Uniqueness for Autonomous Planar Differential Equations and the Lagrangian Formulation of Water Flows with Vorticity

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Abstract

We prove a uniqueness result for autonomous divergence-free systems of ODE's in the plane and give an application to the study of water flows with vorticity.

1 A uniqueness result for differential equations

Consider the system of differential equations

$$\begin{cases} x'(t) = f(x,y) \\ y'(t) = g(x,y) \end{cases}, \qquad (x,y) \in \Omega, \tag{1.1}$$

where f and g are continuous real-valued functions on some open set $\Omega \subset R^2$, satisfying $\nabla \cdot (f,g) = f_x + g_y = 0$ in the sense of distributions.

The fact that $\nabla \cdot (f,g) = 0$ ensures in any simply connected subset $\Omega_0 \subset \Omega$ the existence of a function $\psi \in C^1(\Omega_0, R)$ which satisfies $\psi_x = -g$ and $\psi_y = f$. Note that if $(x(t), y(t))$ is a solution of the system (1.1), we have

$$\frac{d}{dt}\psi(x(t), y(t)) = \psi_x x' + \psi_y y' = -gf + fg = 0.$$

Thus ψ is constant on solutions of (1.1). We can view (1.1) as a Hamiltonian system with Hamiltonian function ψ.

We are now in a position to prove the following

Theorem 1 *Suppose that* $|f(x_0, y_0)| + |g(x_0, y_0)| > 0$*. Then there is a locally unique solution of* (1.1) *with initial data* (x_0, y_0).

Proof. The existence of a local solution follows from Peano's theorem [9]. Note that we can always solve both backwards and forwards in time to get a solution defined on some interval $(-\epsilon, \epsilon)$ with $(x(0), y(0)) = (x_0, y_0)$.

Let ψ be a Hamiltonian function defined in a neighbourhood of (x_0, y_0). We have $(\psi_x, \psi_y) = (-g, f)$. Thus, by assumption, at least one of $\psi_x(x_0, y_0)$ or $\psi_y(x_0, y_0)$ is different from zero. Assume for concreteness that $\psi_y(x_0, y_0) = f(x_0, y_0) \neq 0$. Then by the implicit function theorem, there exist open intervals (a, b) and (c, d) containing x_0 and y_0 respectively, and a C^1 function $\varphi : (a, b) \to (c, d)$, such that on $(a, b) \times (c, d)$ we have that $\psi(x, y) = \psi(x_0, y_0)$ is equivalent to $y = \varphi(x)$. This implies that all solutions are contained in the graph $(x, \varphi(x))$, locally around (x_0, y_0).

To prove that all solutions are in fact locally equal (i.e. equal in a neighbourhood of 0), we proceed as follows. Suppose that (x, y) and $(\tilde{x}, \tilde{y})$ are both solutions of (1.1) defined on some interval $(-\epsilon, \epsilon)$, with $(x(0), y(0)) = (\tilde{x}(0), \tilde{y}(0)) = (x_0, y_0)$. Since $\tilde{x}'(t) = f(\tilde{x}, \tilde{y})$ is nonzero for $t = 0$, we can assume it is nonzero throughout $(-\epsilon, \epsilon)$. We may also assume that $x((-\epsilon, \epsilon)) \subset \tilde{x}(-\epsilon, \epsilon))$ and that $(x, y), (\tilde{x}, \tilde{y}) \in (a, b) \times (c, d)$ for all $t \in (-\epsilon, \epsilon)$.

We then have

$$\begin{aligned} \frac{d}{dt}\tilde{x}^{-1}(x(t)) &= \frac{x'(t)}{\tilde{x}'(\tilde{x}^{-1}(x(t)))} = \frac{f(x(t), y(t))}{f(\tilde{x}((\tilde{x}^{-1} \circ x)(t)), \tilde{y}((\tilde{x}^{-1} \circ x)(t)))} \\ &= \frac{f(x(t), \varphi(x(t)))}{f(x(t), \varphi(x(t)))} = 1. \end{aligned} \tag{1.2}$$

Thus $\tilde{x}^{-1}(x(t)) = t + k$, for some $k \in R$. Since $\tilde{x}^{-1}(x(0)) = 0$, we have $k = 0$ and $\tilde{x}^{-1}(x(t)) = t$. But then

$$(x(t), y(t)) = (x(t), \varphi(x(t))) = (\tilde{x}(t), \varphi(\tilde{x}(t))) = (\tilde{x}(t), \tilde{y}(t)),\ t \in (-\epsilon, \epsilon).$$

This proves the statement for the case $\psi_y(x_0, y_0) \neq 0$. The case $\psi_x(x_0, y_0) \neq 0$ can be handled similarly. □

Remark 1 (i) Our approach uses the Hamiltonian structure of the system (1.1) and does not rely upon the use of integral inequalities or comparison methods. For this reason, it is not contained within the plethora of uniqueness results for ordinary differential equations presented in [1].

