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What Construct of Mathematical Knowledge for Teaching do Mathematics Teachers Need? (A Theoretical Framework and Conceptualization in Geometry)

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***Abstract*—Mathematics is a science related to concepts of knowledge which are very important. In learning mathematics, the conceptual knowledge and mathematics knowledge for teaching (MKT) are the two fundamental things and cannot be separated from one another. This article has reviewed the literature on mathematical knowledge for teaching based conceptual (MKT-C) and developed a theoretical framework for such constructs that Ball and Bloom have developed. MKT-C in the material content knowledge domain includes common content knowledge based conceptual (CCK-C), specialized content knowledge based conceptual (SCK-C) and knowledge at the mathematical horizon based conceptual (KMH-C). While MKT-C in the pedagogic domain includes: knowledge of content and students based conceptual (KCS-C), knowledge of content and teaching based conceptual (KCT-C), and knowledge of content and curriculum-based conceptual (KCC-C), this study also highlights how the interaction of components in the construct is. This theoretical framework is expected to demonstrate how to construct teachers' knowledge of conceptual-based teaching.**

Keywords—conceptual knowledge, mathematics knowledge for teaching, mathematical knowledge for teaching based conceptual

I. INTRODUCTION

Many types of research have explained the constructs of essential knowledge to the development of mathematics teaching. However, teaching will not be effective when the teacher teaches in accordance with the curriculum contained in the textbook; for example, teachers only teach the chapters in the textbook with the limited time allocation. Next, during the twentieth century, conceptual knowledge continued to evolve with increasing knowledge and information available [1]. Conceptual knowledge is an important component in mathematics [2]. Conceptual knowledge involves understanding the relationship between concepts and principles including the existing schema behind concepts [3]. The conceptual knowledge is rich with relationships and refers to the basic constructs of mathematics and the relationship between ideas that describe mathematical procedures and gives meaning [4]. In other words, conceptual knowledge is the knowledge that binds information that was once fragmented into a relatively complete net.

If teachers use conceptual knowledge in teaching, the teachers will be able to see the learning objectives and relationships between concepts clearly. However, there are still gaps that teachers experience in transferring and using their conceptual knowledge to teach math materials. It means there is no need to follow and repeat the old (traditional) pattern of using only the usual teaching model. It is said so because it is less benefit in the process of improving the quality of mathematics learning. Therefore, a framework model is needed to develop teachers' knowledge of mathematics learning potentially.

The concept of mathematics has its own substance and reasoning model. Therefore the designed mathematical learning framework should take account of these characteristics. The question is what knowledge do the teachers need in helping students to learn math? [5] suggest a combination of material content knowledge and pedagogical knowledge. Furthermore, Ball et al. call it a mathematical knowledge framework for teaching (MKT).

There are several reasons why this is so important to study. Relating to this conceptual knowledge, it should place a great focus on improving teachers' knowledge. Therefore, the theoretical framework that synergizes conceptual knowledge by Bloom and mathematical knowledge for teachers to teach by [6] will add the development of theories on teacher professional development. This theoretical framework model is expected to be adopted as a representation of mathematical knowledge models for teaching based conceptual by mathematics teachers.

The utilization of Ball and Bloom theory is very important in answering the following problems: (1) why mathematical knowledge is required for teaching based conceptual, and (2) how is the formulation of mathematical knowledge for teaching based conceptual?

II. What Is Mathematical Knowledge for Teaching (MKT) and Conceptual Knowledge?

A. *Mathematical Knowledge for Teaching (MKT)*

Every talk about teachers will be related to the ability of material mastery and pedagogic mastery. Teachers without mastering the material will make students' understanding minimal, and the delivery of material to students will not run smoothly when teachers do not have the good pedagogic ability. Thus, any discussion of teachers will be related to the ability of matter and pedagogy mastery. The ability of matter and pedagogic mastery is like two sides of a coin and is one unit that is inseparable from one another.

