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The Classification of the Bifurcations Emerging in the case of an Integrable Hamiltonian System with Two Degrees of Freedom when an Isoenergetic Surface is Non-Compact

Galina GOUJVINA

Department of Differential Geometry and Applications
Faculty of Mechanics and Mathematics
Moscow State University
119899 Vorobiovy Gory, Moscow, Russia.

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Abstract

On a symplectical manifold M^4 consider a Hamiltonian system with two degrees of freedom, integrable with the help of an additional integral f. According to the well-known Liouville theorem, non-singular level surfaces of the integrals H and f can be represented as unions of tori, cylinders and planes. The classification of bifurcations of the compact level surfaces was given by Professor A. Fomenko and his school. This paper generalizes this result to the non-compact surfaces.

1 Introduction

Let M^4 be a smooth symplectical manifold. We consider a Hamiltonian system on it with the Hamilton function of H, which is integrable with the help of the additional integral f. Let

$$Q^3 = \Big\{x \in M^4 : H(x) = h, sgrad\, H(x) \neq 0\Big\}.$$

The notion of skew gradient is equivalent to the notion of gradient in case of symplectic structure instead of Riemannian metric.

It is well known that the non-singular level surface of the integrals H and f can be represented as the union of tori (T), cylinders (C) and planes (P) of Liouville. Let us consider the function of the moment $\Phi = (H, f)$ defined on M^4 with the values in R^2 .

Definition 1. Let us call a point $y \in \operatorname{Im}\Phi$ not belonging to the bifurcational diagram Σ of the function of the moment Φ if there exists a neighborhood $O(y)$ such as $\Phi^{-1}(O(y)) \cong \Phi^{-1}(y) \times I$. All the other points belong to the bifurcational diagram.

Remark 1. *The definition of the bifurcational diagram is modified with respect to the compact case [1]*

In the case of compact Q^3 this definition is equivalent to the fact that $\mathrm{grad}\, f = 0$ *on* $\Phi^{-1}(y)$.

In the non-compact case there can occur a situation when $\mathrm{grad}\, f \to 0$ *on some sequence of points. These considerations let us distinguish two types of bifurcations:*

1. *those taking place on the manifolds on which* $\mathrm{grad}\, f = 0$,
2. *those taking place on the manifolds without singular points.*

Recall that in the case of a compact isoenergetic surface the additional integral f was chosen in the Bott representation, i.e. the singular points of it were organized into undegenerating manifolds N such that

1. *in some equidistant punctured ϵ-neighborhood of N there are no other singular points;*
2. *co-rank of the Hess matrix coincides with the dimension of N in all points.*

Let us modify the notion of the Bott-characteristics. We will demand additionally the existence of such metric (g_{ij}) on the Q^3 that $|d^2 f\,\big| > \epsilon_0 > 0$ on the planes which are normal to N.

Definition 2. Let us call the pair (g_{ij}, f) the integral in the equi-Bott representation.

Let us suppose that the Hamiltonian system on M^4 is complete with respect of the given metric, i.e. on the common level surfaces of f and H there is a Poisson action of the group R^2.

2 The first type of the bifurcations.

Let us concentrate on the case of saddle trajectories.

Definition 3. We will call $R^1 \times D^2$ with two noted arcs γ_1 and γ_2 on δD^2 the long handle and $S^1 \times D^2$ with two noted arcs γ_1 and γ_2 the round handle. $\mu \widetilde{\times} I$ where the wave means semi-direct product is the "thick" Möbius list.

Let

$$C^a = \Big\{x \in Q^3 | f(x) \le a\Big\} \qquad B^a = \Big\{x \in Q^3 | f(x) = a\Big\}.$$

Lemma 1. *Let the separatrix diagram $sd\,\gamma$ of the saddle trajectory γ be oriented. Then $C^{a+\epsilon}$ is constructed from $C^{a-\epsilon}$ by gluing the long (round) handle to $C^{a-\epsilon}$. If $sd\,\gamma$ is not oriented $C^{a+\epsilon}$ is constructed from $C^{a-\epsilon}$ by gluing the "thick" Möbius list to $B^{a-\epsilon}$.*

Proof. Let us consider the case of a saddle singular circle having the index equal to 1 on a cylinder. As the system is supposed to be complete on this cylinder the separatrix lines entering into S^1 form the smooth 2-dimensional manifold with the edge which is homeomorphic to one circle or two not-connecting circles. The first case satisfies the gluing of the "thick" Möbius list and the second - the gluing of a round handle. If we consider the saddle critical line the entering separatrix lines form the smooth 2-dimensional manifold with the edge which is homeomorphic to two lines thus representing a cylinder. ■

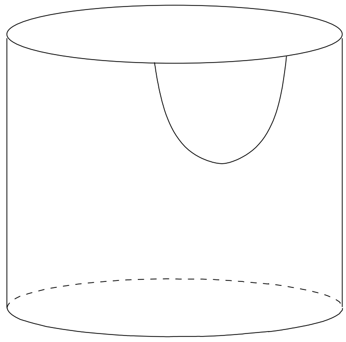

Figure 1. Gluing of the long handle to a cylinder.