(ii) If there are no critical points throughout Ω, global uniqueness holds. I.e. two solutions with the same initial data are equal on their common domain of definition.

(iii) Suppose that the functions u, v extend continuously to the closure $\overline{\Omega}$ of Ω, and that Γ is some part of the boundary which does not contain critical points of (1.1), given by the graph of a C^1-function $\eta : (a, b) \to R$. Furthermore suppose that we know that a solution with initial data $(x_0, y_0) \in \Gamma$ stays in Γ. Then uniqueness holds also at (x_0, y_0). Indeed using the relation (1.2) with η instead of φ proves our claim. □

The following example shows that if we drop the condition that (x_0, y_0) is not a critical point of (1.1), uniqueness is no longer guaranteed.

Example 1 Take $\Omega = R^2$, $\Big(f(x, y), g(x, y)\Big) = \Big(\sqrt{|y|}, \sqrt{|x|}\Big)$. In this case we have that (up to a constant) $\psi(x, y) = -\frac{2}{3}\,\mathrm{sign}(x)\,|x|^{3/2} + \frac{2}{3}\,\mathrm{sign}(y)\,|y|^{3/2}$. Clearly $f(0, 0) = g(0, 0) = 0$. Other than the trivial solution $(x(t), y(t)) = (0, 0),\ t \in R$, we also have the solution $(x(t), y(t)) = (\frac{1}{4}t|t|, \frac{1}{4}t|t|),\ t \in R$. □

Despite the previous example, there are situations when a result similar to Theorem 1 holds even in the case when critical points of (1.1) are allowed.

Theorem 2 *Let ψ be a Hamiltonian function for* (1.1)*. Assume that all critical points of* (1.1) *are isolated and for each critical point* $(x_0, y_0) \in \Omega$*, there is a neighbourhood of it in which the equation $\psi(x, y) = \psi(x_0, y_0)$ has the unique solution $(x, y) = (x_0, y_0)$. Then there is a unique solution of* (1.1) *for any initial data* $(x(0), y(0)) \in \Omega$*.*

Proof. It is enough to prove that locally all solutions are unique. If the initial data $(x(0), y(0)) \in \Omega$ is not a critical point of (1.1), then the argument used in the proof of Theorem 1 is valid. On the other hand, our hypotheses ensure that if the initial data is a critical point (x_0, y_0) of (1.1), then the unique solution with this data is $(x(t), y(t)) = (x_0, y_0)$, $t \in R$. □

We now show the applicability of Theorem 2.

Example 2 Let $\Omega = R^2$ and let $g : R \to R$ be a continuous function satisfying $g(0) = 0$ and $s\,g(s) < 0$ for $s \neq 0$. Then for any initial data $(x_0, y_0) \in R^2$, there is a unique solution of the system

$$\begin{cases} x' = y, \\ y' = g(x). \end{cases} \tag{1.3}$$

Indeed, note that $\psi(x, y) = \frac{y^2}{2} - \int_0^x g(s)\,ds$ is a Hamiltonian function for (1.3). Our assumptions guarantee that the only critical point of (1.3) is $(0, 0)$ and, moreover, the equation $\psi(x, y) = \psi(0, 0)$ has the unique solution $(x, y) = (0, 0)$. Hence the above uniqueness statement is a consequence of Theorem 2. □

Remark 2 As already pointed out in Remark 1, our approach does not depend upon the use of integral inequalities. This fact is emphasized by Example 2, where except the sign condition $s\,g(s) < 0$, $s \neq 0$, no further properties of g are used. □

As a concrete case to which the considerations made in Example 2 apply, consider the function $g(s) = -\text{sign}(s)\sqrt{|s|}$, $s \in R$. The next example shows that with a slight modification of g uniqueness no longer holds.

