On the adjacent vertex distinguishing proper edge colorings of several classes of complete 4-partite and 5-partite graphs

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Abstract–**A proper k-edge coloring of a graph G is an assignment of k colors, $1, 2, \cdots, k$, to edges of G. For a proper edge coloring f of G and any vertex x of G, we use $S(x)$ denote the set of the colors assigned to the edges incident to x. If for any two adjacent vertices u and v of G, we have $S(u) \neq S(v)$, then f is called the adjacent vertex distinguishing proper edge coloring of G (or AVDPEC of G in brief). The minimum number of colors required in an AVDPEC of G is called the adjacent vertex distinguishing proper edge chromatic number of G, denoted by $\chi'_{a}(G)$. In this paper, adjacent vertex distinguishing proper edge chromatic numbers of several classes of complete 4-partite and 5-partite graphs are obtained.** *(Abstract)*

Keywords-complete 4-partite graphs; complete 5-partite graphs; proper edge coloring; adjacent vertex-distinguishing proper edge coloring (key words)

I. INTRODUCTION

The graph coloring has a wide range of applications in real life. In [1]-[6], the vertex distinguishing proper edge coloring of graphs is introduced and investigated; In [7], the adjacent vertex distinguishing proper edge coloring of graphs is introduced and investigated and the adjacent vertex distinguishing proper edge chromatic number of a graph G has been obtained for certain graphs, such as cycles, complete graphs. We use the usual notation as can be found in any book on graph theory, see e.g.[9].

Let G be a simple finite graph with maximum degree $\Delta(G)$, k be a positive integer and f be an edge coloring (i.e., f be an assignment of k colors, $1, 2, \cdots, k$, to the edges of G). For each element $z \in E(G)$, we use $f(z)$ denote the color of z. For every vertex $x \in V(G)$ the set of colors of edges incident with x is denoted by $S(x)$ and is called the color set of x. If f is proper and $S(u) \neq S(v)$ for each edge $uv \in E(G)$, then f is called a $k-$*adjacent vertex distinguishing proper edge coloring* of G (or a $k-$AVDPEC). The minimum number of colors required in an AVDPEC of G is called the *adjacent vertex distinguishing proper edge chromatic number* of G, denoted by $\chi'_{a}(G)$.

Let $G(V, E)$ be a simple graph and the vertex set $V(G)$ can be partitioned into n stable sets $V_1, V_2, \cdots, V_n$, where $V_i = \{v_{ij} \mid j = 1, 2, \cdots, m_i\}$, $m_i = \mid V_i \mid, i = 1, 2, \cdots, n$. If each vertex in V_i is adjacent to any one vertex in V_j, where $i \neq j$, $i, j = 1, 2, \cdots, n$, then G is called a complete n-partite graph and denoted by $K_{m_1, m_2, \cdots, m_n}$.

Lemma 1.1 and Lemma 1.2 are obvious.

Lemma 1.1 For any graph G that has no isolated edge and at most 1 isolated vertex , we have $\chi'_{a}(G) \leqslant \chi'_{s}(G)$.

Lemma 1.2 If $m \geqslant n \geqslant p \geqslant q \geqslant 1$, then $\Delta(K_{m,n,p,q}) = m + n + p$.

Lemma 1.3[7] Let G be a simple connected graph with order at least 3, if G has 2 adjacent vertices of maximum degree, then $\chi'_{a}(G) \geqslant \Delta(G) + 1$.

Lemma 1.4[10] Let $G[\{v \mid d(v) = \Delta(G)\}]$ be an induced subgraph by all vertices of maximum degree in G. If $G[\{v \mid d(v) = \Delta(G)\}]$ is a forest, then $\chi'(G) = \Delta(G)$.