Mathematics is a universal science that has an important role in various disciplines and advances the power of human thought. In mathematics, the basic object studied is abstract. These objects are objects of thought, including facts, concepts, operations, and principles, so that teachers must master the content material taught and how to teach mathematics in learning activities. For this reason, teachers must continue to integrate their experience with everything they understand, including an understanding of teaching. This means that whatever a teacher knows about teaching is important and will provide new experiences and influence all understanding.

Shulman [7] and Ball & Bass [8] state that being a teacher is not enough to have only content knowledge to produce effective mathematics teaching. Shulman [7] describes three main components of knowledge that must be possessed by the teacher namely subject-matter knowledge, pedagogical knowledge, and curricular knowledge. Furthermore, subject-matter knowledge is the number of knowledge and organization that is in the teachers' mind. Pedagogical knowledge is the most widely used in presenting ideas, the most powerful analogy, illustrations, examples, explanations, and demonstrations, or in other words a way of presenting and formulating a subject so that the subject is unity with others. Meanwhile, curricular knowledge is a set of characteristics that serve indications and contraindications to be used in a particular curriculum or program material in a certain condition. Fennema & Franke [9] present models that focus on teacher knowledge in the context of the class including context interaction with knowledge of mathematics, knowledge of learners' cognition and pedagogical knowledge.

The purpose of learning mathematics is to train and improve a systematic, logical, critical, creative, and consistent way of thinking and to develop a persistent and confident attitude in solving problems, so the duty that is assumed by the mathematics teachers is to develop students 'thinking creativity that can improve students' thinking ability and can improve the ability to construct a new knowledge as an effort to improve a good mastery of mathematics materials.

According to [5] mathematical knowledge for teachers to teach is focused on two domains, namely material content knowledge and pedagogical content knowledge. The formulation of material content knowledge lies in three subdomains namely, CCK, SCK, and KMH while pedagogic content knowledge is formulated on three subdomains namely, KCS, KCT, and KCC.

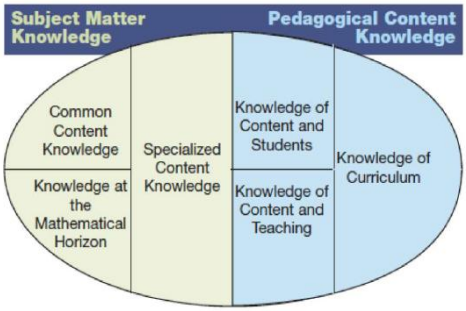


Fig. 1. Mathematical Knowledge for Teaching (MKT) [5], [8]

Common content knowledge is the knowledge used in teaching work in the same way as it is used in other professions or jobs that also use mathematics. Specialized Content Knowledge is a knowledge that allows teachers to engage in specific tasks to teach knowledge that is not needed in a profession outside of teaching. Knowledge at the Mathematical Horizon is knowledge of mathematics outside the taught curriculum. Knowledge of content and student is a knowledge that integrates knowledge of students and knowledge of mathematics. This knowledge is a mathematical knowledge and knowledge of student's thinking, indicated by the teacher's ability to recognize improper student's responses and to anticipate student's misconceptions [10], [11]. Knowledge of content and teaching is a knowledge that integrates the knowledge of learning and mathematical knowledge. This knowledge is the knowledge of mathematics and teaching knowledge, indicated by the teacher's decision in terms of sequencing the learning activities and material examples presented, the teacher's awareness of the strengths and weaknesses of representation used at the time of teaching, or using the student's opinions to propose a mathematical opinion [11], [12]. Knowledge of content and curriculum is a knowledge that integrates teacher's knowledge about mathematical content and curriculum. This knowledge is knowledge of learning objectives for different grade levels and knowledge of teaching materials [11].

However, there are still some problems in the unfinished categorization of Ball's framework under review [13]–[16]. First, the problem of separation between components of knowledge that allows overlap [16]. Second, does MKT accommodate the sections between sets of content knowledge or a combination of knowledge content with other knowledge? MKT is finally a framework for researchers and education practitioners, in an effort to package and develop learning models in order to achieve better learning goals through the process. Knowledge of material content and pedagogical content knowledge should be collected in a teacher. In addition, MKT is no longer considered merely a theory of education but is a way of looking at how teachers can develop professional knowledge in their practice.

