Existence of high energy solutions for Kirchhoff-type equations

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Abstract. In this paper, by applying the fountain theorems, we study the existence of infinitely many high energy solutions for the nonlinear kirchhoff nonlocal equations under the Ambrosetti-Rabinowitz type growth conditions or no Ambrosetti-Rabinowitz type growth conditions, infinitely many high energy solutions are obtained.

Introduction and Main Results

Recently, many authors studied Kirchhoff type problems, some important and interesting results can be found in [1-8]. In this paper, we study the following Kirchhoff-type problems

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2 dx)\Delta u + q(x)u = f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Where Ω is a smooth bounded domain in $R^N (N=1,2 \text{ or } 3)$, $a,b>0$, and $f:\Omega\times R\to R$ is continuous function.

Set $F(x,u)=\int_0^u f(x,s)ds$, Then a weak solution of problem (1) is a critical point of the following functional :

$$\Phi(u)=\frac{a}{2}\int_\Omega |\nabla u|^2 dx+\frac{b}{4}(\int_\Omega |\nabla u|^2 dx)^2+\frac{1}{2}\int_\Omega q(x)u^2 dx-\int_\Omega F(x,u)dx.$$

In order to establish multiple solutions for problem (1), we make the following assumptions:

(L_1) $\Omega\subset R^N$ is bound and open and, $q\in L^\infty(\Omega)$ and $q(x)\ge 0$ a.e. in Ω.

(L_2) $f\in C(\overline{\Omega}\times R,R)$ and for some $4<p<2^*=\begin{cases} 2n/n-2 & n\ge 3 \\ +\infty & n=1,2, \end{cases}$ such that $|f(x,u)|\le C(1+|u|^{p-1})$

(L_3) $\lim_{|u|\to\infty}(f(x,u)u/|u|^4)=+\infty.$

(L_4) There exists $\mu>4,\ R>0$, such that $|u|\ge R\Rightarrow 0<\mu F(x,u)\le uf(x,u).$

(L'_4) There is $c_*\ge 0,\theta\ge 1, H(x,t)\le\theta H(x,s)+c_*.$ for all $0<t<s,\ \forall\ x\in\Omega$, where $H(x,u)=uf(x,u)-4F(x,u)$

(L_5) $f(x,-u)=-f(x,u).$

Before stating our main results, we first introduce some preliminary nations. Let E be a Banach space with the norm $\|\bullet\|$ and $E=\overline{\bigoplus_{j\in N} X_j}$ with $\dim X_j<\infty$ for any $j\in N$ .Set $Y_k=\bigoplus_{j\in 0}^{k} X_j$; $Z_k=\overline{\bigoplus_{j=k}^{\infty} X_j}$ and $B_k=\{u\in Y_k:\|u\|\le\rho_k\}, N_k=\{u\in Y_k:\|u\|\le\gamma_k\}$ for $\rho_k>\gamma_k>0.$

- theorem**1** ([1] Fountain theorem). Let $\varphi\in C^1(E,R)$ be a even functional. If for every $k\in N$, there exist $\rho_k>\gamma_k>0.$ such that

(A_1) $a_k = \max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \le 0$; (A_2) $b_k = \inf_{u\in Z_k, \|u\|=\gamma_k} \varphi(u) \to \infty, k \to \infty$; (A_3) φ satisfies the $(C)_c$ condition for every $c > 0$,then φ has an unbounded sequence of critical values.

Definition 1 Let $\Phi \in C^1(E,R)$,we say that Φ satisfies the cerami condition at the level $c \in R$,if any sequence $\{u_n\} \subset X$ .along with

$\Phi(u_n) \to c$ and $(1+\|u_n\|)\Phi'(u_n) \to 0,$ as $n \to \infty.$

possesses a convergent subsequence; Φ satisfies the (C) condition if Φ satisfies $(C)_c$ for all $c \in R$.

In this paper, we consider $E = H_0^1(\Omega)$ endowed with the norm $\|u\|_E = (\int_\Omega |\nabla u|^2 dx + \int_\Omega q(x)u^2 dx)^{\frac{1}{2}}$.

