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Traveling Wave Solutions of the Camassa-Holm and Korteweg-de Vries Equations

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Abstract

We show that the smooth traveling waves of the Camassa-Holm equation naturally correspond to traveling waves of the Korteweg-de Vries equation.

1 Introduction

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x + 2ku_x = 2u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

and the Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

are models for the propagation of shallow water waves, $u(x,t)$ representing the water's free surface in non-dimensional variables, and $k \in \mathbb{R}$ being a parameter related to the critical shallow water speed [3, 25]. The well-known KdV equation - the simplest equation embodying both nonlinearity and dispersion - has attracted enormous attention over the years and has served as the model equation for the development of soliton theory (see [25]). Equation (1.1) was derived as a model for shallow water waves [3], and was subsequently recognized as having been included implicitly as an abstract equation in [30] via the method of recursion operators (see [29]). For a derivation of the Camassa-Holm equation using Kodama's normal form transformations we refer to [26, 27] and for discussions of its relation to shallow water waves see [26, 27, 10, 33, 35]. Note also that (1.1) models the propagation of nonlinear waves in a cylindrical axially symmetric hyperelastic rod, $u(x,t)$ representing the radial stretch relative to a prestressed state [23].

The equations (1.1) and (1.2) have plenty of structure tied into them. KdV is a completely integrable bi-Hamiltonian equation (see [25, 41]), its solitary waves are solitons [25], and the Cauchy problem for (1.1) is globally well-posed even for rough initial data [5]. On the other hand, the bi-Hamiltonian structure and the isospectral problem for (1.2) were found in [3]. The Camassa-Holm equation is completely integrable: for the periodic case see [7, 19], and for aspects of the direct/inverse scattering see [2, 9, 18, 22, 34, 36]. The solitary waves of (1.2) for $k > 0$ are smooth solitons [22, 34] while in the limiting case $k = 0$ they are peaked solitons (peakons) [3] which have to be understood as weak solutions [15, 20, 44]. The peakons are stable wave patterns [21, 37]. The Camassa-Holm equation models wave breaking [3, 4, 6, 8, 11, 13, 14, 24, 39, 42, 45] and admits wave solutions that exist indefinitely in time [6, 8, 12, 13]. Additionally, (1.1) can be interpreted as Euler's equation for the geodesic flow on the Lie group of compressible diffeomorphisms of the circle [40] (see also [32] for a discussion). This can be used to prove that (1.1) satisfies the Least Action Principle [16, 17]: solutions move along uniquely determined paths that locally minimize energy. Let us also point out that associated to each of the equations (1.1)-(1.2) there is a whole hierarchy of integrable equations [31, 41].

Of particular interest among the solutions to (1.1) and (1.2) are traveling wave solutions $u(x, t) = \varphi(x - ct)$ for some function φ. Through a connection between the isospectral problems associated to (1.1) and (1.2), we will construct a natural mapping taking smooth traveling waves of (1.1) into traveling waves of (1.2).

We need two lemmas on the existence of traveling waves of (1.1) and (1.2). In [38] the traveling waves of (1.1) are classified according to their minimum, maximum, and speed.

Lemma 1. *Fix $k \in \mathbb{R}$ and let $z = c - 2k - M - m$. Then*

(a) (Smooth periodic) If $z < m < M < c$, there is a smooth periodic traveling wave $\varphi(x - ct)$ of (1.1) with $m = \min_{x\in\mathbb{R}} \varphi(x)$ and $M = \max_{x\in\mathbb{R}} \varphi(x)$.

(b) (Smooth with decay) If $z = m < M < c$, there is a smooth traveling wave $\varphi(x - ct)$ of (1.1) with $m = \inf_{x\in\mathbb{R}} \varphi(x)$, $M = \max_{x\in\mathbb{R}} \varphi(x)$, and $\varphi \downarrow m$ exponentially as $x \to \pm\infty$.

(a′) (Smooth periodic) If $z > m > M > c$, there is a smooth periodic traveling wave $\varphi(x - ct)$ of (1.1) with $M = \min_{x\in\mathbb{R}} \varphi(x)$ and $m = \max_{x\in\mathbb{R}} \varphi(x)$.

(b′) (Smooth with decay) If $z = m > M > c$, there is a smooth traveling wave $\varphi(x - ct)$ of (1.1) with $M = \min_{x\in\mathbb{R}} \varphi(x)$, $m = \sup_{x\in\mathbb{R}} \varphi(x)$, and $\varphi \uparrow m$ exponentially as $x \to \pm\infty$. Moreover, these are all bounded smooth traveling waves of the Camassa-Holm equation.