B. *Conceptual KnowledgeMaintaining the Integrity of The Specifications*

Conceptual knowledge is rich with relationships and refers to the basic constructs of mathematics and the relationship between ideas that describe mathematical procedures and gives meaning [4]. In other words, conceptual knowledge is the knowledge that binds information that once was fragmented into the relatively

complete net. Thus, the unit of conceptual knowledge is not stored in isolated information but is part of a net.

The study of teachers' knowledge shows findings that they do not have conceptual knowledge about many mathematical topics at school [8]. The study shows that teachers are able to carry out learning and mathematical procedural tasks well, but they do not understand the underlying concepts and principles. It means that teachers' conceptual knowledge of mathematical topics is low.

According to the author, learning mathematics is not only transferring knowledge, but the students must also be the subjects in the learning process. In learning mathematics, there must be an interconnection between one concept to another, and a mastery of a concept becomes a prerequisite for other concepts. A student must master one material then (s) he can continue the other material because if a student has not mastered the previous material, then the student will have difficulty in following the next material. For that reason, conceptual knowledge is very important knowledge for developing mathematical knowledge to teach.

In the theoretical context of Anderson & Krathwohl [1], conceptual knowledge is the knowledge that shows the interconnection between the basic elements in the larger structure and everything functions together. Conceptual knowledge consists of three types, namely category and classification, principle and generalization, as well as theory, model and structure. Category is a system that helps explore related topics while classification is a division according to certain classes, for instance: facts, concepts, principles, and operations.

Classification is also the process of grouping objects based on similarities and differences. Furthermore, knowledge of category and classification includes knowledge of category, class, section or arrangement that is applicable in a particular field of knowledge. Knowledge of category and classification is very important knowledge because this knowledge also becomes the basis for teachers to classify information and knowledge, for instance: example and counter example, partitional and hierarchical, etc. Without the ability to categorize and classify the teachers will have difficulty in learning.

In the structure of conceptual knowledge, aspects of category and classification are the foundations for principles and generalizations and theories, models and structures. Furthermore, conceptual knowledge is closely related to cognitive. Cognitive is a mental activity in obtaining, processing, organizing and using knowledge. Cognitive process is a mental activity in obtaining, processing, organizing and using knowledge.

Anderson & Krathwohl [1] explain that the cognitive process in conceptual knowledge includes: remembering, understanding, applying, analyzing, evaluating and creating as in Figure 2.

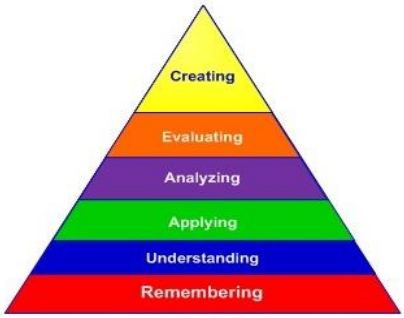
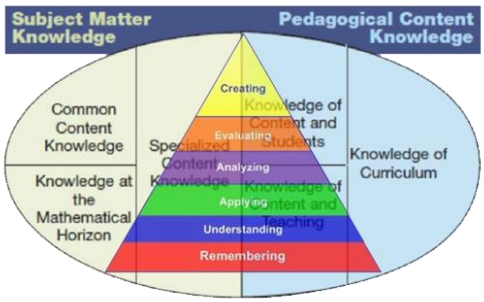


Fig. 2. The dimension of the cognitive process of taxonomy Bloom [1]

