

# The Hamacher Aggregation Operators and their Application to Decision Making with Hesitant Pythagorean Fuzzy Sets

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**Abstract.** As a generalization of intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (IPFSs) can deal with uncertain information more flexibly for its relaxing condition that the square sum of the membership and non-membership is less than 1. While in real world, the uncertainty and fuzziness universally exist, hesitant fuzzy sets (HFSs) is more efficient in expressing the hesitant situation for assigning a set of memberships and non-memberships instead of one. This article deals with the group decision-making problem based on Hesitant Pythagorean fuzzy sets (HPFSs). Firstly, with Hamacher aggregation operations, the Hamacher aggregation hesitant Pythagorean weighted averaging operator (HHPWA) is constructed. Then, we aggregate the information with the collective ones. Finally, an illustrative numerical example is presented to test the effectiveness of the proposed approach.

## Introduction

Multi-criteria decision making (MCDM) is a problem concerning selecting suitable alternatives or getting their ranking orders according to some criteria [1]. Nowadays, it has been applied in many practical research fields. In expressing the data, as the limitation of field knowledge of the decision makers and the imprecise and uncertain evaluations that are associated with the real applications, it is impossible to express the data in the exact information. Therefore, some fuzzy objects such as fuzzy sets (FSs) [2], intuitionistic fuzzy sets (IFSs) [3] and hesitant fuzzy sets (HFSs) [4, 5, 6, 7, 8, 21] are developed. By generalization of the above concepts, Pythagorean fuzzy sets (PFSs) are proposed by Yager [9, 10]. PFSs are characterized both by a membership degree and by a non-membership degree which satisfy the condition that the square sum of its membership degree and non-membership degree is equal to or less than 1. It is clear that every intuitionistic fuzzy set (IFS) is a hesitant fuzzy set (HFS).

The rest of the paper is organized as follows. Section 2 recalls some basic notions and concepts about Pythagorean fuzzy sets, Hesitant fuzzy sets and Hamacher operations. In Section 3, an approach to decision-makings under Pythagorean fuzzy settings is proposed, a new score function of alternatives is developed. A numerical example is given in section 4.

## Preliminaries

Concepts and properties about PFSs and PHFSs can be found in [9, 11, 12, 13]. Here we list basic concepts about them.

**Definition 1.** [14] Let  $X$  be a set, a PFS in  $X$  is defined as

$$P = \{ \langle x, P(\mu_p(x), \nu_p(x)) \rangle \mid x \in X \}$$

Where  $\mu_p(x) \in [0, 1]$  and  $\nu_p(x) \in [0, 1]$  are the degree of membership and nonmembership of an element  $x \in U$ . respectively, satisfying  $0 \leq \mu_p^2(x) + \nu_p^2(x) \leq 1, \forall x \in X, \pi_p(x) = \sqrt{1 - \mu_p^2(x) - \nu_p^2(x)}$  is called the intuitionistic fuzzy index of  $x \in U$ . For simplicity, we call  $\alpha = P(\mu_\alpha, \nu_\alpha)$ ,  $0 \leq \mu_\alpha^2 + \nu_\alpha^2 \leq 1$ , a Pythagorean fuzzy number (PFN)

**Definition 2** [14] Let  $\alpha = P(\mu_\alpha, \nu_\alpha)$  be Pythagorean fuzzy number (PFN), the score function of  $\alpha$  is defined as

$$S(\alpha) = \frac{1}{2}(\mu_\alpha^2 - \nu_\alpha^2)$$

The accuracy function is defined as

$$H(\alpha) = \frac{1}{2}(\mu_{\alpha}^2 + \nu_{\alpha}^2)$$

HFSs allow the membership degrees of an element to be a set to be presented as some possible values in the interval in  $[0, 1]$ [4, 5]. The concept of HFSs and its operations are briefly recalled as follows:

**[8, 15]** Let  $S$  be a fixed set. A hesitant fuzzy set (HFS)  $H$  on  $S$  is in forms of a function that when applied to  $S$  returns a subset of  $[0, 1]$ . To be specific, Xia and Xu[12] represented the HFS in form of the following mathematical symbol:

$$H = \{\langle s, h_H(s) \rangle \mid s \in S\}$$

Where  $h_H(s)$  is a set of values in  $[0, 1]$ , denoting the possible membership degrees of the element  $s \in S$  to the set  $H$ . For convenience, we call  $h_H(s)$  as a hesitant fuzzy element(HFE).