Lemma 2.

1. *The axis of the foot of any round handle realizes a non-trivial cycle on T or C;*
2. *The axis of the foot of any long handle realizes a non-trivial cycle on C;*
3. *No round handle can be glued to a plane.*

Proof. As it is well seen of all three items only 2. and 3. need to be explained, because the case 1. was totally handled by A. Fomenko. [2]
2. Let us suppose that a long handle is glued to a cylinder as it is shown in Fig. 1. According to the Liouville theorem all the vector fields straighten on the cylinder. Let us cut it by the cycle γ. As γ is the trajectory of a vector field on C the property of straightening will not get broken on the parts. Let us parameterize γ. All the trajectories close to γ when $t = t_0$ must be close to it as well when $t > t_0$. But it is impossible because among them there are circles.
3. Is proved similarly. ∎

Lemma 3. *With respect to the ways of gluing handles to T, C, P we have the following types of bifurcations:*

A. The gluing of a long handle:

1. $C \to 2C$;
2. $2C \to C$;
3. $P \to P + C$;
4. $P + C \to P$;
5. $2P \to 2P$;

B. The gluing of a round handle:

6. $C \to C + T$;
7. $C + T \to C$;
8. $2C \to 2C$;
9. $2T \to T$;
10. $T \to 2T$;

C. The gluing of the "thick" Möbius list:

11. $T \to T$;
12. $C \to C$;

Proof. The round handle can be glued to one torus (cylinder) or to two tori (cylinders) or to one torus and one cylinder. The long handle can be glued to one cylinder (plane) or to two cylinders (planes) or to one cylinder and one plane. The "thick" Möbius list can be glued either to the torus or to the cylinder. ∎

Let M^1 be a disc with two holes, M_1^1 be the cut disc with two holes, the edges of the section declared to approach infinity, M_1^{11} be the disc cut twice with two holes, M_2 be the disc with one hole, M_1^2 be the cut disc with one hole.

Lemma 4. *The connected component of the layer which contains a critical saddle trajectory is one of the manifolds of the following types:*

1. $M^1 \times R^1$;

2. $M^1 \times S^1$;

3. $M_1^1 \times R^1$;

4. $M_1^1 \times S^1$;

5. $M_1^{11} \times R^1$;

6. $M_1^{11} \times S^1$;

7. $M_2 * S^1$;

8. $M_1^2 * R^1$.

The star $*$ *here means Seifert foliation over* M_2 *or* M_1^2.

Proof. Let us consider for the example the case number 3. When passing through the saddle line a cylinder and a plane transform into a plane. Let us note on the layer $a - \epsilon \le f \le a + \epsilon$ the connected component $U(R^1)$ which is a 3-dimentional manifold. As the border we have a cylinder and two planes. ■

3 Bifurcations without critical points.

Definition 4. Let us denote a singular point in the sense of **Definition 1** where *grad* f *is not equal to zero the* ∞- critical point.

Lemma 5. ∞*-critical points form a curve without intersections on the level surface* B^a.

Proof. The set of ∞-critical points is closed because its supplementary set consists of Liouville planes and cylindersthus being open. As we consider the integral f in the natural equi-Bott representation, there are no ∞-critical points which have a neighborhood completely consisting of ∞-critical points. A ∞-critical point can not be isolated. Really, if we suppose that then B^a is a cylinder and $B^{a+\epsilon}$ is a torus, a cylinder or a plane on which all the trajectories are directed into the center of some disc which is impossible. It means that a ∞-critical point belongs to a curve. Let us suppose that it lies on the intersection of two curves. Then, as it is seen in Fig. 2 either the continuity of $d^2 f$ is lost on B^a or this differential equals to zero. If the mentioned curve is of finite length then after bifurcation we have got a torus, a cylinder or a plane with a hole which is in contradiction with the Liouville theorem. If the length of the curve is half-infinite there is no bifurcation. ■

Definition 5. Let us call $R^1 \times D^2 \times R_0^1$ the long ∞-handle with the foot $R^1 \times D^1 \times 0$. Let us call $S^1 \times D^1 \times R_0^1$ the round ∞-handle with the foot $S^1 \times D^1 \times 0$.

