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Perturbation Methods for Maximum Principle in Optimal Control Problems

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***Abstract*—Iterative methods for searching of extreme controls (satisfying the maximum principle), which are based on the theory and methods of perturbation of necessary conditions for optimality, are suggested. The methods are characterized by computationally stable alternating solution of phase and conjugate systems of variables and the absence of labor-intensive operations of convex or needle variation of the improving control.**

Keywords—optimal control problem; conditions of the maximum principle; perturbation methods

I. INTRODUCTION

We consider the optimal control problem

$$\Phi(u) = \varphi(x(t_1)) + \int_T F(x(t),u(t),t)dt \to \min_{u\in V} \tag{1}$$

$$\dot{x}(t) = f(x(t),u(t),t),\ x(t_0) = x^0 \tag{2}$$

$$u(t)\in U,\ t\in T=[t_0,t_1],$$

wherein $x(t)=(x_1(t),...,x_n(t))$ - state vector, $u(t)=(u_1(t),...,u_m(t))$ - control vector. As admissible controls, we consider the set V of piecewise continuous functions on T with values in a convex compact set $U\subset R^m$: $V=\{u\in PC(T): u(t)\in U, t\in T\}$. The initial state x^0 and the control interval T are fixed.

The following conditions are assumed:

1) function $\varphi(x)$ continuously differentiable on R^n , function $F(x,u,t)$, vector function $f(x,u,t)$ and their derivatives $F_x(x,u,t)$, $F_u(x,u,t)$, $f_x(x,u,t)$, $f_u(x,u,t)$ continuous in terms of the variables (x,u,t) on the set $R^n\times U\times T$;

2) function $f(x,u,t)$ satisfies the Lipschitz condition by x at $R^n\times U\times T$ with constant $L>0$

$$\|f(x,u,t)-f(y,u,t)\|\le L\|x-y\|.$$

For admissible control $v\in V$ denote $x(t,v),\ t\in T$ - solution of system (2). Conditions guarantee the existence and uniqueness of the solution $x(t,v)$, $t\in T$ system (2) for any admissible control $v\in V$.

We consider the Pontryagin function with conjugate variable $\psi\in R^n$

$$H(\psi,x,u,t)=\langle f(x,u,t),\psi\rangle - F(x,u,t).$$

For admissible control $v\in V$ denote $\psi(t,v),\ t\in T$ - solution of standard conjugate system

$$\dot{\psi}(t) = -H_x(\psi(t),x(t),u(t),t),\quad t\in T \tag{3}$$

$$\psi(t_1) = -\varphi_x(x(t_1))$$

for $u(t)=v(t),\ \ x(t)=x(t,v)$, $t\in T$.

Using mapping

$$u^*(\psi,x,t)=\arg\max_{w\in U} H(\psi,x,w,t),\ \psi\in R^n,\ x\in R^n,\ t\in T,$$

Pontryagin's famous maximum principle [1] for control $v\in V$ is represented in a fixed point problem

$$v(t)=u^*(\psi(t,v),x(t,v),t),\ t\in T. \tag{4}$$

Boundary-value problem of maximum principle has the form

$$\dot{x}(t) = f(x(t),u^*(\psi(t),x(t),t),t),\ x(t_0)=x^0 \tag{5}$$

$$\dot{\psi}(t) = -H_x(\psi(t),x(t),u^*(\psi(t),x(t),t),t)$$

$$\psi(t_1) = -\varphi_x(x(t_1)). \tag{6}$$

The boundary-value problem (5), (6) in the state space is reduced to the fixed point problem (4) on the set of admissible controls. In the general case, the right-hand sides of the boundary-value problem are discontinuous and multi-valued in phase variables x , ψ .