The next lemma classifies the traveling waves of the KdV equation [25]. Henceforth, to distinguish between solutions of the two equations, we let y be the space variable for the KdV equation and x the space variable for the Camassa-Holm equation.

Lemma 2. *Let $\bar{z} = -\bar{c}/2 - \bar{m} - \bar{M}$. Then*

(i) (Cnoidal waves) If $\bar{m} < \bar{M} < \bar{z}$ there is a smooth periodic traveling wave $Q(y - \bar{c}t)$ of the KdV equation with $\bar{m} = \min_{y\in\mathbb{R}} Q(y)$ and $\bar{M} = \max_{y\in\mathbb{R}} Q(y)$.

(ii) (sech^2-profiles) If $\bar{m} < \bar{M} = \bar{z}$ there is a smooth traveling wave $Q(y - \bar{c}t)$ of the KdV equation with $\bar{m} = \min_{y\in\mathbb{R}} Q(y)$, $\bar{M} = \sup_{y\in\mathbb{R}} Q(y)$ and $Q \uparrow \bar{M}$ exponentially as $y \to \pm\infty$. Moreover, these are all bounded traveling waves of the KdV equation.

Before stating our main result we introduce some notation. Let $(m, M, c) \in \mathbb{R}^3$ be coordinates in $\mathbb{R}^3$ and let $z = c - 2k - M - m$. Define $\Gamma_+ = \{z < m < M < c\}$ and $\Gamma_- = \{c < M < m < z\}$, where we write $\{z < m < M < c\}$ instead of $\{(m, M, c) \in \mathbb{R}^3 : z < m < M < c\}$. Note that Γ_+ and Γ_-, being enclosed by the three planes $\{z = m\}$, $\{m = M\}$, and $\{M = c\}$, are shaped as tetrahedrons with common corner at $(-k, -k, -k)$. Also, $\Gamma = \Gamma_+ \cup \Gamma_-$ is symmetric around $(-k, -k, -k)$. For each c, the section Γ_c of all

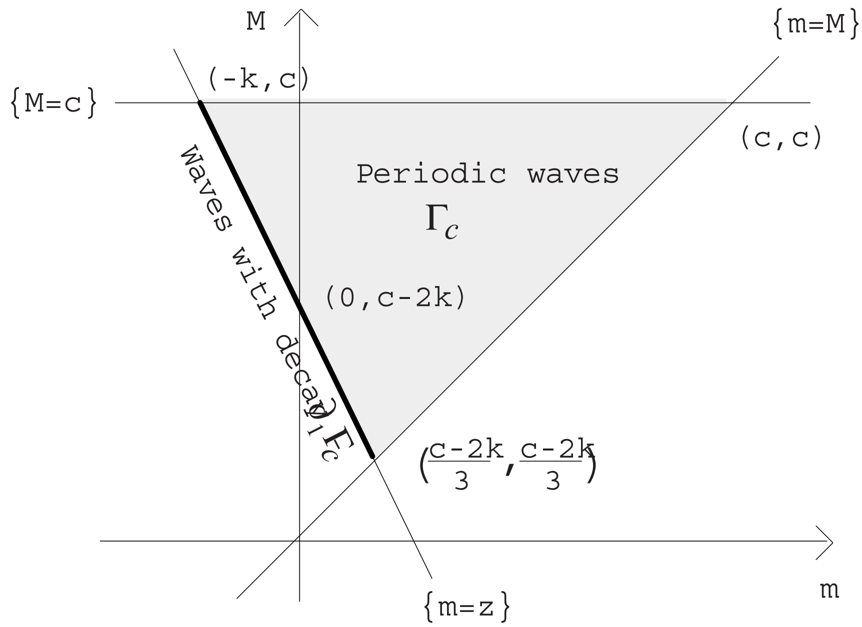

Figure 1. The sets Γ_c and $\partial_1\Gamma_c$ for $c > -k$.

points in Γ with third coordinate c is a triangle - see Figure 1. In fact, $\Gamma_+ \subset \{c > -k\}$ and $\Gamma_- \subset \{c < -k\}$. The cases (a) respectively (a') of Lemma 1 occur exactly when $(m, M, c) \in \Gamma_+$ respectively $(m, M, c) \in \Gamma_-$. Moreover, if $\partial_1\Gamma = \{z = m < M < c\}$ is one of the three edges of Γ, then the cases (b) and (b') of Lemma 1 occur whenever $(m, M, c) \in \partial_1\Gamma$.