In [8] Zhao Xinmei et al have obtained the adjacent vertex distinguishing proper edge chromatic numbers of some complete 4-partite graphs. The results in [8] are listed in the following table:

TABLE1: The adjacent vertex distinguishing proper edge chromatic numbers of some complete 4-partite graphs given in [8]

graph G	condition	$\chi'_{a}(G)$
$K_{m,n,p,q}$	$m > n > p > q \geqslant 1$	$m + n + p$
$K_{n,1,1,1}$	$n \geqslant 2$	$n + 3$
$K_{m,n,1,1}$	$m > n \geqslant 2$	$m + n + 2$
$K_{n,n,n,p}$	$n \geqslant p + 2$ and $p \geqslant 2$	$3n$
$K_{n,n,p,p}$	$n \geqslant p + 1$ and $p \geqslant 1$	$2n + p + 1$
$K_{n,p,p,p}$	$n \geqslant p + 2$ and $p \geqslant 2$	$n + 2p + 1$
$K_{n,n,n,n-1}$	$n \geqslant 2$	$3n$
$K_{n,n-1,n-1,n-1}$	$n \geqslant 3$	$3n - 1$

Based on the above work we will obtain the adjacent vertex distinguishing proper edge chromatic numbers of other several classes of complete 4-partite and 5-partite graphs in this paper.

II. Main Results

Suppose k, l are integers, $l \geqslant 1$, we use symbol $(k)_l$ to denote the number in $\{1, 2, \cdots, l\}$ which is congruent to k. For example, $(-3)_8 = 5$, $(9)_8 = 1$, $(20)_8 = 4$.

Theorem 2.1 If $m \geqslant n > p \geqslant 1$ and $m + p \leqslant 2n$, then $\chi'_a(K_{m,n,n,p}) = m + 2n$.

Proof Obviously we have $\chi'_a(K_{m,n,n,p}) \geqslant \Delta(K_{m,n,n,p}) = m + 2n$. In order to prove $\chi'_a(K_{m,n,n,p}) = m + 2n$, we need only to give the $(m+2n)$-AVDPEC of $K_{m,n,n,p}$. Suppose the colors that we will use are $1, 2, \cdots, m + 2n$.

Let edge $v_{4i}v_{3j}$ receive color $(i + j - 1)_{m+2n}$ and edge $v_{4i}v_{2j}$ receive color $(n+i+j-1)_{m+2n}$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, n$.

Let edge $v_{4i}v_{1j}$ receive color $(2n + i + j - 1)_{m+2n}$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, m$.

Let edge $v_{3i}v_{2j}$ receive color $(m + p + i + j - 1)_{m+2n}$, $i, j = 1, 2, \cdots, n$.

Let edge $v_{3i}v_{1j}$ receive color $(p + i + j - 1)_{m+2n}$ and edge $v_{2i}v_{1j}$ receive color $(m + n + p + i + j - 1)_{m+2n}$, $i = 1, 2, \cdots, n; j = 1, 2, \cdots, m$.

We construct a cycle C such that vertex set of C is $m+2n$ colors $\{1, 2, \cdots, m + 2n\}$ and edge set of C is $\{(i)_{m+2n}(i + 1)_{m+2n} | \; i = 1, 2, \cdots, m + 2n\}$.

The color set $S(v_{4i})$ contains all $m + 2n$ colors, $i = 1, 2, \cdots, p$.

The color set $S(v_{3i})$ contains just $m + n + p$ colors from $(i)_{m+2n}$ to $(m + n + p + i - 1)_{m+2n}$ along cycle C, $i = 1, 2, \cdots, n$.

The color set $S(v_{2i})$ contains just $m + n + p$ colors which are divided into two parts, the first part is from $(n + i)_{m+2n}$ to $(n+p+i-1)_{m+2n}$ along cycle C, the second part is from $(m + p + i)_{m+2n}$ to $(2m + n + p + i - 1)_{m+2n}$ along cycle C, $i = 1, 2, \cdots, n$.

Note that $S(v_{2i})$ contains colors from $(n+i)_{m+2n}$ to $(3n+p+i-1)_{m+2n}$ when $m = n$, $i = 1, 2, \cdots, n$.