The differential maximum principle follows from the maximum principle (4) in the form

$$\left\langle H_u(\psi(t,v),x(t,v),v(t),t),w-v(t)\right\rangle \le 0 \qquad (7)$$

$$w\in U,\ t\in T$$

We define the mapping w^α, $\alpha>0$ using the ratio

$$w^\alpha(\psi,x,u,t)=P_U(u+\alpha H_u(\psi,x,u,t)), \qquad (8)$$

$$\psi\in R^n,\ x\in R^n,\ u\in U,\ t\in T,$$

wherein P_U - projection operator on set U in Euclidean norm.

Based on the Lipschitz condition for the operator P_U function w^α is continuous in terms $(\psi,x,u,t)\in R^n\times R^n\times U\times T$. It is fulfilled inequality

$$\left\langle H_u(\psi,x,u,t),w^\alpha(\psi,x,u,t)-u\right\rangle \ge \frac{1}{\alpha}\left\|w^\alpha(\psi,x,u,t)-u\right\|^2 \qquad (9)$$

Assessment (9) is determined by the properties of the projection operation.

Differential maximum principle (7) for control $v\in V$ using mapping (8) is represented in the form of a fixed point problem

$$v(t)=w^\alpha(\psi(t,v),x(t,v),v(t),t),\ t\in T,\ \alpha>0 \qquad (10)$$

We note that to fulfill (7), it suffices to check condition (10) for at least one $\alpha>0$. Conversely, condition (7) implies (10) for all $\alpha>0$.

In the linear control problem (1), (2) (functions $f(x,u,t)$, $F(x,u,t)$ linear in u) the differential maximum principle (10) is equivalent to the maximum principle (4).

Standard methods for the numerical solution of boundary problem (5), (6) (shooting method, linearization method, finite difference method) even in the case of smoothness and uniqueness of the right-hand sides of problems, as a rule, are computationally unstable due to the presence of positive real values of the eigenvalues of the corresponding Jacobi matrix. These difficulties can be circumvented by going to solving equivalent operator equations in the space of controls, interpreted as fixed-point problems of the constructed control operators.

It is proposed to apply perturbation methods to implement the maximum principle (4) and the differential maximum principle in projection form (10).

II. Perturbation Methods

Parameterize the maximum principle condition (4) using the perturbation parameter $\varepsilon\in[0,1]$ as follows.

To do this, we represent the problem (1), (2), highlighting in it a special linear-state part with separated variables on state and control, in the following form

$$\Phi(u)=\left\langle c_0,x(t_1)\right\rangle+\varphi_1(x(t_1))+$$

$$+\int_T(\left\langle a_0(t),x(t)\right\rangle+d_0(u(t),t)+F_1(x(t),u(t),t))dt\to\min_{u\in V} \qquad (11)$$

$$\dot x(t)=A_0(t)x(t)+b_0(u(t),t)+f_1(x(t),u(t),t)$$

$$x(t_0)=x^0,\ u(t)\in U,\ t\in T=[t_0,t_1] \qquad (12)$$

in which the matrix function $A_0(t)$ and vector function $a_0(t)$ continuous on T, vector function $b_0(u,t)$ and function $d_0(u,t)$ continuous in terms of the variables u, t on the set $U\times T$, c_0 - constant vector .

On the basis of the representation (11), (12) we introduce a perturbed optimal control problem with a perturbation parameter $\varepsilon\in[0,1]$

$$\Phi_\varepsilon(u)=\left\langle c_0,x(t_1)\right\rangle+\varepsilon\varphi_1(x(t_1))+$$

$$+\int_T(\left\langle a_0(t),x(t)\right\rangle+d_0(u(t),t)+\varepsilon F_1(x(t),u(t),t))dt\to\min_{u\in V} \qquad (13)$$

$$\dot x(t)=A_0(t)x(t)+b_0(u(t),t)+\varepsilon f_1(x(t),u(t),t),$$

$$x(t_0)=x^0,\ u(t)\in U,\ t\in T=[t_0,t_1]. \qquad (14)$$

Problem (13), (14) corresponds to the perturbed Pontryagin function

$$H_\varepsilon(\psi,x,u,t)=\left\langle\psi,A_0(t)x+b_0(u,t)\right\rangle-\left\langle a_0(t),x\right\rangle-d_0(u,t)+$$
$$+\varepsilon(\left\langle\psi,f_1(x,u,t)\right\rangle-F_1(x,u,t)),$$