Henceforth s_k will be defined as

$$s_k = \begin{cases} 1 & k \neq 0, \\ 0 & k = 0. \end{cases} \tag{1.3}$$

With $(\bar{m}, \bar{M}, \bar{c})$ as coordinates in $\mathbb{R}^3$ and $\bar{z} = -\bar{c}/2 - \bar{m} - \bar{M}$, we define $\Pi = \{\bar{m} < \bar{M} < \bar{z}\} \cap \{\bar{m} + \bar{M} > -\frac{s_k}{2k}\} \subset \mathbb{R}^3$. Π is a tetrahedron with corner at $(\bar{m}, \bar{M}, \bar{c}) = (-\frac{s_k}{4k}, -\frac{s_k}{4k}, -\frac{s_k}{4k})$, enclosed by the three planes $\{\bar{m} = \bar{M}\}$, $\{\bar{M} = \bar{z}\}$, and $\{\bar{m} + \bar{M} = -\frac{s_k}{2k}\}$. Note that $\Pi \subset \{\bar{c} < -\frac{s_k}{4k}\}$. To any point in Π, a periodic traveling wave of KdV is associated according to Lemma 2. Similarly, every point in $\partial_1\Pi = \{\bar{m} < \bar{M} = \bar{z}\} \cap \partial\Pi$ has a corresponding traveling wave of KdV with decay at infinity.

Now we are ready to state our main result: a bijective correspondence between traveling waves of (1.1) corresponding to $(m, M, c) \in \Gamma_+ \cup \partial_1\Gamma_+$, and traveling waves of (1.2) corresponding to $(\bar{m}, \bar{M}, \bar{c}) \in \Pi \cup \partial_1\Pi$.

Theorem *Fix $k \in \mathbb{R}$. Let $(m, M, c) \in \Gamma_+ \cup \partial_1\Gamma_+$, and let φ be the corresponding smooth periodic traveling wave of the Camassa-Holm equation (1.1). Similarly, let $(\bar{m}, \bar{M}, \bar{c}) \in$*

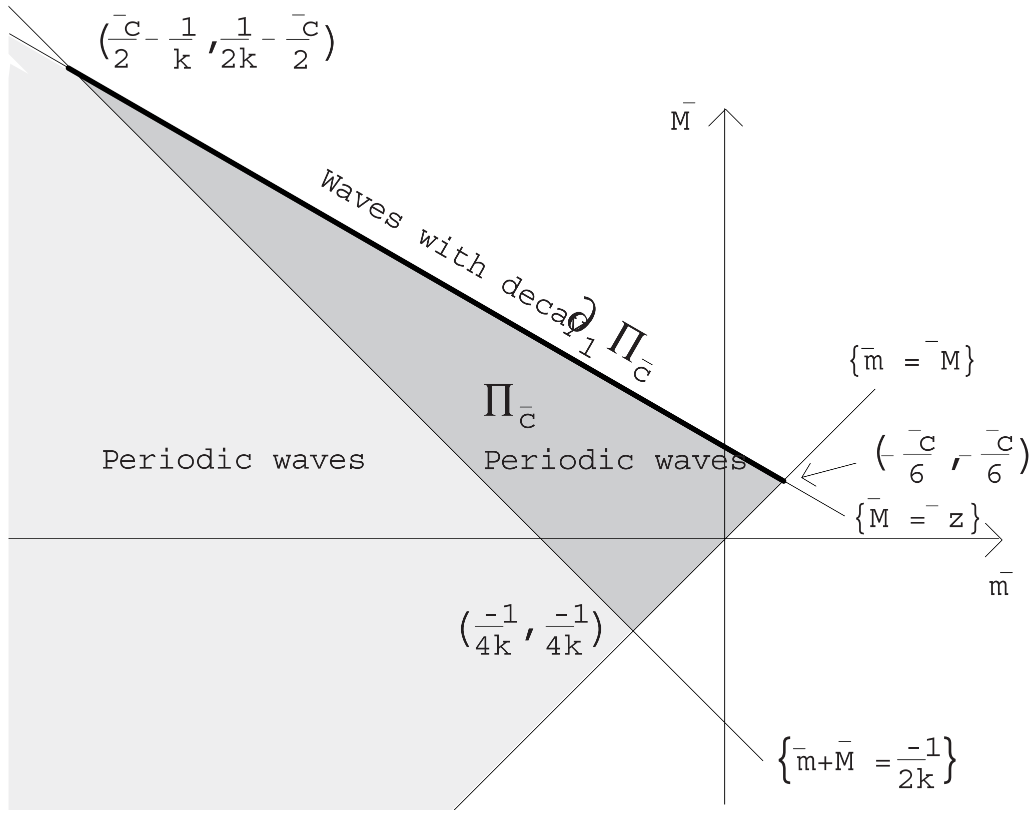

Figure 2. For each $\bar c < -\frac{s_k}{4k}$, the section $\Pi_{\bar c}$ is a triangle.