The color set $S(v_{1i})$ contains just $2n + p$ colors which are divided into three parts, the first part is from $(p + i)_{m+2n}$ to $(n + p + i - 1)_{m+2n}$ along cycle C, the second part is from $(2n + i)_{m+2n}$ to $(2n+p+i-1)_{m+2n}$ along cycle C and the third part is from $(m + n + p + i)_{m+2n}$ to $(m + 2n + p + i - 1)_{m+2n}$ along cycle C, $i = 1, 2, \cdots, m$.

From the above discussions we can see that:

1. The number of the colors in each color set $C(x)$ is equal to the degree of vertex x;

2. The color sets of any two adjacent vertices with the same degree are different.

Thus the resulting coloring is a $(m + 2n)$-AVDPEC of $K_{m,n,n,p}$. The proof is completed.

Theorem 2.2 If $m \geqslant n > p \geqslant 1$, then $\chi'_a(K_{m,m,n,p}) = 2m + n$.

Proof Obviously we have $\chi'_a(K_{m,m,n,p}) \geqslant \Delta(K_{m,m,n,p}) = 2m + n$. In order to prove $\chi'_a(K_{m,m,n,p}) = 2m + n$, we need only to give the $(2m + n)$-AVDPEC of $K_{m,m,n,p}$. Suppose the colors that we will use are $1, 2, \cdots, 2m + n$.

We assign color $(i + j - 1)_{2m+n}$ to edge $v_{4i}v_{3j}$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, n$.

We assign color $(n + i + j - 1)_{2m+n}$ to edge $v_{4i}v_{2j}$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, m$.

We assign color $(n + m + i + j - 1)_{2m+n}$ to edge $v_{4i}v_{1j}$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, m$.

We assign color $(m + p + i + j - 1)_{2m+n}$ to edge $v_{3i}v_{2j}$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$.

We assign color $(p + i + j - 1)_{2m+n}$ to edge $v_{3i}v_{1j}$, $i = 1, 2, \cdots, n; j = 1, 2, \cdots, m$.

We assign color $(m+n+p+i+j-1)_{2m+n}$ to edge $v_{2i}v_{1j}$, $i, j = 1, 2, \cdots, m$.

We construct a cycle C such that vertex set of C is $2m+n$ colors $\{1, 2, \cdots, 2m + n\}$ and edge set of C is $\{(i)_{2m+n}(i + 1)_{2m+n} | \; i = 1, 2, \cdots, 2m + n\}$.

The color set $S(v_{4i})$ contains all $2m + n$ colors, $i = 1, 2, \cdots, p$.

The color set $S(v_{3i})$ contains just $2m + p$ colors from $(i)_{2m+n}$ to $(2m + p + i - 1)_{2m+n}$ along cycle C, $i = 1, 2, \cdots, n$.

The color set $S(v_{2i})$ contains just $m + n + p$ colors which are divided into two parts, the first part is from $(n + i)_{2m+n}$ to $(n+p+i-1)_{2m+n}$ along cycle C, the second part is from $(m + p + i)_{2m+n}$ to $(2m + n + p + i - 1)_{2m+n}$ along cycle C, $i = 1, 2, \cdots, m$.

Note that $S(v_{2i})$ contains colors from $(n+i)_{2m+n}$ to $(3n+p+i-1)_{2m+n}$ when $m = n$, $i = 1, 2, \cdots, m$.

The color set $S(v_{1i})$ contains just $m + n + p$ colors which are divided into two parts, the first part is from $(p + i)_{2m+n}$ to $(n + p + i - 1)_{2m+n}$ along cycle C, the second part $(m + n + i)_{2m+n}$ to $(2m + n + p + i - 1)_{2m+n}$ along cycle C, $i = 1, 2, \cdots, m$.

From the above discussions we can see that:

1. The number of the colors in each color set $C(x)$ is equal to the degree of vertex x;

2. The color sets of any two adjacent vertices with the same degree are different.

Thus the resulting coloring is a $(2m + n)$-AVDPEC of $K_{m,m,n,p}$. The proof is completed.