perturbed mapping

$$u_\varepsilon^*(\psi,x,t)=\arg\max_{w\in U}H_\varepsilon(\psi,x,w,t),$$

$$\psi\in R^n,\ x\in R^n,\ t\in T$$

and perturbed conjugate system

$$\dot\psi(t)=-A_0^T(t)\psi(t)+a_0(t)-\varepsilon(f_{1x}^T(x(t),u(t),t)\psi(t)-$$
$$-F_{1x}(x(t),u(t),t)),$$

$$\psi(t_1)=-c_0-\varepsilon\varphi_{1x}(x(t_1)),\ t\in T. \qquad (15)$$

We denote $x_\varepsilon(t,v)$, $t \in T$ - solution of the perturbed phase system (14) with $u(t) = v(t)$; $\psi_\varepsilon(t,v)$, $t \in T$ - solution of the perturbed conjugate system (15) with $u(t) = v(t)$, $x(t) = x_\varepsilon(t,v)$.

The condition of the maximum principle for the perturbed problem (13), (14)

$$v(t) = u_\varepsilon^*(\psi_\varepsilon(t,v), x_\varepsilon(t,v), t), \quad t \in T \tag{16}$$

define as a perturbed condition of the maximum principle with the parameter $\varepsilon \in [0,1]$.

The initial problem in the form (11), (12), the Pontryagin function H, mapping u^*, the conjugate system (3) and the condition of the maximum principle (4) are obtained, respectively, from the perturbed problem (13), (14), the perturbed Pontryagin function H_ε, perturbed mapping u_ε^*, perturbed conjugate system (15), and a perturbed condition (16) for $\varepsilon = 1$.

The unperturbed condition of the maximum principle corresponds to the unperturbed optimal control problem

$$\Phi_0(u) = \langle c_0, x(t_1) \rangle +$$

$$+\int_T (\langle a_0(t), x(t) \rangle + d_0(u(t),t))dt \to \min_{u \in V}, \tag{17}$$

$$\dot{x}(t) = A_0(t)x(t) + b_0(u(t),t) \tag{18}$$

$$x(t_0) = x^0, \quad t \in T = [t_0, t_1]$$

with unperturbed Pontryagin function

$$H_0(\psi, x, u, t) = \langle \psi, A_0(t)x + b_0(u,t) \rangle - \langle a_0(t), x \rangle - d_0(u,t)$$

unperturbed mapping

$$u_0^*(\psi, x, t) = \arg\max_{w \in U} H_0(\psi, x, w, t), \quad \psi \in R^n, \quad x \in R^n, \quad t \in T$$

unperturbed conjugate system

$$\dot{\psi}(t) = -A_0^T(t)\psi(t) + a_0(t), \quad t \in T, \quad p(t_1) = -c_0 \tag{19}$$

For $v \in V$ denote $x_0(t,v)$, $t \in T$ - solution of the unperturbed phase system (18); $\overline{\psi}_0(t)$, $t \in T$ - solution of the unperturbed conjugate system (19). The unperturbed condition of the maximum principle is obtained from (16) for $\varepsilon = 0$ and has the form

$$v(t) = u_0^*(\overline{\psi}_0(t), x_0(t,v), t), \quad t \in T \tag{20}$$

Unperturbed phase and conjugate systems, Pontryagin function H_0, mapping u_0^* are obtained from the corresponding perturbed for $\varepsilon = 0$.

We note that the unperturbed problem (17), (18) is a linear-convex, for which the maximum principle (20) is a necessary and sufficient condition for optimal control [1].