$\Pi \cup \partial_1 \Pi$ *and let* Q *be the corresponding periodic traveling wave of KdV. Then*

$$Q(y) = \gamma - \frac{1}{c - \varphi(x)}, \quad y = \int_{x_0}^{x} \sqrt{\varphi(\xi) - \varphi_{xx}(\xi) + k}\, d\xi, \tag{1.4}$$

where

$$\bar{m} = \gamma - \frac{1}{c - M}, \quad \bar{M} = \gamma - \frac{1}{c - m}, \quad \bar{c} = \frac{s_k}{k} - \gamma, \tag{1.5}$$

s_k *is given by* (1.3), *and*

$$\gamma = \frac{1}{2}\left(\frac{1}{2k + M + m} + \frac{1}{c - m} + \frac{1}{c - M} - \frac{s_k}{2k}\right). \tag{1.6}$$

The mapping $(m, M, c) \mapsto (\bar{m}, \bar{M}, \bar{c})$ *given by (1.5) is a bijection, mapping* Γ_+ *onto* Π *and* $\partial_1\Gamma_+$ *onto* $\partial_1\Pi$*, with inverse*

$$m = c - \frac{1}{\bar{m} + \bar{z} + s_k/2k}, \quad M = c - \frac{1}{\bar{M} + \bar{z} + s_k/2k}, \tag{1.7}$$

$$c = \frac{1}{2}\left(\frac{1}{\bar{m} + \bar{z} + s_k/2k} + \frac{1}{\bar{M} + \bar{z} + s_k/2k} + \frac{1}{\bar{m} + \bar{M} + s_k/2k} - 2k\right).$$

Notice that if $\varphi(x)$ is a traveling wave of (1.1), then also $x \mapsto -\varphi(-x)$ is a traveling wave of (1.1) with k, respectively c, replaced by $-k$, respectively $-c$. In particular, replacing

$$(\varphi(x), c, M, m, k) \mapsto (-\varphi(-x), -c, -M, -m, -k), \tag{1.8}$$

transforms the statements $(a'), (b')$ of Lemma 1 into $(a), (b)$. In the geometrical interpretation the replacements (1.8) amount to mapping Γ_- corresponding to k, into $\Gamma+$ corresponding to $-k$. Combining (1.8) with our theorem we get a bijection between Γ_- (corresponding to k) and Π (corresponding to $-k$).

Remark The Camassa-Holm equation admits peaked and cusped weak traveling wave solutions (see [3, 4], respectively [38]), whereas all traveling waves of (1.2) are smooth. As we approach these peaked and cusped solutions the correspondence between solutions of (1.1) and (1.2) breaks down as the amplitude of the corresponding traveling waves of (1.2) blows up.

2 Proof of the Theorem

Let $u(x, t) = \varphi(x - ct)$ be a traveling wave solution of (1.1) corresponding to

$$(m, M, c) \in \Gamma_+ \cup \partial_1\Gamma_+ = \{z \le m < M < c\}. \tag{2.1}$$

We get from (1.1) that

$$-c\varphi_x + c\varphi_{xxx} + 3\varphi\varphi_x + 2k\varphi_x = 2\varphi_x\varphi_{xx} + \varphi\varphi_{xxx}. \tag{2.2}$$

Integrate to get

$$-c\varphi + c\varphi_{xx} + \frac{3}{2}\varphi^2 + 2k\varphi = \varphi\varphi_{xx} + \frac{1}{2}\varphi_x^2 + \frac{a}{2}, \tag{2.3}$$

for some constant a. We multiply by $2\varphi_x$ and integrate again to obtain

$$-c\varphi^2 + c\varphi_x^2 + \varphi^3 + 2k\varphi^2 = \varphi\varphi_x^2 + a\varphi + b, \tag{2.4}$$

where b is another constant. We write this as

$$\varphi_x^2(c-\varphi) = P(\varphi), \tag{2.5}$$

where P is the third order polynomial

$$P(\varphi) = \varphi^2(c - 2k - \varphi) + a\varphi + b. \tag{2.6}$$

Since $m = \min_{x\in\mathbb{R}} \varphi(x)$ and $M = \max_{x\in\mathbb{R}} \varphi(x)$, by the smoothness of φ it is easy to see that $\varphi_x \to 0$ as $\varphi \to m$ or $\varphi \to M$. Therefore, m and M are zeros of P. We denote the third zero by z so that

$$P(\varphi) = (M-\varphi)(\varphi-m)(\varphi-z). \tag{2.7}$$

Hence (2.6) becomes

$$\varphi_x^2(c-\varphi) = (M-\varphi)(\varphi-m)(\varphi-z). \tag{2.8}$$

Now we differentiate and divide by φ_x to find

$$2\varphi_{xx}(c-\varphi) - \varphi_x^2 = -(\varphi-m)(\varphi-z) + (M-\varphi)(\varphi-z) + (M-\varphi)(\varphi-m).$$