Theorem 2.3 If $n > 1$, then $\chi'_a(K_{n,1,1,1,1}) = n + 4$.

Proof Obviously we have $\Delta(K_{n,1,1,1,1}) = n+3$. By Lemma 1.3, $\chi'_a(K_{n,1,1,1,1}) \geqslant \Delta(K_{n,1,1,1,1}) + 1 = n + 4$. In order to prove $\chi'_a(K_{n,1,1,1,1}) = n+4$, we need only to give the $(n+4)$-AVDPEC of $K_{n,1,1,1,1}$. Suppose the colors that we will use are $1, 2, \cdots, n + 4$.

We assign color $1, 2, 3$ respectively to edges $v_{51}v_{41}$, $v_{51}v_{31}$ and $v_{51}v_{21}$.

We assign color $(i+3)_{n+4}$ to edge $v_{51}v_{1i}$ $(i = 1, 2, \cdots, n)$.

We assign color $3, 4$ respectively to edges $v_{41}v_{31}$ and $v_{41}v_{21}$.

We assign color $(i + 4)_{n+4}$ to edge $v_{41}v_{1i}$, $i = 1, 2, \cdots, n$.

We assign color 5 to edge $v_{31}v_{21}$.

We assign color $(i + 5)_{n+4}$ to edge $v_{31}v_{1i}$, $i = 1, 2, \cdots, n$.

We assign color $(i + 6)_{n+4}$ to edge $v_{21}v_{1i}$, $i = 1, 2, \cdots, n$.

We construct a cycle C such that vertex set of C is $n+4$ colors $\{1, 2, \cdots, n+4\}$ and edge set of C is $\{(i)_{n+4}(i+1)_{n+4} | \ i = 1, 2, \cdots, n+4\}$.

The color set $S(v_{51})$ contains just $n+3$ colors from $(1)_{n+4}$ to $(n+3)_{n+4}$ along cycle C.

The color set $S(v_{41})$ contains just $n+3$ colors which are divided into two parts, the first part is $\{1\}$, the second part is from $(3)_{n+4}$ to $(n+4)_{n+4}$ along cycle C.

The color set $S(v_{31})$ contains just $n+3$ colors which are divided into two parts, the first part is from 2 to 3 along cycle C, the second part is from $(5)_{n+4}$ to $(n+5)_{n+4}$ along cycle C.

The color set $S(v_{21})$ contains just $n+3$ colors which are divided into two parts, the first part is from 3 to 5 along cycle C, the second part is from $(7)_{n+4}$ to $(n+6)_{n+4}$ along cycle C.

The color set $S(v_{1i})$ contains just 4 colors from $(i+3)_{n+4}$ to $(i+6)_{n+4}$ along cycle C, $i = 1, 2, \cdots, n$.

From the above discussions we can see that:

1. The number of the colors in each color set $C(x)$ is equal to the degree of vertex x;

2. The color sets of any two adjacent vertices with the same degree are different.

Thus the resulting coloring is a $(n+4)$-AVDPEC of $K_{n,1,1,1,1}$. The proof is completed.

Theorem 2.4 If $m > n > 1$, then $\chi_{\mathrm{a}}^{'}(K_{m,n,1,1,1}) = m+n+3$.

Proof Obviously we have $\Delta(K_{m,n,1,1,1}) = m+n+2$. By Lemma 1.3, $\chi_{\mathrm{a}}^{'}(K_{m,n,1,1,1}) \geqslant \Delta(K_{m,n,1,1,1}) + 1 = m+n+3$. In order to prove $\chi_{\mathrm{a}}^{'}(K_{m,n,1,1,1}) = m+n+3$, we need only to give the $(m+n+3)$-AVDPEC of $K_{m,n,1,1,1}$. Suppose the colors that we will use are $1, 2, \cdots, m+n+3$.

We assign color $1, 2$ respectively to edges $v_{51}v_{41}$, $v_{51}v_{31}$.

We assign color $(i+2)_{m+n+3}$ to edge $v_{51}v_{2i}$, $i = 1, 2, \cdots, n$.