Lemma 6. *Let* B^a *be a singular surface without critical points. Then* $C^{a+\epsilon}$ *is constructed out of* $C^{a-\epsilon}$ *by gluing to* B^a *the long (round)* ∞*-handle to* B^a.

Proof. On every singular surface without critical points there is a sequence of points by which *grad* $f \to 0$ (otherwise the section in its neighborhood would trivially transit along the lines of the gradient). It means that the length of the integral curves of the gradient approaches infinity by these points. ■

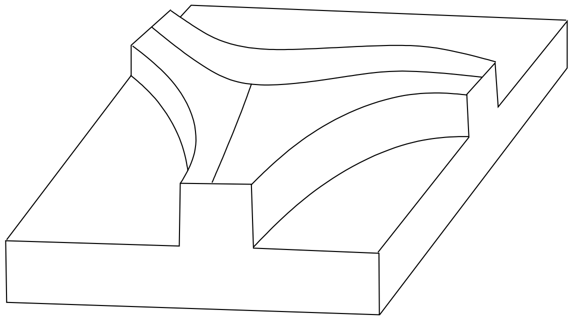

Figure 2. Gluing of the long ∞-handle to a level surface.

Lemma 7.

A. The gluing of a long ∞-handle:

1. $C \to C + P$;
2. $P \to P + P$;

B. The gluing of a round ∞-handle:

1. $T \to C$;
2. $C \to C + C$; *and the axis of the handle realizes a non-trivial cycle on the torus or the cylinder;*

C. A round ∞-handle can not be glued to a plane.

Proof. An infinite curve without points of accumulation cuts a plane into two connected pieces (two discs), a cylinder - into one piece (a disc). A circle cuts a torus into a ring, a cylinder - into two cylinders. If some plane could be cut by circle (and either a torus or a cylinder - by a trivial circle) we could have chosen a cycle transversal to the trajectories of $sgrad\, H$ receiving thus a disc such that all the trajectories are directed inside of it which is impossible. ∎

4 The critical levels of minimum and maximum.

The critical trajectories of the index 0 or 2 give two types of bifurcations: $T \to S^1 \to 0$ or $C \to R^1 \to 0$. Due to the equi-Bott properties of the integral they are the connected pieces

P of the critical level B^a. If the border of P consists of two non-connected components we will consider that there is no bifurcation (in fact, there is just a reflection from the singular level). Only the case when the connected component of $B^{a\pm\epsilon}$ covers P twice is of interest for us.

Lemma 8.

1. *Let the border of the tube neighborhood of P is a cylinder. Then P is a Möbius list;*
2. *A plane cannot cover a 2-dimensional level manifold twice.*

Proof. 1. A cylinder can cover a cylinder or a Möbius list twice as vector fields which are nowhere equal to zero exist on these types of 2-dimensional manifolds only. But going along the curve which connects two points on the covering cylinder that lie on different pre-images of a point from P we see that on P we have a cycle on which the vector of $grad\, f$ changes its direction. Thus we understand that only the Möbius list satisfies this demands.
2. If we suppose that such a covering is possible we will have an involution on the covering plane without unintact points which is impossible. ■

And now we are ready to formulate and prove the basic result of this paper:

Theorem 1. *Let M^4 be a smooth symplectical manifold and $v = sgrad\, f$ is the Hamiltonian system integrable in the sense of Liouville on some non-singular 3-dimentional manifold of the constant energy Q^3 with the help of an equi-Bott integral f. Then Q^3 can be represented as the union of the following simple sets (see Fig. 3):*

1. $M^1 \times R^1$*; the long oriented saddle, the edge - 3 cylinders;*
2. $M^1 \times S^1$*; the round oriented saddle, the edge - 3 tori;*
3. $M_1^1 \times R^1$*; the cut long oriented saddle, the edge - a cylinder and 2 planes;*
4. $M_1^1 \times S^1$*; the cut round oriented saddle, the edge - a torus and 2 cylinders;*
5. $M_1^{11} \times R^1$*; the long oriented saddle cut twice, the edge - 4 planes;*
6. $M_1^{11} \times S^1$*; the round oriented saddle cut twice, the edge - 4 cylinders;*
7. $M_2 * S^1$*; the non-oriented saddle, the edge - 2 tori;*
8. $M_1^2 * R^1$*; the cut non-oriented saddle, the edge - 2 cylinders;*
9. $T \times D^1$*; the torus-like cylinder, the edge - 2 tori;*
10. $C \times D^1$*; the cylinder-like cylinder, the edge - 2 cylinders;*
11. $P \times D^1$*; the plane-like cylinder, the edge - 2 planes;*
12. *the sub-cut cylinder-like cylinder of the first type, the edge - a cylinder and a plane;*
13. *the sub-cut cylinder-like cylinder of the second type, the edge - 3 cylinders;*
14. *the sub-cut plane-like cylinder , the edge - 3 planes;*
15. *the sub-cut torus-like cylinder , the edge - a torus and a cylinder;*
16. *the full torus; the edge - a torus;*