Identifying the coefficients of φ^2 in (2.6) and (2.7), we get $z = c - 2k - M - m$. Hence

$$\begin{aligned} 2\varphi_{xx}(c-\varphi) - \varphi_x^2 &= (\varphi-m)(c-\varphi) - (\varphi-m)(2k+M+m) - (M-\varphi)(c-\varphi) \\ &\quad +(M-\varphi)(2k+M+m) + (M-\varphi)(\varphi-m). \end{aligned} \tag{2.9}$$

We rearrange terms to infer that

$$\begin{aligned} 2(\varphi_{xx} - \varphi + \frac{M+m}{2})(c-\varphi) &= \varphi_x^2 + (M+m-2\varphi)(2k+M+m) \\ &+ (M-\varphi)(\varphi-m). \end{aligned}$$

Using (2.8) we rewrite this as

$$\begin{aligned} (2\varphi_{xx} - 2\varphi + M + m)(c-\varphi) &= \frac{(M-\varphi)(\varphi-m)(\varphi-z)}{c-\varphi} \\ &\quad +(M+m-2\varphi)(2k+M+m) + (M-\varphi)(\varphi-m). \end{aligned}$$

Multiply by $c-\varphi$ and recall that $z = c - 2k - M - m$, to obtain

$$\begin{aligned} (2\varphi_{xx} - 2\varphi + M + m)(c-\varphi)^2 &= (M-\varphi)(\varphi-m)(2k+M+m) \\ &+ (M+m-2\varphi)(2k+M+m)(c-\varphi). \end{aligned}$$

Note that the right hand side can be rewritten as

$$(2k+M+m)\big((M-c)(c-m)+(c-\varphi)^2\big),$$

and so

$$2(\varphi_{xx}-\varphi-k)(c-\varphi)^2=(2k+M+m)(M-c)(c-m).$$

We conclude that

$$\varphi-\varphi_{xx}+k=\frac{(2k+M+m)(c-M)(c-m)}{2(c-\varphi)^2}.$$

With $w=\varphi-\varphi_{xx}+k$ and $\alpha=(2k+M+m)(c-M)(c-m)>0$ we can write this as

$$w=\frac{\alpha}{2(c-\varphi)^2}. \tag{2.10}$$

Physically w has the interpretation of momentum [3, 32].

Notice that (2.10) and (2.1) imply $w(x)>0$ for all x. Therefore we may perform the following Liouville transformation. Let

$$y=\int_{x_0}^x\sqrt{w(\xi)}d\xi,\qquad dy=\sqrt{w(x)}dx. \tag{2.11}$$

The substitution

$$\phi(y)=w(x)^{1/4}\psi(x)$$

converts the isospectral problem for the Camassa-Holm equation

$$\psi_{xx}=\frac{1}{4}\psi+\lambda w\psi,$$

into

$$-\phi_{yy}+Q(y)\phi=\mu\phi,$$

where

$$Q(y)=\frac{1}{4q(y)}+\frac{q_{yy}(y)}{4q(y)}-\frac{3q_y^2(y)}{16q^2(y)}-\frac{s_k}{4k},\quad q(y)=w(x),\quad \mu=-\frac{s_k}{4k}-\lambda,$$

and s_k is defined in (1.3). As a function of x, we have

$$Q(y)=\frac{1}{4w(x)}+\frac{w_{xx}(x)}{4w^2(x)}-\frac{5w_x^2(x)}{16w^3(x)}-\frac{s_k}{4k}. \tag{2.12}$$

Differentiating (2.10) we find

$$w_x=\frac{\alpha\varphi_x}{(c-\varphi)^3},\qquad w_{xx}=\alpha\left(\frac{\varphi_{xx}}{(c-\varphi)^3}+\frac{3\varphi_x^2}{(c-\varphi)^4}\right).$$

Putting this into (2.12) yields

$$Q(y)+\frac{s_k}{4k}=\frac{1}{2\alpha}\big((c-\varphi)^2+2\varphi_{xx}(c-\varphi)+\varphi_x^2\big).$$