$L^P(\Omega)$ denotes the usual Lebesgue space with the norm $|u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$. Since Ω is a bounded domain, it is well known that $E \hookrightarrow L^p(\Omega)$ continuously for $p \in [1, 2^*]$,and compactly for $p \in [1, 2^*]$. Hence,for $p \in [1, 2^*]$, there exists γ_p such that $|u|_p \le \gamma_p \|u\|_E$, $\forall u \in E.$

The main results of this paper are the following:

Theorem 2 Suppose that $(L_1)(L_2)(L_4)(L_5)$ hold, Then problem (1) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \to \infty,$ as $n \to \infty.$

Theorem 3 Suppose that $(L_1)-(L_3)(L'_4)(L_5)$ hold, Then problem (1) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \to \infty,$ as $n \to \infty.$

Proofs of Theorems

Proof of Theorem 2 (i) Let $\{u_n\} \subset E$ be a $(C)_c$ sequence of $\Phi(u)$. Then for n large enough, we have

$$C(1+\|u_n\|) \ge \Phi(u_n) - \frac{1}{\mu}\Phi'(u_n)u_n \qquad = a(\frac{1}{2}-\frac{1}{\mu})\int_\Omega |\nabla u_n|^2 dx + b(\frac{1}{4}-\frac{1}{\mu})(\int_\Omega |\nabla u_n|^2 dx)^2$$

$$+(\frac{1}{2}-\frac{1}{\mu})\int_\Omega q(x){u_n}^2 dx - \int_\Omega F(x,u_n)dx + \frac{1}{\mu}\int_\Omega f(x,u_n)u_n dx.$$

$$\ge \min\{a,1\}(\frac{1}{2}-\frac{1}{\mu})\|u\|_E^2 + \frac{1}{\mu}[\int_\Omega -\mu F(x,u_n)dx + \int_\Omega f(x,u_n)u_n dx].$$

we obtain from (L_2) that

$$\int_\Omega [-\mu F(x,u_n) + f(x,u_n)u_n]dx \ge \int_{|u_n|\le R} [-\mu F(x,u_n) + f(x,u_n)u_n dx \ge -C_1.$$

we have $C(1+\|u_n\|) \ge \min\{a,1\}(\frac{1}{2}-\frac{1}{\mu})\|u_n\|_E^2 - C_1.$

which is course implies that $\{u_n\}$ is bounded. We have $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $u_n \to u$ in $L^p(\Omega)$. Then Theorem A.2[1] (willem 1996) implies that $f(x,u_n) \to f(x,u)$ in $L^q(\Omega), q = p/p-1$ .we next prove that $\{u_n\}$ has a convergent subsequence obverse that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = a\int_\Omega |\nabla(u_n-u)|^2 dx + b\int_\Omega |\nabla u_n|^2 dx \int_\Omega \nabla u_n \nabla(u_n-u)dx + b\int_\Omega |\nabla u|^2 dx \int_\Omega \nabla u \nabla(u_n-u)dx$$

$$+b\int_\Omega |\nabla u|^2 dx \int_\Omega \nabla u \nabla(u-u_n)dx + \int_\Omega q(x)|u_n-u|^2 dx - \int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx$$

$$\ge \min\{a,1\}\|u_n-u\|_E^2 + b\int_\Omega (|\nabla u_n|^2 - |\nabla u|^2)dx \int_\Omega \nabla u \nabla(u_n-u)dx$$

$$-\int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx$$

using the boundedness of $\{u_n\}$ and $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, one has

$$b\int_\Omega (|\nabla u_n|^2 - |\nabla u|^2)dx \int_\Omega \nabla u \nabla (u_n - u)dx \to 0, \text{ as } n \to \infty.$$

It follows from the Höld inequality that

$\int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \le |f(x,u_n) - f(x,u)|_q |u_n - u|_p$ .as $n \to \infty$.