The complexity of solving the unperturbed relation (20) is determined by solving the Cauchy problem for the conjugate system (19) and solving the Cauchy problem for the phase system

$$\dot{x}(t) = A_0(t)x(t) + b_0(u_0^*(\overline{\psi}_0(t), x(t), t), t) \tag{21}$$

$$x(t_0) = x^0, \quad t \in T.$$

Let $\overline{x}_0(t)$, $t \in T$ - solution of the problem (21) and output control $\overline{v}_0(t) = u_0^*(\overline{\psi}_0(t), \overline{x}_0(t), t)$, $t \in T$ is a piecewise continuous function. Then $\overline{x}_0(t) = x_0(t, \overline{v}_0)$, $t \in T$ and, therefore, $\overline{v}_0(t)$, $t \in T$ is a solution of the unperturbed conditions (20).

The representation of the maximum principle in the form of a fixed point problem allows one to apply the developed theory and fixed point methods to the search for extremal controls. For a example, to solve the perturbed condition (16) with the perturbation parameter $\varepsilon \in (0,1]$, the simple iteration method [2] can be used

$$v^{k+1}(t) = u_\varepsilon^*(\psi_\varepsilon(t, v^k), x_\varepsilon(t, v^{k+1}), t), \quad t \in T, \quad k \geq 0 \tag{22}$$

The unperturbed solution $\overline{v}_0$ can be chosen as the initial approximation $v^0 \in V$ of process (22) for $k = 0$.

The complexity of each iteration of process (22) constitutes two Cauchy problems, similarly to the complexity of solving the unperturbed condition (20).

As a criterion for stopping the iteration process (22), the achievement of a predetermined small value of the residual index [1] to satisfy the perturbed maximum principle (16) can be specified.

Under some assumptions, it is possible to justify the convergence of the iterative process (22) in the space of measurable functions $L_\infty(T)$, similar to [3].

The calculation of the perturbed conditions of the maximum principle is repeated with a gradual increase in the perturbation parameter $\varepsilon \in (0,1]$. In this case, the control obtained in the problem with a smaller value of $\varepsilon > 0$ is taken as the initial approximation of the iterative process. Reaching the value of $\varepsilon = 1$ we obtain the solution of the original problem.

We illustrate another perturbation method for the fixed point problem (10).

The projection parameter $\alpha > 0$ will be considered as a perturbation parameter, condition (10) is called perturbed. The unperturbed condition is obtained from (10) with $\alpha = 0$ and any admissible control $v(t)$, $t \in T$ satisfies it.

The iterative process for solving problem (10) has the form

$$v^{k+1}(t) = P_U(v^k(t) + \alpha H_u(\psi(t, v^k), x(t, v^k), v^k(t), t)), \quad t \in T \quad (23)$$

At the initial (zero) iteration, the initial approximation $v^0 \in V$ is set.

The convergence of iterative process (23) to solutions of perturbed problems in the space of continuous functions with a uniform norm is justified similarly to [3].

The projection perturbation method (23) with the parameter $\alpha > 0$ favorably differ from the perturbation method with the artificial perturbation parameter $\varepsilon \in [0,1]$ in that the solution to the perturbed problem obtained for any parameter $\alpha > 0$ is extreme. Solving perturbed problems in methods with the parameter $\varepsilon < 1$ do not provide, in the general case, obtaining extreme control.

The perturbation approach based on fixed point methods to search for extreme controls can be extended to other optimal control problems. In particular, for problems of parametric optimization of dynamic systems [4], [5].

III. Conclusion

The proposed perturbation methods do not guarantee relaxation over the target function at each iteration, in contrast to gradient methods. Perturbation methods are characterized by the absence of a convex or a needle variation operation of the control, computational stability, and obtaining computational controls that do not contain frequent sections of sharp amplitude control changes that are difficult to implement in practice.

These properties are essential factors for improving the efficiency and quality of solving optimal control problems.

The proposed approach opens up new possibilities for the effective application of the perturbation method in optimal control problems, when it is proposed to use the necessary optimality conditions as objects of parametrization.

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