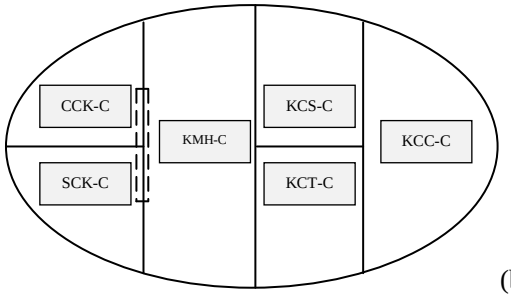
Remembering an aspect of thinking is the ability to recall information/ knowledge stored in memory. Understanding is the ability to understand the instructions and affirm the meaning of the ideas or concepts that have been described in the form of either oral, written, or graph / diagram. Applying is the ability to do something and apply the concept in a certain situation. Analyzing is the ability to separate concepts into some components and connect with one another to gain an understanding or concept as a whole. Evaluating is the ability to set the degree of things based on certain norms, criteria or benchmarks. Creating is the ability to integrate elements into a new form that is intact and coherent or make something original.

C. A Theoretical Framework and Conceptualization of MKT Based Conceptual (MKT-C)

Conceptual knowledge includes a subset of mathematical knowledge for teaching. Thus, if the mathematical knowledge for teaching uses a conceptual knowledge base, then the components should pay attention to and lead to both knowledges. This relation can be described schematically as shown in Figure 3 (a) and 3 (b).



(a)



(b)

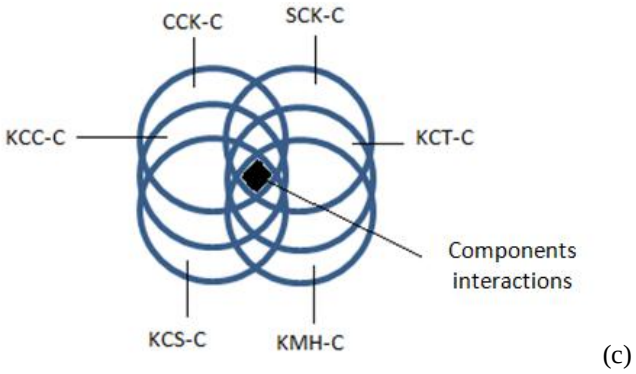


Fig. 3. The proposed MKT-C hypothetical framework (a) MKT-C pada aspek proses kognitif(a) MKT-C on aspects of cognitive process; (b) MKT-C pada aspek kategori dan klasifikasi(b) MKT-C on aspects of category and classification; (c) Interaction of MKT-C components

CCK-C : common content knowledge based conceptual
SCK-C : specialized content knowledge based conceptual
KMH-C : knowledge at the mathematical horizon based conceptual
KCS-C : knowledge of content and students based conceptual
KCT-C : knowledge of content and teaching based conceptual
KCC-C : knowledge of content and curriculum based conceptual
⊏⊐ : interaction of each component CCK-C, SCKC, KMH-C, KCS-C, KCT-C, KCC-C

CCK-C is a mathematical knowledge and skill possessed by teachers that are used for various situations or not specific to teach related to classification, category, and cognitive process. SCK-C is a unique mathematical knowledge and skill used by teachers for teaching objectives related to classification, category, and cognitive process. KMH-C is a teacher's knowledge of advanced mathematical topics related to classification, category, and cognitive process. KCS-C is a knowledge that integrates knowledge of students and knowledge of mathematics related to classification, category, and cognitive process. KCT-C is a knowledge that integrates learning knowledge and mathematical knowledge related to classification, category, and cognitive process. KCC-C is a knowledge that integrates teachers' knowledge of mathematical content and curriculum related to classification, category, and cognitive process.

The interaction of each component (Figure 3c) should support the learning objectives of mathematics and be done with an integrated approach. Each component (Figure 3a) is hierarchical, and there are interactions, whereas (Figure 3b) components of the category and classification aspects are nonlinear and between those components, there is an interaction. This is in line with Lappan's (2000) view that the (domain) component of the teacher's knowledge is nonlinear. Furthermore, Lappan [17] also claims that effective learning occurs in the interaction of knowledge domains.