**Definition 3 [16]** Let  $S$  be a fixed set, a hesitant fuzzy set (HFS)  $H$  on  $S$  is in forms of a function that when applied to  $S$  returns a subset of  $[0, 1]$ . To be specific, Xia and Xu[12] represented the HFS in forms of the following mathematical Symbol:

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Suppose there exist hesitant fuzzy elements (HFEs)  $h, h_1, h_2$ , then some operations[8,15] are defined as follows

- (1)  $h^c = \cup_{\gamma \in h} \{1 - \gamma\}$
- (2)  $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$
- (3)  $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$
- (4)  $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$
- (5)  $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}$
- (6)  $\lambda h = \cup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}, \lambda \geq 0$

**Definition 4 [16]** Let  $S$  be a set, a hesitant Pythagorean fuzzy set (HPFS)  $F$  on  $S$  is defined as:

$$H = \{\langle s, F(h_H(s), g_H(s)) \rangle \mid s \in S\}$$

$h_H(s)$  and  $g_H(s)$  are two sets of some values in  $[0,1]$ , denoting the possible Pythagorean membership degrees and the Pythagorean non-membership degrees of the elements to the set  $F$  respectively, with the condition:

$$0 \leq m_h^2 + n_g^2 \leq 1,$$

Where  $m_h = \max_{m \in h_H(s)} \{m\}, n_g = \max_{n \in g_H(s)} \{n\}$

The valuation of a HPFN is computed from the following score function as follows

**Definition 5 [19]** Let  $f = F(h, g)$  be HPFE, then the score function  $f$  is

$$s(f) = (1/l(h)) \sum_{\gamma \in h} \gamma^2 - (1/l(g)) \sum_{\eta \in g} \eta^2$$

where  $l(h)$  and  $l(g)$  are the numbers of the elements in  $h$  and  $g$ , respectively.

**Example 1** Let  $f$  be a hesitant Pythagorean fuzzy set,  $f = F(h, g) = F(\{0.5, 0.6, 0.8\}, \{0.3, 0.4, 0.5\})$ , since  $0.8^2 + 0.5^2 = 0.89 < 1$ , therefore  $f$  is a HPFN.  $l(h) = l(g) = 3$ , thus,

$$s(f) = (0.5^2 + 0.6^2 + 0.8^2) / 3 - (0.3^2 + 0.4^2 + 0.5^2) / 3 = 0.25$$

## Hamacher operations and Hamacher Hesitant Pythagorean fuzzy weighted averaging operators(HHPFWA)

Union and intersection are two basic operations of fuzzy set, as a generalization of the two operations,  $t$ -norm and  $t$ -conorm are developed in fuzzy set theory[17]. As a special cases of  $t$ -conorm and  $t$ -norm,

Hamacher operations[18] which consist of Hamacher sum and Hamacher product are introduced by Hamacher, they are put as follows

$$T(x, y) = \frac{xy}{r + (1-r)(x + y - xy)}, S(x, y) = \frac{x + y - (2-r)xy}{1 - (1-r)xy}, r > 0$$

Suppose that three hesitant Pythagorean fuzzy numbers(HPFNs) are represented by  $f = F(h, g)$ ,  $f_1 = F(h_1, g_1)$ ,  $f_2 = F(h_2, g_2)$ , then the operations among them are defined as follows

