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On Well-Posedness Results for Camassa-Holm Equation on the Line: A Survey

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Abstract

We survey recent results on well-posedness, blow-up phenomena, lifespan and global existence for the Camassa-Holm equation. Results on weak solutions are also considered.

1 Introduction

The Camassa-Holm equation reads

$$\text{(C-H)} \qquad \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & (t,x) \in \mathbb{R}^2 \\ u(0) = u_0 \end{cases}$$

It has been derived by Camassa and Holm [3] as a model for the evolution of a shallow water layer under the influence of gravity. The idea is to write the Green-Naghdi equations in Lie-Poisson Hamiltonian form and then make an asymptotic expansion which keeps the Hamiltonian structure. The validity of this model for water waves is addressed in [20]. Note that (C-H) was also found independently by Dai [15] as a model for nonlinear waves in cylindrical hyperelastic rods and was, in fact, first discovered by the method of recursive operator by Fokas and Fuchsteiner [18] as an example of bi-Hamiltonian equation. Finally, (C-H) is also a re-expression of geodesic flow on the diffeomorphism group of the line, see [26], [10], [11].

(C-H) possesses a Lax pair and is thus, at least formally, completely integrable, see [3],[4]. For a discussion on the direct/inverse scattering approach we refer to [6], [23]. The first three invariants of the motion are :

$$I(v) = \int_{\mathbb{R}} u(x)\,dx, \quad E(u) = \int_{\mathbb{R}} u^2(x) + u_x^2(x)\,dx$$

$$\text{and} \quad F(u) = \int_{\mathbb{R}} u^3(x) + u(x)u_x^2(x)\,dx \quad .$$

Camassa and Holm exhibited solitary waves of the form

$$u_c(x,t) = c\varphi(x-ct), \quad x \in \mathbb{R} \quad , \tag{1.1}$$

where $\varphi(x) = e^{-|x|}$, $x \in I\!R$. These solitary waves are orbitally stable ([12], [14]). They are solitons in the sense that they retain their individuality under interaction and eventually emerge with their original shapes and speeds, see [3], [2]. They are peaked waves and can only be understood as weak solutions of (C-H), i.e. solutions of the following weaker form of (CH)

$$u_t + uu_x + \partial_x p \star [u^2 + u_x^2/2] = 0 \quad , \tag{1.2}$$

with $p(x) = \frac{1}{2} e^{-|x|}$. Note that (1.2) has a conservation law structure.

In this paper we would like to present an overview of the available results on the following problems :

1. Local well-posedness of the Cauchy problem for (C-H).
2. Blow-up criteria and blow-up phenomena.
3. Lifespan and global existence.
4. Weak solutions.

2 Local well-posedness results

Since the Hamiltonian E is nothing else but the H^1-norm of the solution, the Sobolev spaces are natural spaces for the Cauchy problem associated with (C-H).

A first result is due to Constantin and Escher [7] who proved the local well-posednes of (C-H) in H^s, $s \geq 3$, by applying Kato's theory for hyperbolic quasi-linear PDE to $y = u - u_{xx}$. The equation satisfied by y reads

$$\begin{cases} y_t + (Q^{-2}y)y_x + 2y\partial_x(Q^{-2}y) = 0, \quad (t,x) \in I\!R^2 \\ y(0) = u_0 - u_{0,xx} \end{cases} \tag{2.1}$$

where $Q = (I - \partial_x^2)^{1/2}$.
This result was improved to $s > 3/2$ by Li and Olver [24] and Rodriguez-Blanco [27]. A Besov spaces approach can also be found in [16]. The statement of the local existence theorem is the following :

Theorem 1. *Given $u_0 \in H^s(I\!R)$, with $s > 3/2$, there exists $T = T(\|u_0\|_{H^{\frac{3}{2}+}})$[1] > 0 and a unique associated solution*

$$u \in C([0,T]; H^s(I\!R)) \cap C^1([0,T]; H^{s-1}(I\!R))$$

to (CH).Moreover, the flow-map is continuous from $H^s(I\!R)$ to the class defined above.

[1] $\frac{3}{2}+$ denotes any real number larger but close enough to $\frac{3}{2}$.