III. Description of MKT-C in Geometry

The following is a description of MKT-C construct in the geometry context. Geometry is a material that is difficult for students to understand and is not easily taught by teachers [18]–[22]. For example, the definition is as an expression that limits a concept. There are basically two challenges for teachers: (1) how to define the concept of a parallelogram, rectangle, rhombus, square, trapezoid, and kite correctly, and (2) how to create a rectangular concept map showing the relationship among parallelogram, rectangular, rhombus, square, trapezoid and kite. From the teacher's perspective, this is a problem that is not easy for teachers. To overcome this obstacle, teachers need to have mathematical knowledge based on conceptual knowledge.

A. CCK-C

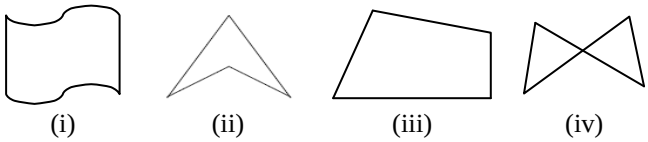
In learning quadrangle material, Mr. Rudi as a mathematics teacher of junior high school taught about the elements of the quadrangle. The elements were: side, angle and diagonal. At the end of the lesson, Mr. Rudi gave the task of measuring students' understanding of the characteristics of parallelogram, rectangle, rhombus, square, trapezoid and kite based on quadrangle elements. According to your knowledge, write down the characteristics of parallelogram, rectangle, rhombus, square, trapezoid, kite.

B. SCK-C

Mr. Baco as a mathematics teacher of junior high school would teach a quadrangle. Quadrangle in question was: parallelogram, rectangle, rhombus, square, trapezoid, and kite. Write the definition of a quadrangle according to your knowledge?

C. KMH-C

In the reference books, it can be found images of the following geometry structures.



Check each image above, which includes a quadrangle by giving a tick (√) in the image. Give your reasons.

In the class, Mrs. Wa Ode asked students to define rectangles. Students were expected to generalize the rectangle as a parallelogram. Four students gave the following answers.

Student	Response
Rezkiyanti	The rectangle is a quadrangle in which two pairs of opposite sides have the same length, and all angles are right angles
Agung	Rectangle is a quadrangle in which two pairs of opposite sides have length, and one angle is a right angle
Fita	The rectangle is a quadrangle in which a pair of opposite sides are parallel and have the same length, and one angle is a right angle
Muaz	Rectangle is a quadrangle in which a pair of opposite sides are parallel and have the same length, and all angles are right angles

How is the answer given by each student?

D. KCT-C

Mr. Amrin taught quadrangle topic. As an exercise, he gave questions and asked the students to answer by ticking (√) on the existing box.

Statement	Correct	Wrong
Square is a rectangle	□	□
Square does not include rectangles	□	□
Square is a rectangle whose four sides are the same length, and all angles are right angles	□	□

Which statement is correct and which statement is wrong? Tell your reasons.

E. KCC-C

According to their experience teaching quadrangle material, create a schema or concept map of learning objectives for quadrangle material according to school curriculum!

The examples of CCK-C, SCK-C, and KMH-C above show that mathematics teachers need well-organized knowledge between mathematical content taught and mathematical content that the teachers have. For this purpose, they must be able to understand the concepts of mathematics and make connections between elements and definitions of mathematical concepts and vice versa. The examples of KCS-C, KCT-C, and KCC-C show that mathematics teachers should also be able to teach mathematical materials.

IV. SUMMARY

Conceptual knowledge is encompassed in facts, concepts, principles, and skills so that conceptual knowledge needs to be integrated into learning. The integration of conceptual knowledge through mathematical knowledge for teaching is called MKT-C. MKT-C is needed by mathematics teachers because teachers are the main determinants of the success of students' mathematics learning. Theoretical frameworks of MKT-C developed include material content domains consisting of CCK-C, SCK-C, KMH-C, while the pedagogical domains consist of KCS-C, KCT-C, and KCC-C. The theoretical framework of MKT-C is a very important element for teachers' professional development, so it needs to be applied and studied further. This theoretical framework can contribute to the science of mathematical knowledge for teaching based conceptual for a theoretical model.

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