We may simplify this further by noting that (2.9) can be written as

$$\begin{aligned}2\varphi_{xx}(c-\varphi) &= \varphi_x^2 + (M+m-2\varphi)(2k+M+m)\\ &+ (2\varphi - M - m)(c-\varphi) + (M-\varphi)(\varphi - m).\end{aligned}$$

Therefore

$$Q(y) + \frac{s_k}{4k} = \frac{1}{2\alpha(c-\varphi)}\big((c-\varphi)^3 + 2\varphi_x^2(c-\varphi) + (M+m-2\varphi)(2k+M+m)(c-\varphi)$$

$$+ (2\varphi - M - m)(c-\varphi)^2 + (M-\varphi)(\varphi-m)(c-\varphi)\big).$$

Since $z = c - 2k - M - m$, (2.8) gives us

$$\varphi_x^2(c-\varphi) = -(M-\varphi)(\varphi-m)(c-\varphi) + (2k+M+m)(M-\varphi)(\varphi-m).$$

Therefore

$$Q(y) + \frac{s_k}{4k} = \frac{1}{2\alpha}\big((c-\varphi)^2 - 2(M-\varphi)(\varphi-m) + (2\varphi-M-m)(c-\varphi) + (M-\varphi)(\varphi-m)\big)$$

$$+ \frac{2k+M+m}{2\alpha(c-\varphi)}\big((M+m-2\varphi)(c-\varphi) + 2(M-\varphi)(\varphi-m)\big).$$

This can be simplified to

$$Q(y) + \frac{s_k}{4k} = \frac{(c-M)(c-m)}{2\alpha}\frac{(2k+M+m)(c-M)(m-\varphi)}{2\alpha(c-\varphi)}$$

$$+ \frac{(2k+M+m)(c-m)(M-\varphi)}{2\alpha(c-\varphi)}.$$

Recall that $\alpha = (2k+M+m)(c-M)(c-m)$. Hence

$$Q(y) + \frac{s_k}{4k} \quad = \quad \frac{1}{2(2k+M+m)}\frac{(m-\varphi)}{2(c-m)(c-\varphi)} + \frac{(M-\varphi)}{2(c-M)(c-\varphi)}.$$

We conclude

$$2Q(y) = -\frac{2}{c-\varphi} + \frac{1}{(2k+M+m)}\frac{1}{(c-m)} + \frac{1}{(c-M)} - \frac{s_k}{2k}. \tag{2.13}$$

To establish (1.4) of the Theorem we have to show that $Q(y)$ is a traveling wave of (1.2) with minimum $\bar{m}$, maximum $\bar{M}$ and speed $\bar{c}$ as in (1.5). We first prove a lemma.

Lemma 3. *Let $Q(y)$ be a smooth function with $\bar{m} = \min_{y\in\mathbb{R}} Q(y)$ and $\bar{M} = \max_{y\in\mathbb{R}} Q(y)$. Then $u(y,t) = Q(y - \bar{c}t)$ is a traveling wave of KdV with speed $\bar{c}$ if and only if*

$$\frac{1}{2}Q_y^2 = (\bar{M} - Q)(Q - \bar{m})(\bar{z} - Q), \tag{2.14}$$

for $\bar{z} = -\frac{\bar{c}}{2} - \bar{M} - \bar{m}$.

Proof. $Q(y-\bar{c}t)$ is a traveling wave if and only if

$$-\bar{c}Q_y - 6QQ_y + Q_{yyy} = 0. \tag{2.15}$$

We integrate to get

$$-\bar{c}Q - 3Q^2 + Q_{yy} = A, \tag{2.16}$$

for a constant A. Multiply by Q_y and integrate to find that

$$-\frac{\bar{c}}{2}Q^2 - Q^3 + \frac{1}{2}Q_y^2 = AQ + B, \tag{2.17}$$

for some constant B. Thus Q satisfies the equation

$$\frac{1}{2}Q_y^2 = Q^3 + \frac{\bar{c}}{2}Q^2 + AQ + B. \tag{2.18}$$

Let $\bar{m} = \min_{y\in\mathbb{R}} Q(y)$ and $\bar{M} = \max_{y\in\mathbb{R}} Q(y)$. Since Q is smooth, $Q_y \to 0$ as $Q \to \bar{m}$ or $Q \to \bar{M}$. Therefore the polynomial $Q^3 + \frac{\bar{c}}{2}Q^2 + AQ + B$ has zeros at $\bar{m}$ and $\bar{M}$. Let $\bar{z}$ be the third zero, so that

$$Q^3 + \frac{\bar{c}}{2}Q^2 + AQ + B = (\bar{M} - Q)(Q - \bar{m})(\bar{z} - Q).$$