2.1 Sketch of the proof of Theorem 1

Step 1. We look at the following parabolic regularization

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} u_t^\varepsilon - \varepsilon\,\partial_x^2 u_t = -u^\varepsilon u_x^\varepsilon - p_x \star [(u^\varepsilon)^2 + \frac{1}{2}(u_x^\varepsilon)^2], & (t,x) \in I\!R^2 \\ u(0) = u_0^\varepsilon \end{cases}$$

where $u_0^\varepsilon \to u_0$ in $H^s(I\!R)$ as $\varepsilon \searrow 0$. Since

$$G \mapsto (Id - 1/\varepsilon\,\partial_x^2)^{-1}\Big(-uu_x - p_x \star [u^2 + \frac{1}{2}u_x^2]\Big)$$

is locally Lipschitz on $H^s(I\!R)$, $s > 3/2$, by the Cauchy-Lipschitz-Picard theorem, there exists $T = T(\|\phi\|_{H^s}) > 0$ and a unique solution $u^\varepsilon \in C^1([0,T];H^s)$ to $(\mathcal{P}_\varepsilon)$.
Step 2. Taking the H^s-scalar product of $(\mathcal{P}_\varepsilon)$ with u^ε, using Kato-Ponce commutator estimates and that

$$\|f\,g\|_{H^s} \le C\,\|f\|_{H^s}\,\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}, \quad s > 1/2,$$

we get the following uniform estimate for $s > 3/2$,

$$\frac{d}{dt}\|u^\varepsilon\|_{H^s}^2 \le C\,\|u_x^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^s}^2 \quad . \tag{2.2}$$

This obviously yields

$$\|u^\varepsilon(t)\|_{H^s} \le C\,\Big(\frac{1}{\|\phi\|_{H^s}^{1/2}} - t\Big)^{-2} \quad . \tag{2.3}$$

(2.2) and (2.3) force $\|u^\varepsilon\|_{H^s}$ to be uniformly bounded on $[0,T]$ with $T = T(\|u_0\|_{H^{\frac{3}{2}+}})$. It then follows from $(\mathcal{P}_\varepsilon)$ that $\{u_t^\varepsilon\}$ is bounded on $L^\infty(0,T;L^2)$ and using the Aubin-Lions compactness lemma (see [25]) we can pass to the limit in the nonlinear terms of (1.2). We thus obtain a solution $u \in L^\infty(0,T;H^s(I\!R))$ to (C-H). Moreover, $\{u^\varepsilon\}$ and $\{u_t^\varepsilon\}$ being bounded in respectively $L^\infty(0,T;H^s)$ and $L^\infty(0,T;L^2)$, for any $v \in C_0^\infty(I\!R)$ fixed, the family $t \mapsto (u^\varepsilon(t),v)_{H^s}$ is equicontinuous on $[0,T]$. It follows from the Arzela-Ascoli theorem that $t \mapsto (u(t),v)_{H^s}$ is continuous on $[0,T]$ and from the density of $C_0^\infty(I\!R)$ in $H^s(I\!R)$ we deduce that $u \in C_w([0,T];H^s)$.
Step 3. The uniqueness follows directly by writing the equation for the difference $w = u_1 - u_2$ of two solutions to (1.2) and taking the scalar product in H^{s-1} with w. Indeed, using that

$$\|f\,g\|_{H^t} \le C\,\|f\|_{H^r}\,\|g\|_{H^t}, \quad r > 1/2,\ -r < t < r,$$

to treat the convolution term, we get

$$\frac{d}{dt}\|w\|_{H^{s-1}}^2 \le C\,\Big(\|u_1\|_{H^s} + \|u_2\|_{H^s}\Big)\,\|w\|_{H^{s-1}}^2$$

which permits to conclude thanks to Gronwall lemma.
Step 4. The fact that u belongs to $C([0,T];H^s(I\!R)) \cap C^1([0,T];H^{s-1}(I\!R))$ and the continuity of the flow-map in $H^s(I\!R)$ follows from Bona-Smith type argument (see, for instance, [1]) : For a good choice of the approximative sequence $\{u_0^\varepsilon\} \subset H^\infty$ to u_0, the sequence of associated solutions $\{u^\varepsilon\}$ to (1.2) is shown to be a Cauchy sequence in $C([0,T];H^s(I\!R))$. Finally, it is easy to see that these solutions satisfies (C-H) at least in the distributional sense and the continuity with respect to initial data clearly implies that $E(u(t))$ is conserved in time.