We assign color $(n+i+2)_{m+n+3}$ to edge $v_{51}v_{1i}$, $i = 1, 2, \cdots, m$.

We assign color 3 to edge $v_{41}v_{31}$.

We assign color $(i+3)_{m+n+3}$ to edge $v_{41}v_{2i}$, $i = 1, 2, \cdots, n$.

We assign color $(n+i+3)_{m+n+3}$ to edge $v_{41}v_{1i}$, $i = 1, 2, \cdots, m$.

We assign color $(i+4)_{m+n+3}$ to edge $v_{31}v_{2i}$, $i = 1, 2, \cdots, n$.

We assign color $(n+i+4)_{m+n+3}$ to edge $v_{31}v_{1i}$, $i = 1, 2, \cdots, m$.

We assign color $(n+i+j+4)_{m+n+3}$ to edge $v_{2i}v_{1j}$, $i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m$.

We construct a cycle C such that vertex set of C is $m+n+3$ colors $\{1, 2, \cdots, m+n+3\}$ and edge set of C is $\{(i)_{m+n+3}(i+1)_{m+n+3} | \ i = 1, 2, \cdots, m+n+3\}$.

The color set $S(v_{51})$ contains just $m+n+2$ colors from $(1)_{m+n+3}$ to $(m+n+2)_{m+n+3}$ along cycle C.

The color set $S(v_{41})$ contains just $m+n+2$ colors which are divided into two parts, the first part is $\{1\}$, the second part is from $(3)_{m+n+3}$ to $(m+n+3)_{m+n+3}$ along cycle C.

The color set $S(v_{31})$ contains just $m+n+2$ colors which are divided into two parts, one part is from 2 to 3 along cycle C, other part is from $(5)_{m+n+3}$ to $(m+n+4)_{m+n+3}$ along cycle C.

The color set $S(v_{2i})$ contains just $m+3$ colors which are divided into two parts, one part is from $(i+2)_{m+n+3}$ to $(i+4)_{m+n+3}$ along cycle C, other part is from $(n+i+5)_{m+n+3}$ to $(m+n+i+4)_{m+n+3}$ along cycle C, $i = 1, 2, \cdots, n$.

The color set $S(v_{1i})$ contains just $n+3$ colors from $(n+i+2)_{m+n+3}$ to $(2n+i+4)_{m+n+3}$ along cycle C, $i = 1, 2, \cdots, m$.

From the above discussions we can see that:

1. The number of the colors in each color set $C(x)$ is equal to the degree of vertex x;

2. The color sets of any two adjacent vertices with the same degree are different.

Thus the resulting coloring is a $(m+n+3)$-AVDPEC of $K_{m,n,1,1,1}$. The proof is completed.

Theorem 2.5 If $m > n > p > 1$, then $\chi_{\mathrm{a}}^{'}(K_{m,n,p,1,1}) = m+n+p+2$.

Proof Obviously we have $\Delta(K_{m,n,p,1,1}) = m+n+p+1$. By Lemma 1.3, $\chi_{\mathrm{a}}^{'}(K_{m,n,p,1,1}) \geqslant \Delta(K_{m,n,p,1,1}) + 1 = m+n+p+2$. In order to prove $\chi_{\mathrm{a}}^{'}(K_{m,n,p,1,1}) = m+n+p+2$, we need only to give the $(m+n+p+2)$-AVDPEC of $K_{m,n,p,1,1}$. Suppose the colors that we will use are $1, 2, \cdots, m+n+p+2$.

We assign color 1 to edge $v_{51}v_{41}$.

We assign color $(i+1)_{m+n+p+2}$ to edge $v_{51}v_{3i}$, $i = 1, 2, \cdots, p$.

We assign color $(p+i+1)_{m+n+p+2}$ to edge $v_{51}v_{2i}$, $i = 1, 2, \cdots, n$.

We assign color $(p+n+i+1)_{m+n+p+2}$ to edge $v_{51}v_{1i}$, $i = 1, 2, \cdots, m$.

