A Note on Fermionic Flows of the N=(1|1) Supersymmetric Toda Lattice Hierarchy

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Abstract

We extend the Sato equations of the N=(1|1) supersymmetric Toda lattice hierarchy by two new infinite series of fermionic flows and demonstrate that the algebra of the flows of the extended hierarchy is the Borel subalgebra of the N=(2|2) loop superalgebra.

In this note, we consider the integrable N=(1|1) supersymmetric generalization [1] of the two-dimensional bosonic Toda lattice hierarchy (2DTL hierarchy) [7]. It is given by an infinite system of evolution equations (flows) for an infinite set of bosonic and fermionic lattice fields evolving in two bosonic and two fermionic infinite "towers" of times. A subsystem of the 2DTL hierarchy involves an N=(1|1) supersymmetric integrable generalization of the 2DTL equation, which is called the N=(1|1) 2DTL equation.

Two new infinite series of fermionic flows of the N=(1|1) 2DTL hierarchy were constructed in [3] in a heuristic way by solving symmetry equations corresponding to the N=(1|1) 2DTL equation derived in [5]. This hierarchy was shown to actually have a higher symmetry, namely an $N = (2|2)$ supersymmetry [4, 2]. Together with the previously known bosonic flows of the N=(1|1) 2DTL hierarchy, these flows are symmetries of the N=(1|1) 2DTL equation.

The existence of those additional fermionic symmetries implies that the Lax pair or Sato equations proposed in [1] are incomplete because they lack the fermionic flows corresponding to these symmetries. Therefore, the following task arises: Can one construct a complete description of this hierarchy including the additional fermionic flows? Let us mention that a similar problem was partly discussed in [6] in a slightly different context.

This brief note addresses the above question. We extend the Sato equations of the paper [1] by new equations which indeed describe the additional fermionic flows constructed in [3]. With these, the complete algebra of flows of the extended hierarchy is confirmed.

Our starting point is the Sato equations of the N=(1|1) 2DTL hierarchy [1]

$$D_n^{\pm} W^{\mp} = \Big((L^{\pm})_*^n\Big)_{\pm} W^{\mp} - W^{\mp *(n)}(\Lambda^{\pm})^n,$$
$$D_n^{\pm} W^{\pm} = \Big((L^{\pm})_*^n\Big)_{\pm} W^{\pm}, \quad n \in \mathbb{N}, \tag{1}$$

where Lax operators $L^{\pm}$ and dressing operators $W^{\mp}$ are given by

$$L^{\pm} = (W^{\mp})^* \Lambda^{\pm} (W^{\mp})^{-1}, \quad \Lambda^{\pm} \equiv (\pm 1)^{j+1} e^{\pm\partial} \tag{2}$$

and

$$W^{\pm} = \sum_{k=0}^{\infty} w_{k,j}^{\pm} e^{\pm k\partial}, \quad w_{0,j}^{-} = 1, \tag{3}$$

respectively. Here, $w_{2k,j}^{\pm}$ ($w_{2k+1,j}^{\pm}$) are bosonic (fermionic) lattice fields ($j \in \mathbb{Z}$). The operator $e^{l\partial}$ ($l \in \mathbb{Z}$) is the discrete lattice shift which acts according to the rule

$$e^{l\partial} w_{k,j}^{\pm} \equiv w_{k,j+l}^{\pm} e^{l\partial}, \tag{4}$$

and the subscript + (−) distinguishes the part of an operator which includes operators $e^{l\partial}$ at $l \geq 0$ ($l < 0$). The symbols $D_{2n}^{\pm}$ and $D_{2n+1}^{\pm}$ denote bosonic and fermionic evolution derivatives, respectively. The subscripts $*$ and superscripts $*(n)$ and $*$ are defined according to the rules

$$(L^{\pm})_*^{2n} := ((L^{\pm})^* L^{\pm})^n, \quad (L^{\pm})_*^{2n+1} := L^{\pm}((L^{\pm})^* L^{\pm})^n,$$
$$(W^{\pm})^{*(2n)} := W^{\pm}, \quad (W^{\pm})^{*(2n+1)} := (W^{\pm})^*,$$
$$(W^{\pm}[w_{k,j}^{\pm}])^* := W^{\pm}[w_{k,j}^{\pm\,*}], \quad (w_{k,j}^{\pm})^* := (-1)^k w_{k,j}^{\pm}. \tag{5}$$