2.2 Some Remarks on well-posedness results

We just saw that (C-H) and (1.2) are locally well-posed (in the Hadamard sense[2]) in $H^s(\mathbb{R})$ for $s > 3/2$. (C-H) does not make sense in H^s for $s < 3/2$ (this can be easily seen using the solitons) but one can ask the following question:*what is the minimum Sobolev index s for (1.2) to be well-posed ?* This is certainly a hard question. In this direction, in [19], by using the specific form of the solitons, it is shown that the flow-map associated to (1.2) is not uniformly continuous in $H^s(\mathbb{R})$ for $s < 3/2$. Of course it is not reasonable to require the flow map to be uniformly continuous in $H^s(\mathbb{R})$, even for $s > 3/2$. So this remark does not seem to be very pertinent. Actually, since (C-H) has much in common with the Burgers equation, it is likely that *the flow-map for (C-H) is not locally uniformly continuous in $H^s(\mathbb{R})$ even for large s* . It would be very interesting to prove this result (if it is true !) as it was done recently for the Benjamin-Ono equation [22]. Finally, note that in [17] it is shown that the flow-map cannot be continuous in the Besov space $\mathcal{B}_2^{3/2,\infty}$ (see [17] for the definition of this space). But again this does not have too much interest since it is not reasonable to ask for a flow-map to be continuous in $\mathcal{B}_2^{s,\infty}$ whatever the value of s is.

3 Blow-up criteria and blow-up results

Even though the Camassa-Holm equation is integrable, unlike the KdV equation, there are no invariants which control the H^s-norm higher than H^1-one. This does not permit to extend automatically the above local solutions to the whole line and, actually, we will see in this section that blow-up in finite time can occur.

3.1 Blow-up criteria

For $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, we call $T^*_{u_0} > 0$ the maximal time of existence in $H^s(\mathbb{R})$ of the associated solution u to (C-H). Recall that according to the local well-posedness theory, the maximal time of existence of u in $H^s(\mathbb{R})$ does not depend on $s > 3/2$ but only on u_0. First from (2.2) one can easily check that for any $u_0 \in H^s(\mathbb{R})$, $s > 3/2$,

$$T^*_{u_0} < \infty \Leftrightarrow \int_0^{T^*_{u_0}} \|u_x(t)\|_{L^\infty}\, dt = +\infty \tag{3.1}$$

Actually, one can be more precise and prove that the singularities can arise only in the form of wave breaking, i.e. the solution remains bounded while its slope goes to $-\infty$ in finite time. Setting $m(t) = \inf_{x\in\mathbb{R}} u(t,x)$, the following blow-up criterion holds :

$$T^*_{u_0} < \infty \Leftrightarrow \int_0^{T^*_{u_0}} m(t)\, dt = -\infty \quad . \tag{3.2}$$

Indeed, taking the L^2-scalar product of (2.1) with y, one gets

$$\frac{d}{dt}\|y\|^2_{L^2} = -2\int_{\mathbb{R}} u y_x y \; - \; 4\int_{\mathbb{R}} u_x y^2 = -3\int_{\mathbb{R}} u_x y^2 \le -3m(t)\|y\|^2_{L^2} \quad .$$

[2]This ensures that the trajectory is a continuous curve in $H^s(\mathbb{R})$ as well as the continuity of the flow-map in $H^s(\mathbb{R})$.

Hence,

$$\|u\|_{L^\infty_T H^2} \le C\,\|y\|_{L^\infty_T L^2} \le \exp\Big(-3\int_0^T m(t)\,dt\Big)\|y(0)\|_{L^2} \quad ,$$

which proves (3.2) for $u_0 \in H^s(I\!\!R)$ with $s \ge 2$. Following Danchin [16], the assumption $s \ge 2$ can be weakened to $s > 3/2$. The key point is to prove the following a priori estimate for any $u_0 \in H^s(I\!\!R)$, $s > 3/2$,

$$\sup_{t\in[0,T^*_{u_0}[} M(t) \le \max\Big(\|u_0\|_{H^1}/\sqrt{2}\,,\, M(0)\Big) \quad , \tag{3.3}$$

where

$$M(t) = \sup_{x\in I\!\!R} u_x(t,x) = u_x(t,\xi(t)) \quad .$$

To prove (3.3), we first assume that $u_0 \in H^3(I\!\!R)$. Differentiating (1.2) with respect to the space variable, one gets

$$\begin{aligned} u_{tx} + u u_{xx} + u_x^2 &= -\partial_x^2 p * (u^2 + u_x^2/2) \\ &= u^2 + u_x^2/2 - p*(u^2+u_x^2/2) \quad . \end{aligned} \tag{3.4}$$