We assign color $(i+2)_{m+n+p+2}$ to edge $v_{41}v_{3i}$, $i = 1, 2, \cdots, p$.

We assign color $(p+i+2)_{m+n+p+2}$ to edge $v_{41}v_{2i}$, $i = 1, 2, \cdots, n$.

We assign color $(p+n+i+2)_{m+n+p+2}$ to edge $v_{41}v_{1i}$, $i = 1, 2, \cdots, m$.

We assign color $(p+i+j+2)_{m+n+p+2}$ to edge $v_{3i}v_{2j}$, $i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, n$.

We assign color $(p+n+i+j+2)_{m+n+p+2}$ to edge $v_{3i}v_{1j}$, $i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, m$.

We assign color $(2p+n+i+j+2)_{m+n+p+2}$ to edge $v_{2i}v_{1j}$, $i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m$.

We construct a cycle C such that vertex set of C is $m+n+p+2$ colors $\{1, 2, \cdots, m+n+p+2\}$ and edge set of C is $\{(i)_{m+n+p+2}(i+1)_{m+n+p+2} | \ i = 1, 2, \cdots, m+n+p+2\}$.

The color set $S(v_{51})$ contains just $m+n+p+1$ colors from $(1)_{m+n+p+2}$ to $(m+n+p+1)_{m+n+p+2}$ along cycle C.

The color set $S(v_{41})$ contains just $m+n+p+1$ colors which are divided into two parts, the first part is $\{1\}$, the second part is from $(3)_{m+n+p+2}$ to $(m+n+p+2)_{m+n+p+2}$ along cycle C.

The color set $S(v_{3i})$ contains just $m+n+2$ colors which are divided into two parts, the first part is from $(i+1)_{m+n+p+2}$

to $(i+2)_{m+n+p+2}$ along cycle C, the second part is from $(p+i+3)_{m+n+p+2}$ to $(p+m+n+i+2)_{m+n+p+2}$ along cycle C, $i=1,2,\cdots,p$.

The color set $S(v_{2i})$ contains just $m+p+2$ colors which are divided into two parts, the first part is from $(p+i+1)_{m+n+p+2}$ to $(2p+i+2)_{m+n+p+2}$ along cycle C, the second part is from $(2p+n+i+3)_{m+n+p+2}$ to $(2p+m+n+i+2)_{m+n+p+2}$ along cycle C, $i=1,2,\cdots,n$.

The color set $S(v_{1i})$ contains just $n+p+2$ colors from $(p+n+i+1)_{m+n+p+2}$ to $(2p+2n+i+2)_{m+n+p+2}$ along cycle C, $i=1,2,\cdots,m$.

From the above discussions we can see that:

1. The number of the colors in each color set $C(x)$ is equal to the degree of vertex x;

2. The color sets of any two adjacent vertices with the same degree are different.

Thus the resulting coloring is a $(m+n+p+2)$-AVDPEC of $K_{m,n,p,1,1}$. The proof is completed.

Theorem 2.6 If $m>n>p>q>1$, then $\chi_{\mathrm{a}}^{'}(K_{m,n,p,q,1}) = m+n+p+q$.

Proof Obviously we have $\chi_{\mathrm{a}}^{'}(K_{m,n,p,q,1}) \geqslant \Delta(K_{m,n,p,q,1}) = m+n+p+q$. In order to prove $\chi_{\mathrm{a}}^{'}(K_{m,n,p,q,1}) = m+n+p+q$, we need only to give the $(m+n+p+q)$-AVDPEC of $K_{m,n,p,q,1}$. Suppose the colors that we will use are $1,2,\cdots,m+n+p+q$.

We assign color $(i)_{m+n+p+q}$ to edge $v_{51}v_{4i}$, $i=1,2,\cdots,q$.

We assign color $(q+i)_{m+n+p+q}$ to edge $v_{51}v_{3i}$, $i=1,2,\cdots,p$.