$$\begin{aligned} (1) f_1 \oplus f_2 &= \cup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \left\{ \sqrt{\frac{\gamma_1^2 + \gamma_2^2 - (2-r)\gamma_1^2\gamma_2^2}{1 - (1-r)\gamma_1^2\gamma_2^2}} \right\}, \left\{ \frac{\eta_1\eta_2}{\sqrt{r + (1-r)(\eta_1^2 + \eta_2^2 - \eta_1^2\eta_2^2)}} \right\} \right\} \\ (2) f_1 \otimes f_2 &= \cup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \left\{ \frac{\gamma_1\gamma_2}{\sqrt{r + (1-r)(\gamma_1^2 + \gamma_2^2 - \gamma_1^2\gamma_2^2)}} \right\}, \left\{ \sqrt{\frac{\eta_1^2 + \eta_2^2 - (2-r)\eta_1^2\eta_2^2}{1 - (1-r)\eta_1^2\eta_2^2}} \right\} \right\} \\ (3) \lambda f &= \cup_{\gamma \in h, g \in h} \left\{ \left\{ \sqrt{\frac{(1 + (r-1)\gamma^2)^\lambda - (1 - \gamma^2)^\lambda}{(1 + (r-1)\gamma^2)^\lambda + (r-1)(1 - \gamma^2)^\lambda}} \right\}, \left\{ \frac{\sqrt{r}\eta^\lambda}{\sqrt{(1 + (r-1)(1 - \eta^2))^\lambda + (r-1)\eta^{2\lambda}}} \right\} \right\} \end{aligned}$$

Now we construct the Hamacher Hesitant Pythagorean fuzzy weighted averaging operators (HHPFWA). Let  $f_i = F(h_i, g_i) (i = 1, 2, \dots, n)$  be a collection of Hesitant Pythagorean Fuzzy numbers (HPFNs), and

$$HHPFWA_w(f_1, f_2, \dots, f_n) = w_1 f_1 \oplus w_2 f_2 \oplus \dots \oplus w_n f_n$$

It is easy to verify that

$$\begin{aligned} HHPFWA_w(f_1, f_2, \dots, f_n) &= \cup_{\gamma_i \in h_i, g_i \in h_i} \left\{ \left\{ \sqrt{\frac{\prod_{i=1}^n (1 + (r-1)\gamma_i^2)^{w_i} - \prod_{i=1}^n (1 - \gamma_i^2)^{w_i}}{\prod_{i=1}^n (1 + (r-1)\gamma_i^2)^{w_i} + (r-1) \prod_{i=1}^n (1 - \gamma_i^2)^{w_i}}} \right\}, \right. \\ &\quad \left. \left\{ \frac{\sqrt{r} \prod_{i=1}^n \eta_i^{w_i}}{\sqrt{\prod_{i=1}^n (1 + (r-1)(1 - \eta_i^2))^{w_i} + (r-1) \prod_{i=1}^n \eta_i^{2w_i}}} \right\} \right\} \end{aligned}$$

Where  $w = (w_1, w_2, \dots, w_n)^T$  is the weight vector of  $f_i (i = 1, 2, \dots, n)$ ,  $w_i \in [0, 1]$  and  $\sum_{i=1}^n w_i = 1$ .

### Algorithm

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a finite set of alternatives,  $C = \{c_1, c_2, \dots, c_m\}$  be the set of decision makers, whose weight vector is  $w = (w_1, w_2, \dots, w_m)^T$  and  $\sum_{i=1}^m w_i = 1$ , for any pair  $x_i$  and  $c_j$ , the expert inputs value in the form of HPFN  $f_{ij} = F(h_i, g_j)$ .

**Step 1:** Utilize the HHPFWA operator in Eq.(2) to aggregate all the HPFNs into HPFN

$$\bar{x}_i = HHPFWA_w(f_{i1}, f_{i2}, \dots, f_{in})$$

**Step 2:** With definition 5, compute the score function of  $\bar{x}_i$ . Rank all the alternatives  $x_i (i = 1, 2, \dots, n)$ , the larger the  $S(x_i)$ , the better the alternative  $x_i$ .

### Numerical Examples

In this section, an example about a group decision-making problem with intuitionistic fuzzy preference relations is illustrated to show the effectiveness of the proposed approach.

### Example

The application case is from[8]. Energy development strategy is becoming a common topic of concern among the countries in the world[20]. Considering the uncertain and risk environment, the

energy project selection is a critical step for the energy development. In this section, we use our proposed method to support the selection of energy projects and illustrate its decision-making process. During the evaluation of energy projects, four attributes to be considered: (1)  $c_1$  :Economic; (2)  $c_2$  :Technological; (3)  $c_3$  :Environmental; and (4)  $c_4$  :Sociopolitical, i.e.,  $C = \{c_1, c_2, c_3, c_4\}$ . Accordingly, the weight vector of the criteria is given  $w = (0.15, 0.3, 0.2, 0.35)^T$ . With respect to energy projects, we assume that there are five alternatives  $X = \{x_1, x_2, x_3, x_4, x_5\}$ . Under the hesitant Pythagorean fuzzy settings, the experts evaluate these alternatives with HPFEs. Therefore, the hesitant Pythagorean fuzzy decision matrix  $F$  is constructed in Table 1.