On the other hand, according to [8], $M(\cdot)$ is almost everywhere differentiable on $[0,T^*_{u_0}[$ and for almost every $t \in [0,T^*_{u_0}[$,

$$\dot M(t) = u_{tx}(t,\xi(t)) \quad .$$

Therefore, using that clearly $u_{xx}(t,\xi(t)) = 0$, one deduces from (3.4) that

$$\dot M(t) + \frac{M^2}{2}(t) = u^2(t,x) - p*(u^2+u_x^2/2)(t,\xi(t)) \quad .$$

Finally, the following estimates proven in [5],

$$p*(v^2+v_x^2/2) \ge v^2/2 \quad \text{and } 2\,v^2(x) \le \|v\|_{H^1}^2 \quad , \tag{3.5}$$

yield

$$\dot M(t) + \frac{M^2(t)}{2} \le \frac{u^2(t,x)}{2} \le \frac{\|u_0\|_{H^1}^2}{4} \quad \text{a.e. on }]0,T^*_{u_0}[, \tag{3.6}$$

where one makes use of the conservation of the H^1-norm. This proves (3.3) for $u_0 \in H^3(I\!\!R)$. Now, for $u_0 \in H^s(I\!\!R)$, $s > 3/2$, we approximate u_0 by a sequence $\{u_0^n\} \subset H^3(I\!\!R)$ which tends to u_0 in H^s. Since $v \mapsto \sup_{x\in I\!\!R} v_x$ is continuous in $H^s(I\!\!R)$ for $s > 3/2$, (3.3) follows by passing to the limit thanks to the continuity of the flow-map.
One thus deduces that M is uniformly bounded on the maximal interval of existence of u which clearly implies that

$$\int_0^{T^*_{u_0}} |u_x(t)|_{L^\infty}\,dt = +\infty \Leftrightarrow \int_0^{T^*_{u_0}} m(t)\,dt = -\infty \quad . \tag{3.7}$$

(3.2) is of course a direct consequence of (3.1) and (3.7).

3.2 Blow-up results

Two of the main results on blow-up for the Camassa-Holm equation on the line are the following ones :

Theorem 2. *([7]) Assume that $u_0 \in H^3(\mathbb{R})$ is odd with $u_0'(0) < 0$. Then the corresponding solution of (C-H) does not exist globally. The maximal time of existence is estimated above by $1/(2|u_0'(0)|)$.*

Theorem 3. *([8]) Assume that the initial profile $u_0 \in H^3(\mathbb{R})$ has at some point a slope which is less than $-(1/\sqrt{2})\|u_0\|_{H^1}$. Then wave breaking for the corresponding solution of (C-H) occurs.*

Note that Theorem 2 clearly shows that initial data with arbitrary small initial H^s-norm can product solutions that blow-up in finite time. The proof of Theorem 2 is simple and makes use of the conservation of antisymmetry by the flow of (C-H). Indeed, for u_0 odd, setting $g(t) = u_x(t,0)$, one infers from (3.4) and the antisymmetry of $u(t)$ that g satisfies

$$\dot{g}(t) + \frac{1}{2}g^2(t) = -[p * (u^2(t) + u_x^2(t)](0) \le 0 \quad .$$

Therefore,

$$g(t) \le \frac{2g(0)}{2 + tg(0)} \quad , \tag{3.8}$$

and consequently $\quad T^*_{u_0} < -2/u_0'(0)$.

To prove Theorem 3, one uses (3.4)-(3.5) and the fact that, according to [8],

$$m(t) = \inf_{x \in \mathbb{R}} u_x(t,x) = u_x(t, \theta(t))$$

is almost everywhere differentiable on $]0, T^*_{u_0}[$ to obtain as in (3.6),

$$\dot{m}(t) + m^2(t)/2 \le \frac{1}{4}\|u_0\|^2_{H^1} \quad \text{a.e. on }]0, T^*_{u_0}[. \tag{3.9}$$