Thus we deduce that $\|u_n - u\|_E \to 0$ as $n \to \infty$ .Hence Φ satisfies the $(C)_c$ condition for every $c > 0$.

(ii) By (L_4) ,we have $F(x,u) \ge C(|u|^\mu - 1)$.

$$\Phi(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}(\int_\Omega |\nabla u|^2 dx)^2 + \frac{1}{2}\int_\Omega q(x)u^2 dx - \int_\Omega F(x,u)dx$$

$$\le \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}(\int_\Omega |\nabla u|^2 dx)^2 + \frac{1}{2}\int_\Omega q(x)u^2 dx - C\int_\Omega |u|^\mu dx + \int_\Omega C dx$$

$$\le \max\{a,1\}\|u\|_E^2 + \frac{b}{4}\|u\|_E^2 - C\int_\Omega |u|^\mu dx + C\Omega.$$

Since on the finite-dimensional space Y_k all norms are equivalent, there for (A_1) is satisfied for every $\rho_k > 0$ large enough.

(iii) We next verify condition (A_3) ,To this end,from (L_2) ,we have $|F(x,u)| \le C_1(1 + |u|^p)$.

Define $\beta_k = \max_{u \in Z_k, \|u\|_E = 1} |u|^p$. so that on Z_k , we have

$$\Phi(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}(\int_\Omega |\nabla u|^2 dx)^2 + \frac{1}{2}\int_\Omega q(x)u^2 dx - \int_\Omega F(x,u)dx$$

$$\ge \frac{1}{2}\min\{a,1\}\|u\|_E^2 - C_1|u|_p^p - C_2 \ge \frac{1}{2}\min\{a,1\}\|u\|_E^2 - C_1\beta_k^p\|u\|_E^p - C_2.$$

Take $\gamma_k = (C_1\rho\beta_k^p / \sqrt{\min\{a,1\}/2})^{\frac{1}{2-p}}$, ,since $\beta_k \to 0, k \to \infty$. we obtain for $u \in Z_k$, with $\|u\|_E = r_k$.

$$\Phi(u) \ge \frac{1}{2}\min\{a,1\}(C_1\rho\beta_k^p / \sqrt{\min\{a,1\}/2})^{\frac{2}{2-p}} - C_2 \to \infty, \text{ as } k \to \infty.$$

So condition (A_2) is proved. Now all conditions of Theorem 1 hold. Therefore, problem(1) has a sequence of solutions $\{u_n\}$, $\Phi(u_n) \to \infty$, as $n \to \infty$.

Proof of Theorem 3 (i) We verify Φ satisfies $(C)_c$ for all $c \in R$, such that

$$\Phi(u_n) \to c \text{ and } (1 + \|u_n\|)\Phi'(u_n) \to 0, \text{ as } n \to \infty. \quad (2)$$

we first show that $\{u_n\}$ is bound, which is our main propose. Indeed, suppose that $\|u_n\|_E \to \infty$,

setting $v_n = u_n / \|u_n\|_E$,then $\|v_n\|_E = 1$, so we can suppose that

$v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \to v$ in $L^p(\Omega)$, $v_n \to v$ a.e. $x \in \Omega$.

Next we consider the two possible cases:(1) $v \ne 0$, (2) $v = 0$.

In case (1), we have $o(1) = \langle \Phi'(u_n), u_n \rangle$,

$$\int_\Omega (f(x,u_n)u_n / \|u_n\|_E^4)dx \le \max\{a,1\} / \|u_n\|_E^4 + b - o(1) \le C < +\infty. \quad (3)$$

On the other hand,on the set $\sum = \{x \in \Omega \mid v(x) \ne 0\}$. $f(x,u_n)u_n / \|u_n\|_E^4 = (f(x,u_n)u_n / u_n^4)v_n^4 \to +\infty$, as $n \to \infty$.

by (L_2) for any $M > 0$, ,there exist $C_1 > 0$, such that $f(x,u_n) \ge M|u|^3 - C_1$, for a.e. $x \in \Omega$,and $n \in N$.