We identify the coefficients of Q^2 to infer that $\bar{z} = -\frac{\bar{c}}{2} - \bar{M} - \bar{m}$. This shows (2.14). Conversely, if Q satisfies (2.14), then we can trace these steps backwards to deduce that (2.15) holds. Hence $Q(y-\bar{c}t)$ is a traveling wave. This proves the lemma. □

Let

$$\gamma = \frac{1}{2}\left(\frac{1}{(2k+M+m)} + \frac{1}{(c-m)} + \frac{1}{(c-M)} - \frac{s_k}{2k}\right), \tag{2.19}$$

so that (2.13) becomes

$$Q(y) = \gamma - \frac{1}{c-\varphi}.$$

Since $m = \min_{x\in\mathbb{R}} \varphi(x)$ and $M = \max_{x\in\mathbb{R}} \varphi(x)$, it is easy to see that $\bar{m} = \min_{y\in\mathbb{R}} Q(y)$ and $\bar{M} = \max_{y\in\mathbb{R}} Q(y)$, where

$$\bar{m} = \gamma - \frac{1}{c-M}, \quad \bar{M} = \gamma - \frac{1}{c-m}.$$

We will show that Q solves (2.14) for $\bar{c} = \frac{s_k}{k} - \gamma$. By Lemma 3 this will prove (1.4).

Differentiation of (2.12) yields

$$Q_y = -\frac{\varphi_x}{(c-\varphi)^2}\frac{dx}{dy},$$

so in view of (2.11), we obtain

$$Q_y^2 = -\frac{\varphi_x^2}{w(c-\varphi)^4}.$$

Replacing w using (2.10) gives

$$Q_y^2 = -\frac{2\varphi_x^2}{\alpha(c-\varphi)^2}.$$

Furthermore, using (2.8), we arrive at

$$Q_y^2 = -\frac{2(M-\varphi)(\varphi-m)(\varphi-z)}{\alpha(c-\varphi)^3}.$$

The right hand side is actually a third order polynomial in Q. Indeed, we can write

$$\begin{aligned}\frac{1}{2}Q_y^2 &= \frac{\big((c-\varphi)-(c-M)\big)\big((c-\varphi)-(c-m)\big)\big((c-\varphi)-(2k+M+m)\big)}{(2k+M+m)(c-M)(c-m)(c-\varphi)^3}\\ &= \left(\frac{1}{c-M}-\frac{1}{c-\varphi}\right)\left(\frac{1}{c-m}-\frac{1}{c-\varphi}\right)\left(\frac{1}{2k+M+m}-\frac{1}{c-\varphi}\right).\end{aligned}$$

With

$$\bar{m} = \gamma - \frac{1}{c-M}, \quad \bar{M} = \gamma - \frac{1}{c-m}, \quad \bar{z} = \gamma - \frac{1}{2k+M+m},$$

we get

$$\frac{1}{2}Q_y^2 = (\bar{M}-Q)(Q-\bar{m})(\bar{z}-Q). \tag{2.20}$$

This is exactly (2.14), because a calculation shows that

$$\bar{z} = \gamma - \frac{1}{2k+M+m} = -\frac{\bar{c}}{2} - \bar{M} - \bar{m}.$$

This proves (1.4).

It remains to show that the mapping $(m, M, c) \to (\bar{m}, \bar{M}, \bar{c})$ defined in (1.5) is a bijection $\Gamma_+ \leftrightarrow \Pi$ and $\partial_1\Gamma_+ \leftrightarrow \partial_1\Pi$, with inverse as in (1.7). This is straightforward using the definitions and the proof of the main result is complete. □

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References

[1] Beals R, Sattinger D, and Szmigielski J, Multi-peakons and a theorem of Stieltjes, *Inverse Problems* **15** (1999), L1–L4.

[2] Beals R, Sattinger D, and Szmigielski J, Acoustic scattering and the extended Korteweg-de Vries hierarchy, *Adv. Math.* **40** (1998), 190–206.

[3] Camassa R and Holm D, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[4] Camassa R, Holm D and Hyman J, A new integrable shallow water equation, *Adv. Appl. Mech.* **31** (1994), 1-33.

[5] Colliander J, Keel M, Staffilani G, Takaoka H and Tao T, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$. *J. Amer. Math. Soc.* **16** (2003), 705–749.

[6] Constantin A, On the Cauchy problem for the periodic Camassa-Holm equation, *J. Differential Equations* **141** (1997), 218–235.

[7] Constantin A, On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.* **155** (1998), 352–363.

[8] Constantin A, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)* **50** (2000), 321–362.