But by the assumption on u_0, there exists $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\|u_0\|^2_{H^1} \le (2 - 2\varepsilon)(u_0'(x_0))^2 \le (2 - 2\varepsilon)m^2(0)$$

and consequently,

$$\dot{m}(t) + \frac{m^2(t)}{2} \le \frac{1-\varepsilon}{2}\, m^2(0) \quad \text{a.e. on }]0, T^*_{u_0}[. \tag{3.10}$$

Then using that (3.10) forces on $[0, T^*_{u_0}[$,

$$m^2(t) > (1 - \frac{1}{2}\varepsilon)m^2(0)$$

(Assuming that there exists $t_0 > 0$ such that $m^2(t_0) = (1 - \frac{1}{2}\varepsilon)m^2(0)$ leads to a contradiction.). One finally gets

$$\begin{aligned} \dot{m}(t) &\le -\frac{1}{4}\,\varepsilon\, m^2(t) \quad \text{a.e. on }]0, T^*_{u_0}[\\ \Rightarrow \quad m(t) &\le \frac{4m(0)}{4 + \varepsilon t m(0)} \quad . \end{aligned} \tag{3.11}$$

Hence, blow-up in finite time occurs. Note that by the continuity of the flow-map, (3.8) and (3.11) also hold for $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$ and thus Theorems 2-3 also hold for $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$.

4 Lifespan and global existence results

As noticed in [16], one can derive a sharp lower bound for the maximal time of existence $T^*_{u_0}$ in $H^s(\mathbb{R})$, $s > 3/2$, of the solutions to (C-H). Indeed, given $u_0 \in H^3(\mathbb{R})$, according to (3.4) for almost every $t \in]0, T^*_{u_0}[$, one has

$$\dot{m}(t) + \frac{m^2(t)}{2} = u^2(t, \xi(t)) - [p \star (u^2 + \frac{u_x^2}{2})](t, \theta(t)) \qquad \text{a.e. on }]0, T^*_{u_0}[.$$

Since $\|p\|_{L^\infty} \le 1/2$ this implies

$$\dot{m}(t) + \frac{m^2(t)}{2} \ge -1/2\,\|u(t)\|^2_{H^1} = -1/2\,\|u_0\|^2_{H^1} \quad \text{a.e. on }]0, T^*_{u_0}[.$$

So, integration in time yields

$$-m(t) \le \|u_0\|_{H^1} \frac{\|u_0\|_{H^1} \tan\Big(t\,\|u_0\|_{H^1}/2\Big) - m(0)}{\|u_0\|_{H^1} + m(0)\tan\Big(t\,\|u_0\|_{H^1}/2\Big)}, \quad \forall t \in]0, T^*_{u_0}[. \tag{4.1}$$

By continuity with respect to initial data, (4.1) also holds for $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. The criterion (3.2) then gives the following lower bound for the lifespan

$$T^*_{u_0} \ge T_{u_0} := -\frac{2}{\|u_0\|_{H^1}} \arctan\Big(\|u_0\|_{H^1}/m(0)\Big) \quad . \tag{4.2}$$

Now, taking $u_0 = -xe^{-x^2}$ and defining $u_0^n = n^{-1/2}u_0(nx)$, one can easily check that u_0^n satisfies the hypotheses of Theorem 2 and thus

$$T^*_{u_0^n} \le -2(u^n_{0,x}(0))^{-1} \tag{4.3}$$

Since, on the other hand, straightforward calculations yield

$$T_{u_0^n} \to -2(u^n_{0,x}(0))^{-1} \quad , \tag{4.4}$$

the sharpness of (4.2) follows.

4.1 A global existence result

It was first noticed in [7] that the solutions are global in time provided the potential $y_0 = u_0 - u_{0,xx}$ associated with the initial value u_0 is a bounded measure with definitive sign. Let us show this result. So let $u_0 \in H^s(\mathbb{R})$, $s > 3/2$ such that y_0 is a non negative bounded measure. For a classical sequence of mollifiers ρ_n, we define $y_0^n = \rho_n * y_0 \ge 0$, $\|y_0^n\|_{L^1} \le \|y_0\|_{\mathcal{M}}$ and $y_0^n \to y_0$ in $\mathcal{M}$. For the solutions u^n associated with the initial data $u_0^n = \rho_n * u_0$, we define the integral curves $q(t, \cdot)$ by

$$\begin{cases} q_t^n = u^n(t, q) \\ q^n(0, x) = x, \quad x \in \mathbb{R} \end{cases}$$