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)u_n / \|u_n\|_E^4 dx = \lim_{n\to\infty}\int_\Omega (f(x,u_n)u_n / u_n^4)v_n^4 dx \ge \lim_{n\to\infty}(M\int_\Omega v_n^4 dx - (C_1 / \|u_n\|_E^3)\int_\Omega v_n dx = M\int_\Omega v^4 dx.$$

Since $m\sum > 0$, we infer that $\lim_{n\to\infty}\int_\Omega (f(x,u_n)u_n / \|u_n\|_E^4 dx) \to \infty$, as $n \to \infty$.which contradicts to (3).

In case (2),we can define $\Phi(t_n u_n) = \max_{t\in[0,1]} \Phi(tu_n)$, For any give $R > 0$, define $\bar{v}_n = R(u_n / \|u_n\|_E) = Rv_n$, from (L_2),we have

$$\int_\Omega F(x,\bar{v}_n) \le C\int_\Omega |\bar{v}_n| dx + C\int_\Omega |\bar{v}_n|^p dx \to 0. \quad \text{as} \quad n\to\infty.$$

$$\Phi(t_n u_n) \ge \frac{a}{2}\int_\Omega |\nabla \bar{v}_n|^2 dx + \frac{b}{4}(\int_\Omega |\nabla \bar{v}_n|^2 dx)^2 + \frac{1}{2}\int_\Omega q(x)\bar{v}_n^2 dx - \int_\Omega F(x,\bar{v}_n)dx$$

$$\ge \frac{1}{2}\min\{a,1\}\|\bar{v}_n\|_E^2 + \frac{b}{4}(\int_\Omega |\nabla \bar{v}_n|^2 dx)^2 - \int_\Omega F(x,\bar{v}_n)dx \ge CR.$$

which means that $\lim_{n\to\infty}\Phi(t_n u_n) = \infty$. By the definition of t_n ,we see that $o(1) = \langle \Phi'(u_n), u_n\rangle$ consequently, By (L'_4) ,we have

$$\infty \leftarrow 4\Phi(t_n u_n) - \langle \Phi'(t_n u_n), t_n u_n\rangle = a\int_\Omega |\nabla(t_n u_n)|^2 dx + \int_\Omega q(x)(t_n u_n)^2 dx + \int_\Omega [f(x,t_n u_n)t_n u_n - 4F(x,t_n u_n)]dx$$

$$\le a\int_\Omega |\nabla u_n|^2 dx + \int_\Omega q(x)u_n^2 dx + \theta\int_\Omega [f(x,u_n)u_n - 4F(x,u_n)]dx + C_*$$

$$\le \theta[4\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle] + C_* < C.$$

as $n\to\infty$, which is a contradiction. This proves that $\{u_n\}$ is bounded.

$u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $u_n \to u$ in $L^p(\Omega)$, $u_n \to u$ a.e. $x\in\Omega$.

By Theorem 1 (i) We have $\|u_n - u\|_E \to 0$, as $n\to\infty$. Thus $u_n \to u$ strongly in $H_0^1(\Omega)$, which means that Φ satisfies $(C)_c$.

(ii) By (L_3) ,we have $F(x,u) \ge M|u|^4 - C_1|u|$,

$$\Phi(u) \le \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}(\int_\Omega |\nabla u|^2 dx)^2 + \frac{1}{2}\int_\Omega q(x)u^2 dx - M\int_\Omega |u|^4 dx + C_1\int_\Omega |u| dx$$

$$\le \max\{a,1\}\|u\|_E^2 + (b/4)\|u\|_E^2 - MC\|u\|_E^4 + C_1\int_\Omega |u| dx.$$

Since on the finite-dimensional space Y_k all norms are equivalent,we take M large enough, such that $b/4 - MC < 0$, there for (A_1) is satisfied for every $\rho_k > 0$ large enough.

By Theorem 1 (iii), condition (A_2) is proved. Now all conditions of Theorem 1 hold, therefore, problem (1) has a sequence of solutions $\{u_n\}$, such that $\Phi(u_n) \to \infty$, as $n\to\infty$.

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