$, 0 is the general critical point of F_m.

Proposition 1. The general Mandelbrot sets M_m of complex non-analytic iteration $F_m(z)$ with an integer $m \geq 2$ possess the following properties:

1. M_m are axial symmetry by real axis.
2. M_m have (m+1)-fold rotational symmetry.
3. M_m are bounded.

Proof. The first and second properties are trivial. Now we only proof the third property.

Let $|c| > \sqrt[m-1]{2}$, then we have

$$\left|\bar{c}^m + c\right| \geq \left|\bar{c}^m\right| - |c| = |c|(|c|^{m-1} - 1) > |c|. \quad (21)$$

And when $|\omega| > |c| > \sqrt[m-1]{2}$,

$$\left|\bar{\omega}^m + c\right| > |\omega|(\sqrt[m-1]{2})^{m-1} - |c| > 2|\omega| - |\omega| = |\omega|. \quad (22)$$

Equation (22) means that the sequence

$$\{\left|F_m^n(0)\right|\}_{n=1}^{\infty} \quad (23)$$

with $c = \omega \in C$ is strictly monotone increasing. Then the limit of this sequence is infinity. Thus, we obtain

$$M_m \subseteq \{c \in C : |c| \leq \sqrt[m-1]{2}\}. \quad (24)$$

It shows that M_m are bounded. □

C. The General Mandelbrot Sets Constructed by Escaping Time Algorithm

Now we use escape-time algorithm to construct the general Mandelbrot sets generated from complex non-analytic iterations $F_m(z) = \bar{z}^m + c$.

Figure 1 illustrates four general Mandelbrot sets generated from $F_m(z) = \bar{z}^m + c$ for m = 2, 3, 5 and 8, which are denoted by M_2, M_3, M_5 and M_8.

We can find from Fig. 1 that the general Mandelbrot set M_m is axial symmetry by real axis, and has (m+1)-fold rotational symmetry around 0.

IV. JULIA SETS

In complex dynamics, the important object from the dynamics point of view is the Julia set. When we choose a parameter from the Mandelbrot set, then it is known that the dynamics on its Julia set is determined in a sense. In this section, we will use the escape-time algorithm to study the structures of the Julia sets generated from the complex non-analytic iteration $F_m(z) = \bar{z}^m + c$.

Definition 3. For the complex non-analytic iteration $F_m(z) = \bar{z}^m + c$, where $z, c \in C$, its filled-in Julia set $K_{m,c}$ consists of all points of dynamical plane that have bounded orbit with respect to F_m. That is,

$$K_{m,c} = \{z \in C : F_m{}^n(z) \text{ is bounded } \forall n \in N\}. \quad (26)$$

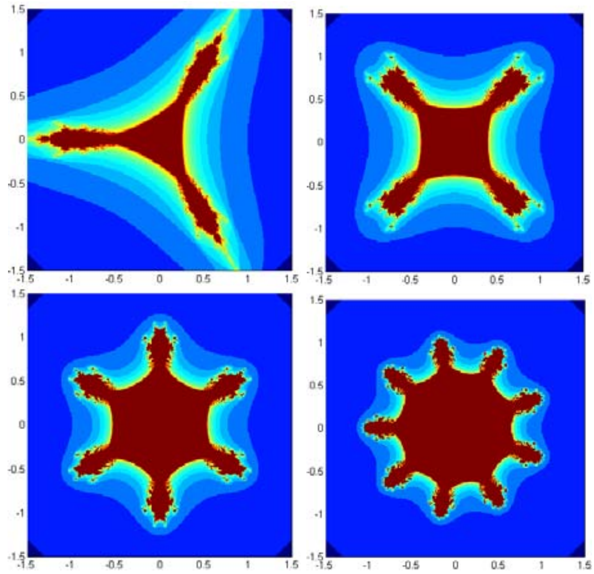
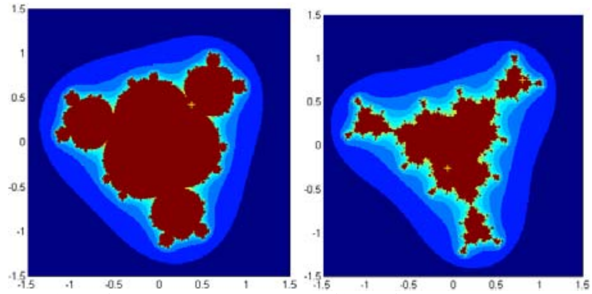
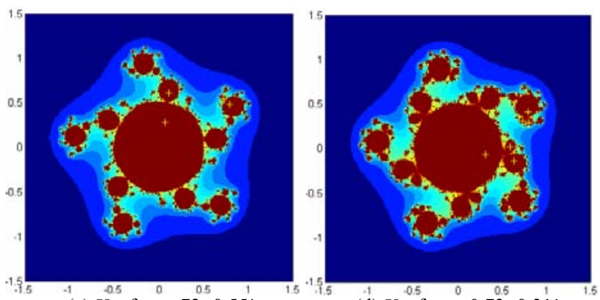