**Step 1:** The hesitant Pythagorean fuzzy decision matrix  $F = (f_{ij})_{5 \times 4}$ ,  $f_{ij} = F(h_i, g_j)$  ( $i = 1, 2, 3, 4, 5$ ;  $j = 1, 2, 3, 4$ )

$$\begin{aligned} f_{11} &= F(h_1, g_1) = F(\{0.3, 0.4, 0.5\}, \{0.7, 0.8\}) & f_{21} &= F(h_2, g_1) = F(\{0.1, 0.7, 0.8, 0.9\}, \{0.2, 0.4\}) \\ f_{12} &= F(h_1, g_2) = F(\{0.3, 0.5\}, \{0.6, 0.7, 0.8\}) & f_{22} &= F(h_2, g_2) = F(\{0.2, 0.5, 0.6, 0.7\}, \{0.6, 0.7\}) \\ f_{13} &= F(h_1, g_3) = F(\{0.6, 0.7\}, \{0.4, 0.5, 0.6\}) & f_{23} &= F(h_2, g_3) = F(\{0.6, 0.9\}, \{0.2, 0.3, 0.4\}) \\ f_{14} &= F(h_1, g_4) = F(\{0.3, 0.4, 0.6, 0.7\}, \{0.5, 0.7\}) & f_{24} &= F(h_2, g_4) = F(\{0.2, 0.4, 0.7\}, \{0.4, 0.5, 0.6\}) \\ f_{15} &= F(h_1, g_5) = F(\{0.1, 0.3, 0.6\}, \{0.4, 0.6\}) & f_{25} &= F(h_2, g_5) = F(\{0.4, 0.6, 0.7, 0.8\}, \{0.1, 0.2, 0.3\}) \\ f_{31} &= F(h_3, g_1) = F(\{0.2, 0.4, 0.5\}, \{0.2, 0.6, 0.8\}) & f_{41} &= F(h_4, g_1) = F(\{0.3, 0.5, 0.6, 0.9\}, \{0.2, 0.3, 0.4\}) \\ f_{32} &= F(h_3, g_2) = F(\{0.1, 0.5, 0.6, 0.8\}, \{0.2, 0.5\}) & f_{42} &= F(h_4, g_2) = F(\{0.3, 0.4, 0.7\}, \{0.4, 0.5\}) \\ f_{33} &= F(h_3, g_3) = F(\{0.3, 0.5, 0.7\}, \{0.4, 0.5, 0.6\}) & f_{43} &= F(h_4, g_3) = F(\{0.4, 0.6\}, \{0.3, 0.4, 0.6\}) \\ f_{34} &= F(h_3, g_4) = F(\{0.1, 0.8\}, \{0.3, 0.4\}) & f_{44} &= F(h_4, g_4) = F(\{0.6, 0.8, 0.9\}, \{0.2, 0.3\}) \\ f_{35} &= F(h_3, g_5) = F(\{0.7, 0.8, 0.9\}, \{0.1, 0.2, 0.4\}) & f_{45} &= F(h_4, g_5) = F(\{0.3, 0.6, 0.7, 0.9\}, \{0.3, 0.4\}) \end{aligned}$$

**Step 2:** With the weights of criteria  $w = (0.15, 0.3, 0.2, 0.35)^T$  and HHPFWA operators, we choose  $r = 3$ , by computing, we get that

$$S(\bar{x}_1) = 0.0281, S(\bar{x}_2) = -0.1610, S(\bar{x}_3) = -0.0607, S(\bar{x}_4) = 0.0411, S(\bar{x}_5) = 0.1871$$

**Step 3:** The ranking order of the five criteria is

$$x_5 \succ x_4 \succ x_2 \succ x_3 \succ x_1$$

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