[9] Constantin A, On the scattering problem for the Camassa-Holm equation, *Proc. Roy. Soc. London* **457** (2001), 953–970.

[10] Constantin A, A Lagrangian approximation to the water-wave problem, *Applied Mathematics Letters* **14** (2001), 789–795.

[11] Constantin A and Escher J, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica* **181** (1998), 229–243.

[12] Constantin A and Escher J, Global existence and blow-up for a shallow water equation, *Annali Sc. Norm. Sup. Pisa* **26** (1998), 303–328.

[13] Constantin A and Escher J, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math* **51** (1998), 475–504.

[14] Constantin A and Escher J, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.* **233** (2000), 75–91.

[15] Constantin A and Escher J, Global weak solutions for a shallow water equation, *Indiana Univ. Math. J.* **47** (1998), 1527–1545.

[16] Constantin A and Kolev B, On the geometric approach to the motion of inertial mechanical systems, *J. Phys. A* **35** (2002), R51–R79.

[17] Constantin A and Kolev B, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* **78** (2003), 787–804.

[18] Constantin A and Lenells J, On the inverse scattering approach to the Camassa-Holm equation, *J. Nonlinear Math. Phys.* **10** (2003), 252–255.

[19] Constantin A and McKean HP, A shallow water equation on the circle, *Comm. Pure Appl. Math.* **52** (1999), 949–982.

[20] Constantin A and Molinet L, Global weak solutions for a shallow water equation, *Comm. Math. Phys.* **211** (2000), 45–61.

[21] Constantin A and Strauss W, Stability of peakons, *Comm. Pure Appl. Math.* **53** (2000), 603–610.

[22] Constantin A and Strauss W, Stability of the Camassa-Holm solitons, *J. Nonlinear Sci.* **12** (2002), 415–422.

[23] Dai HH, Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mech.* **127** (1998), 193–207.

[24] Danchin R, A few remarks on the Camassa-Holm equation, *Differential Integral Equations* **14** (2001), 953–988.

[25] Drazin PG and Johnson RS, *Solitons: an Introduction*, Cambridge University Press, 1989.

[26] Dullin HR, Gottwald G, and Holm DD, An integrable shallow water equation with linear and nonlinear dispersion, *Phys. Rev. Lett.* **87** (2001), 194501–04.

[27] Dullin HR, Gottwald G, and Holm DD, Camassa-Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves, *Fluid Dyn. Res.* **33** (2003), 73–95.

[28] Fokas AS, On a class of physically important integrable equations, *Physica D* **87** (1995), 145–150.

[29] Fuchssteiner B, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, *Physica D* **95** (1996), 229–243.

[30] Fuchssteiner B and Fokas AS, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Physica D* **4** (1981), 47–66.

[31] Gesztesy F and Holden H, Algebro-geometric solutions of the Camassa-Holm hierarchy, *Rev. Mat. Iberoamericana* **19** (2003), 73–142.

[32] Holm D, Marsden J, and Ratiu T, The Euler-Poincaré equations and semidirect products with applications to continuum theories, *Adv. Math.* **137** (1998), 1–81.

[33] Johnson RS, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** (2002), 63–82.

[34] Johnson RS, On solutions of the Camassa-Holm equation, *Proc. Roy. Soc. London A* **459** (2003), 1687–1708.

[35] Johnson RS, The classical problem of water waves: a reservoir of integrable and nearly-integrable equations, *J. Nonlinear Math. Phys.* **10** Suppl. 1 (2003), 72–92.

[36] Lenells J, The scattering approach for the Camassa-Holm equation, *J. Nonlinear Math. Phys.* **9** (2002), 389–393.

[37] Lenells J, Stability of periodic peakons, *Internat. Math. Res. Notices* **10** (2004), 485–499.

[38] Lenells J, Traveling wave solutions of the Camassa-Holm equation, submitted.

[39] Li Y and Olver P, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Diff. Eq.* **162** (2000), 27–63.

[40] Misiolek G, A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.* **24** (1998), 203–208.

[41] McKean HP, Integrable systems and algebraic curves, *Global Analysis, Springer Lecture Notes in Mathematics,* **755** (1979), 83–200.

[42] McKean HP, Breakdown of a shallow water equation, *Asian J. Math* **2** (1998), 867–874.

[43] Rodriguez-Blanco G, On the Cauchy problem for the Camassa-Holm equation, *Nonl. Anal.* **46** (2001), 309–327.

[44] Xin Z and Zhang P, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.* **53** (2000), 1411–1433.

[45] Yin Z, On the blow-up of solutions of the periodic Camassa-Holm equation, *Dyn. Cont. Disc. Imp. Syst.*, to appear.