The flows (1) generates the Borel subalgebra of the N=(1|1) loop superalgebra [1]

$$\big[D_{2n}^{\pm}, D_k^{\pm}\big] = \big[D_n^{+}, D_k^{-}\big\} = 0, \quad \big\{D_{2k+1}^{\pm}, D_{2l+1}^{\pm}\big\} = 2D_{2(k+l+1)}^{\pm}. \tag{6}$$

The $N = (1|1)$ supersymmetric 2DTL equation belongs to the system of equations (1). Indeed, using (1) one can easily derive the following equations

$$D_1^{+} w_{0,j}^{+} = -(w_{1,j}^{-} + w_{1,j+1}^{-}) w_{0,j}^{+},$$
$$D_1^{-} w_{1,j}^{-} = (-1)^{j+1} \frac{w_{0,j}^{+}}{w_{0,j-1}^{+}}. \tag{7}$$

Then, eliminating the field $w_{1,j}^{-}$ from (7) and introducing the notation

$$v_{0,j} := (-1)^{i+1} \frac{w_{0,j}^{+}}{w_{0,j-1}^{+}} \tag{8}$$

we obtain the equation

$$D_1^{+} D_1^{-} \ln v_{0,j} = v_{0,j+1} - v_{0,j-1} \tag{9}$$

which reproduces the $N = (1|1)$ superfield form of the $N = (1|1)$ 2DTL equation.

We propose to extend consistently the flows (1) by the following two new infinite series of fermionic flows,

$$\widehat{\mathcal{D}}^{\pm}_{2k+1} W^{\mp} = \Big((\widehat{L}^{\pm})^{2k+1}_{*} \Big)_{\pm} W^{\mp} - W^{\mp *} (\Pi^{\pm})^{2k+1},$$
$$\widehat{\mathcal{D}}^{\pm}_{2k+1} W^{\pm} = \Big((\widehat{L}^{\pm})^{2k+1}_{*} \Big)_{\pm} W^{\pm}, \quad k \in \mathbb{N}, \tag{10}$$

where

$$\widehat{L}^{\pm} = (W^{\mp})^{*} \Pi^{\pm} (W^{\mp})^{-1}, \quad \Pi^{\pm} \equiv (\mp 1)^{j} e^{\pm \partial} \tag{11}$$

and $\widehat{\mathcal{D}}^{\pm}_{2k+1}$ denote new fermionic evolution derivatives. The calculation of their algebra yields

$$\Big\{ \widehat{\mathcal{D}}^{+}_{2k+1}, \widehat{\mathcal{D}}^{-}_{2l+1} \Big\} = 0, \quad \Big\{ \widehat{\mathcal{D}}^{\pm}_{2k+1}, \widehat{\mathcal{D}}^{\pm}_{2l+1} \Big\} = 2(-1)^{k+l+1} D^{\pm}_{2(k+l+1)},$$
$$\Big[\widehat{\mathcal{D}}^{\pm}_{2k+1}, D^{\pm}_{l} \Big\} = \Big[\widehat{\mathcal{D}}^{\pm}_{2k+1}, D^{\mp}_{l} \Big\} = 0, \tag{12}$$

which, together with (6), form the Borel subalgebra of the N=(2|2) loop superalgebra. Hence, the extended supersymmetric hierarchy with the flows (1) and (10) can be called the N=(2|2) supersymmetric 2DTL hierarchy. Thus, we have finally established the origin of the fermionic symmetries observed in [3]: they are exactly the flows (10) of the extended hierarchy.

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