Example 3 Again let $\Omega = R^2$ and consider the system

$$\begin{cases} x' = y, \\ y' = \sqrt{|x|}. \end{cases}$$

It is easy to check that $(x(t), y(t)) = (0, 0),\ t \in R$ and $(x(t), y(t)) = (t^4/12^2, t^3/6^2),\ t \in R$ are both solutions with initial data $(0, 0)$. □

2 Application to water flows with vorticity

The motivation for the results of the previous section is from the theory of steady two-dimensional water flows. We will now describe the physical setting and also the relevance of our results.

We consider the propagation of two-dimensional periodic inviscid gravity waves at the surface of a layer of water that is either unbounded (deep water waves) or with a flat bottom (shallow water waves). In its undisturbed state the equation of the flat surface is $y = 0$ and the flat bottom is given by $y = -d$ for some $d > 0$ (for infinite depth, we let $d = \infty$). In the presence of waves, let $y = \eta(t, x)$ be the free surface and let $(u(t, x, y), v(t, x, y))$ be the velocity field. Since the motion is identical in any direction orthogonal to the direction of propagation of the wave, a complete description is obtained by analyzing a cross-section of the flow, perpendicular to the crest line. We choose Cartesian co-ordinates (x, y) so that the horizontal x-axis is in the direction of wave propagation and the y-axis points vertically upwards. The governing equations [11] are the equation of mass conservation

$$u_x + v_y = 0, \tag{2.1}$$

and Euler's equation

$$\begin{cases} u_t + uu_x + vu_y = -P_x, \\ v_t + uv_x + vv_y = -P_y - g, \end{cases} \tag{2.2}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration. The boundary conditions for the water wave problem are

$$P = P_0 \quad \text{on} \quad y = \eta(t, x), \tag{2.3}$$

P_0 being the constant atmospheric pressure,

$$v = \eta_t + u\,\eta_x \quad \text{on} \quad y = \eta(t, x), \tag{2.4}$$

and

$$v = 0 \quad \text{on} \quad y = -d. \tag{2.5}$$

Equation (2.3) decouples the motion of the air from that of the water (neglecting the effects of surface tension, as is proper for the description of gravity waves), while (2.4)-(2.5) express the fact that the same particles always form the free surface, respectively, the fact that it is impossible for the water to penetrate the rigid bed. In the deep-water regime, characterized by the fact that the motion is confined to near-surface water layers, the boundary condition (2.5) has to be replaced by

$$(u, v) \to (0, 0) \quad \text{as} \quad y \to -\infty \quad \text{uniformly for } x \in R, \tag{2.6}$$

guaranteeing that at great depths there is practically no motion.

Given $c > 0$, we are considering periodic waves traveling at speed c, that is, the space-time dependence of the free surface, of the pressure and of the velocity field has the form $(x - ct)$. The problem can be simplified by the change of frame $(x - ct, y) \mapsto (x, y)$. In the new reference frame, in which the origin moves in the direction of propagation of the wave with the wave speed c, the wave is stationary and the flow is steady. In this moving reference frame a stream function ψ is determined by requiring

$$\psi_x = -v, \quad \psi_y = u - c. \tag{2.7}$$

The equations of motion (2.2) with the corresponding boundary conditions (2.3)-(2.5) or (2.3)-(2.6) become

$$\begin{cases} (u-c)u_x + vu_y = -P_x, \\ (u-c)v_x + vv_y = -P_y - g, \end{cases} \quad \text{on} \quad -d < y < \eta(x), \tag{2.8}$$

and (for shallow water waves)

$$\begin{cases} v = (u-c)\eta_x \quad \text{at} \quad y = \eta(x), \\ P = P_0 \quad \text{at} \quad y = \eta(x), \\ v = 0 \quad \text{at} \quad y = -d, \end{cases} \tag{2.9}$$

or (for deep water)

$$\begin{cases} v = (u-c)\eta_x \quad \text{at} \quad y = \eta(x), \\ P = P_0 \quad \text{at} \quad y = \eta(x), \\ (u,v) \to (0,0) \quad \text{as} \quad y \to -\infty \quad \text{uniformly for } x \in R. \end{cases}$$

Let

$$\omega = v_x - u_y \tag{2.10}$$

be the vorticity of the flow. The existence of water waves for a given vorticity was recently considered in [7] [8] - see also [6] for symmetry properties of these wave patterns. In these investigations the function ω was assumed to be continuously differentiable. For the study of water flows with a discontinuous vorticity distribution (e.g. taking two different values - a situation encountered in the description of the interaction between swell and a coastal current cf. [12]), it is essential to ensure that the problem is well-posed. That is, adopting the Lagrangian viewpoint, particles do not collide. The location $(x(t), y(t))$ at time $t > 0$ of a particle with position (x_0, y_0) at time $t = 0$ is found by solving the system

$$\begin{cases} x'(t) = u(t, x(t), y(t)), \\ y'(t) = v(t, x(t), y(t)). \end{cases}$$