Figure 1. The general Mandelbrot sets of $F_m(z) = \bar{z}^m + c$ M_2, M_3, M_5 and M_8.



(a) $K_{3,c}$ for c= 0.5234+ 0.5259i. (b) $K_{3,c}$ for c=0.7893+0.7692i.



(c) $K_{5,c}$ for c=72+0.55i. (d) $K_{5,c}$ for c=0.73+0.31i.

Figure 2. The filled-in Julia sets $K_{3,c}$ and $K_{5,c}$ of $F_m(z) = \bar{z}^m + c$ for m=3 and m=5 at different parameter c.

Figure 2 illustrate the filled-in Julia set $K_{3,c}$ and $K_{5,c}$, for c in different stability regions of the general Mandelbrot sets M_3 and M_5.

Fig.2 (a) is $K_{3,c}$ for c = 0.5234+0.5259i in the 1-cycle region of M_3, and the orbits of all the points in $K_{3,c}$ converge to the 1-cycle 0.3716 + 0.4268i; Fig.2 (b) is $K_{3,c}$ for c = 0.7893 + 0.7692i in the 2-cycle region of M_3, and the orbits of the points in $K_{3,c}$ converge to the 2-cycle 0.8026 + 0.7553i, -0.0672 - 0.2594i; Fig.2 (c) is $K_{5,c}$ for c=72 + 0.55i in the 4-cycle region of M_5, and the orbits of the points in $K_{5,c}$ converge to the 4-cycle 0.1098 + 0.6173i, 0.7947 + 0.4882i, 0.0664 + 0.2836i, 0.7219 + 0.5491i; Fig.2 (d) is $K_{5,c}$ for c= 0.73 + 0.31i in the 6-cycle region of M_5, and the orbits of the points in $K_{5,c}$ converge to the 6-cycle 0.8015 + 0.2961i, 0.6402 - 0.1364i, 0.7899 + 0.4142i, 0.3082 - 0.0649i, 0.7316 + 0.3127i, 0.5917 + 0.0228i.

We can find from Fig.2 that the Julia sets $K_{m,c}$ have m-fold structures. Like the case of analytic families, the orbit of the general critical point 0 also plays an important role in determining the structure of general Julia sets.

V. CONCLUSION

The general Mandelbrot set M_m generated from the complex non-analytic iteration $F_m(z)=\bar{z}^m+c$ is axial symmetry by real axis, and has (m+1)-fold rotational symmetry around 0. The Julia sets $K_{m,c}$ generated from the complex non-analytic iteration $F_m(z)=\bar{z}^m+c$ have m-fold structures. Similar to the analytic complex iterated function systems, the general Mandelbrot sets M_m are kinds of mathematical dictionaries or atlases that map out the dynamical behavior of the Julia set $K_{m,c}$ for different values of c.

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