We assign color $(q+p+i)_{m+n+p+q}$ to edge $v_{51}v_{2i}$, $i=1,2,\cdots,n$.

We assign color $(q+p+n+i)_{m+n+p+q}$ to edge $v_{51}v_{1i}$, $i=1,2,\cdots,m$.

We assign color $(q+i+j)_{m+n+p+q}$ to edge $v_{4i}v_{3j}$, $i=1,2,\cdots,q;\ j=1,2,\cdots,p$.

We assign color $(q+p+i+j)_{m+n+p+q}$ to edge $v_{4i}v_{2j}$, $i=1,2,\cdots,q;\ j=1,2,\cdots,n$.

We assign color $(q+p+n+i+j)_{m+n+p+q}$ to edge $v_{4i}v_{1j}$, $i=1,2,\cdots,q;\ j=1,2,\cdots,m$.

We assign color $(2q+p+i+j)_{m+n+p+q}$ to edge $v_{3i}v_{2j}$, $i=1,2,\cdots,p;\ j=1,2,\cdots,n$.

We assign color $(2q+p+n+i+j)_{m+n+p+q}$ to edge $v_{3i}v_{1j}$, $i=1,2,\cdots,p;\ j=1,2,\cdots,m$.

We assign color $(2q+2p+n+i+j)_{m+n+p+q}$ to edge $v_{2i}v_{1j}$, $i=1,2,\cdots,n;\ j=1,2,\cdots,m$.

We construct a cycle C such that vertex set of C is $m+n+p+q$ colors $\{1,2,\cdots,m+n+p+q\}$ and edge set of C is $\{(i)_{m+n+p+q}(i+1)_{m+n+p+q}|\ i=1,2,\cdots,m+n+p+q\}$.

The color set $S(v_{51})$ contains just $m+n+p+q$ colors from $(1)_{m+n+p+q}$ to $(m+n+p+q)_{m+n+p+q}$ along cycle C.

The color set $S(v_{4i})$ contains just $m+n+p+1$ colors which are divided into two parts, the first part is $\{i\}$, the second part is from $(q+i+1)_{m+n+p+q}$ to $(m+n+p+q+i)_{m+n+p+q}$ along cycle C, $i=1,2,\cdots,q$.

The color set $S(v_{3i})$ contains just $m+n+q+1$ colors which are divided into two parts, the first part is from $(q+i)_{m+n+p+q}$ to $(2q+i)_{m+n+p+q}$ along cycle C, the second part is from $(p+2q+i+1)_{m+n+p+q}$ to $(p+2q+m+n+i)_{m+n+p+q}$ along cycle C, $i=1,2,\cdots,p$.

The color set $S(v_{2i})$ contains just $m+p+q+1$ colors which are divided into two parts, the first part is from $(q+p+i)_{m+n+p+q}$ to $(2p+2q+i)_{m+n+p+q}$ along cycle C, the second part is from $(2p+2q+n+i+1)_{m+n+p+q}$ to $(2p+2q+m+n+i)_{m+n+p+q}$ along cycle C, $i=1,2,\cdots,n$.

The color set $S(v_{1i})$ contains just $n+p+q+1$ colors from $(q+p+n+i)_{m+n+p+q}$ to $(2q+2p+2n+i)_{m+n+p+q}$ along cycle C, $i=1,2,\cdots,m$.

From the above discussions we can see that:

1. The number of the colors in each color set $C(x)$ is equal to the degree of vertex x;

2. The color sets of any two adjacent vertices with the same degree are different.

Thus the resulting coloring is a $(m+n+p+q)$-AVDPEC of $K_{m,n,p,q,1}$. The proof is completed.

Since $K_{m,n,p,q,r}$ has no two adjacent vertices of maximum degree when $m>n>p>q>r\geqslant 1$, by Lemma 1.4, we may obtain the following Theorem 2.7.

Theorem 2.7 If $m>n>p>q>r\geqslant 1$, then

$$\chi_{\mathrm{a}}^{'}(K_{m,n,p,q,r}) = m+n+p+q.$$

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