Passing to the moving frame, we see that the Lagrangian formulation is valid if uniqueness holds for the solutions of the differential system

$$\begin{cases} x' = u(x,y) - c, \\ y' = v(x,y). \end{cases} \tag{2.11}$$

A classical result (see [2]) in this direction, for flows without a free boundary (i.e. the water fills the whole of R^2) is due to Yudovich [15]: uniqueness is ensured if $\omega \in L^1(R^2) \cap L^\infty(R^2)$. Yudovich's result actually holds true under the weaker condition $\omega \in L^p(R^2) \cap$

$L^\infty(R^2)$ for some $p \in (1,\infty)$ - see [3]. Note that, setting $f = u - c$ and $g = v$, the results of Section 1 are applicable to (2.11). Critical points of (2.11) correspond to stagnation points of the flow. For waves not near the spilling or breaking state it is natural to assume that the horizontal velocity component, u, of the water particles is considerably smaller than the propagation speed, c, of the surface wave (see [13], [14]). This implies that there are no stagnation points of the flow, so that we can use Theorem 1 to conclude uniqueness within the fluid domain, and Remark 1 (iii) to conclude uniqueness on the free surface, $y = \eta(x)$, or on the flat bottom, $y = -d$.

Example 4 Consider for an arbitrary constant $-1 \le a \le 0$, the velocity field

$$\begin{cases} u(x,y) - c = \sqrt{|y-a|} - 2 \\ v(x,y) = 0 \end{cases}, \qquad (x,y) \in \Omega, \tag{2.12}$$

where $\Omega = \{(x,y) \in R^2 \,|\, -1 < y < 0\}$. It is clear that $(u-c, v)$ is divergence-free and satisfies the Euler equation (2.8) with the hydrostatic pressure $P(x,y) = P_0 - gy$. Furthermore, the boundary conditions (2.9) are satisfied with $\eta(x) = 0$ and $d = 1$. Since the velocity field has no stagnation point we can use Theorem 1 and Remark 1 (iii) to conclude that the flow (2.12) is well-defined. Note that u_y is singular at $y = a$. If $a = 0$ this singularity occurs on the free surface, if $a = -1$ it occurs on the flat bottom and if $-1 < a < 0$ it occurs within the fluid domain. Note also that the equation

$$\begin{cases} x'(t) = \sqrt{|y-a|} - 2, \\ y'(t) = 0, \end{cases}$$

has a unique solution for any initial data $(x_0, y_0) \in \Omega$, given explicitely by $x(t) = (\sqrt{|y_0 - a|} - 2)t + x_0$, $y(t) = y_0$. □

Remark 3 Note that the vorticity $\omega = v_x - u_y = -\frac{1}{2}\,\text{sign}(y-a)\,|y-a|^{-1/2} \notin L^\infty(\Omega)$, so that we cannot use a result analogous to Yudovich's theorem. We would also like to point out that the proof of Yudovich's theorem (see [2], [3]) uses integral inequalities by proving the validity of an Osgood condition. □

In Example 4 we presented a laminar flow i.e. each particle path is parallel to the flat bottom. In particular, the free surface is flat. This leads naturally to the question of whether there is a flow with a non-flat free surface and, say, unbounded vorticity (so that a result analogous to Yudovich's theorem is not applicable) for which the results in Section 1 are conclusive.

Example 5 Gerstner's limiting wave [10] is a periodic deep water wave whose free surface has the profile of a cycloid. The flow has stagnation points at the crest of the wave and the vorticity is unbounded (see [4]). The water flow below the free surface enters therefore into the framework of the uniqueness results presented in Section 1. Note that the presence of stagnation points on the free boundary prevents an application of Remark 1 (iii) to conclude that the uniqueness holds also on the free boundary. As a matter of fact, uniqueness does not hold on the free boundary for the corresponding system (2.11) and it is ensured by requiring that in the original frame, each particle on the surface describes a circle and does not stagnate at the crest. □

Remark 4 Our approach is basically two-dimensional. However, there are special genuine three-dimensional waves where each particle path lies in a plane and our methods are applicable (see e.g. [5]). □

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