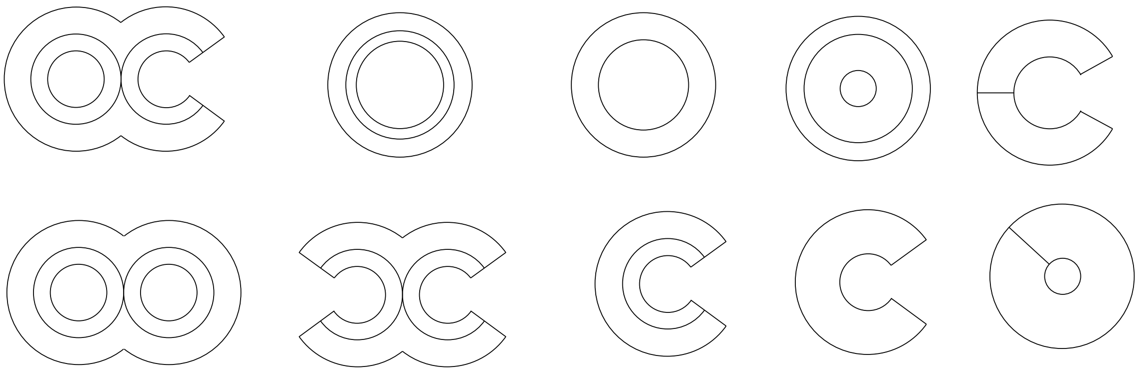

Figure 3. Models of bifurcations in the general case.

17. the full cylinder, the edge - a cylinder;

18. $KL^2\tilde{\times}I$; the edge - a torus;

19. $\mu\tilde{\times}I$; the edge - a cylinder;

20. $T \times R^1_0$; the edge - a torus;

21. $C \times R^1_0$; the edge - a cylinder;

22. $P \times R^1_0$; the edge - a plane.

And all these types are not level-by level equivalent.

Proof. The first part of the theorem follows from Lemmas 1–8. To prove the second part it is sufficient to consider any two of the mentioned manifolds. If their edges are homeomorphic (as in the case 1 and 13 for example) then either the bifurcations take place when passing through the different critical manifolds and have different phase pictures or they represent different Seifert foliations or they are evidently topologically different. ■

Lemma 9. *Let a critical level contains several saddle trajectories. Then their separatrix diagrams intersect the connected components of $B^{a+\epsilon}$ by isotopic non-intersecting cycles or curves.*

Lemma 10. *Let the critical level contain several non-intersecting ∞-curves. Then by a small deformation of f and ω we can leave an only ∞-curve on each level.*

Proof. Let us consider a neighborhood N of the isoenergetic surface Q^3. The ∞-curve transits under the action of $grad\,H \neq 0$ to the surfaces $H = const$. Let $grad^1 f$ be the projection of $grad\,f$ to these surfaces. Let us consider the neighborhoods of the ∞-curves $\Theta_i, |grad^1 f| < \delta, x \in \Theta_i \in N$. It is clear that if there are 2 ∞-curves on B^a then on N we have 2 non-intersecting areas Θ_1 and Θ_2. Let us define the deformation of f and ω on Θ_1. For this let us choose $\Theta_1^1 \in \Theta_1$ where $|grad^1 f| < \delta^1 < \delta$ and let us define a smooth function $e(x)$:

$$\begin{cases} e(x) = 0 \quad \text{when} \quad x \in N \setminus \Theta_1 \\ e(x) = 1 \quad \text{when} \quad x \in \Theta_1^1 \\ 0 \leq e(x) \leq 1 \quad \text{when} \quad x \in \Theta_1. \end{cases}$$

Let us define the transformation in N fulfilling a transit along the lines of the vector field $e(x) = grad^1 f$. Here f remains the integral of the field $sgrad\, H$. ■

Lemma 11. *Let a saddle trajectory and a ∞-curve lie on a singular level surface of the integral f. Then there is a small deformation of the integral f and the simplectical form ω which leaves the Hamiltonian system integrable and leads the trajectories to different levels.*

Proof. As f is represented in the equi-Bott form in the tube neighborhood of the saddle trajectory the angle between the separatrix diagrams does not approach zero that means that there are no critical ∞-points in this neighborhood. We can consider the bifurcations separately. ■

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