According to [5], for all $x \in I\!R$, $t \in [0, T^*_{u_0}[$,

$$y^n_0(x) = y^n(t, q(t,x)) \Big(q^n_x(t,x) \Big)^2 \quad . \tag{4.5}$$

Note that Relation (4.5) has an interesting geometric interpretation, see [11].
In particular $y^n(t)$ stays non negative on the interval of existence of u. On the other hand, note that if $g = f - f_{xx}$ then

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{-\infty}^{x} e^{y-x}\, g(y)\, dy + \frac{1}{2}\int_{x}^{\infty} e^{x-y}\, g(y)\, dy, \quad x \in I\!R, \\ f_x(x) &= -\frac{1}{2}\int_{-\infty}^{x} e^{y-x}\, g(y)\, dy + \frac{1}{2}\int_{x}^{\infty} e^{x-y}\, g(y)\, dy, \quad x \in I\!R. \end{aligned}$$

Hence, if $g(\cdot) \geq 0$ on $I\!R$, then $f_x^2(\cdot) \leq f^2(\cdot)$ on $I\!R$. It thus follows that $u_x^2(t,\cdot) \leq u^2(t,\cdot)$ on $I\!R$ for $t \in [0, T_{u^*_{0,n}}[$. The conservation of the H^1-norm, Sobolev embedding and the blowup criterion (3.1) then clearly imply that $T^*_{u_{0,n}} = +\infty$. Passing to the limit in n, we infer that $\forall t \in [0, T^*_{u_0}[$,

$$u_x^2(t,\cdot) \leq u^2(t,\cdot) \quad \text{on } I\!R$$

which again by the blow-up criterion (3.1) implies the global existence of u.

5 Weak solutions

As indicated in the Introduction the solitons of (C-H) (see (1.1)) are peaked waves. They cannot be seen as solutions of (C-H) but only of the weak formulation (1.2).

One can define two notions of weak solutions. On one hand, there are what we could call "strong" weak solutions. These solutions are unique in their class and exists until the blow-up time. On the other hand, there are "weak" weak solutions. These solutions are always global (they are defined even after the blow-up time). They are not unique and the energy is not known to be preserved for these solutions.

5.1 "Strong" weak solutions

In [9], Constantin and Escher used compensated-compactness arguments in space-time to get the existence of global weak solutions for[3] $u_0 \in H^1(I\!R)$ with $y_0 = u_0 - u_{0,xx} \in \mathcal{M}^+$. In [13], Constantin and the author proved that actually these solutions are not that weak since they are unique in their class and are continuous with values in $H^1(I\!R)$. Moreover they showed that the functionals I, E and F are conserved along the trajectory. Finally, in [16], Danchin noticed that the time of existence of a solution $u \in H^s(I\!R)$, $s > 3/2$, is bounded below by a positive real number $T = T(\|y_0\|_{\mathcal{M}})$ which permits to obtain local weak solutions without the positivity assumption on y_0.
We summarize all these results in the following theorem :

[3] $\mathcal{M}$ is the space of Radon measures on $I\!R$ with bounded total variation and $\mathcal{M}$ is the subset of positive measures.

Theorem 4. *Let* $u_0 \in H^1(I\!R)$ *with* $y_0 := u_0 - u_{0,xx} \in \mathcal{M}(I\!R)$ *then there exists* $T = T(\|y_0\|_{\mathcal{M}}) > 0$ *and a unique solution*[4] *to (1.2),*

$$u \in C([0,T]; H^1(I\!R)) \cap L^\infty(0,T; W^{1,1}(I\!R)), \quad u_x \in L^\infty(0,T; BV(I\!R)) \tag{5.1}$$

with initial data u_0*. Moreover, the functionals* $E(\cdot)$ *and* $F(\cdot)$ *are constant along the trajectory and if* y_0 *has a definite sign then* u *is global in time.*

Sketch of the proof. It is convenient to first prove the uniqueness result since this will be useful for the proof of the continuity of the solution in $H^1(I\!R)$.
Uniqueness. Let u and v be two solution of (1.2) within the class defined by (5.1). The main idea is to use exterior regularization techniques to get the following integral inequality for the $W^{1,1}$-norm of the difference $u - v$ (see [13] for details) :

$$\int_{I\!R} |u - v|(t,x) + |u_x - v_x|(t,x)\, dx \le e^{6Mt} \int_{I\!R} |u_0 - v_0|(x) + |u_{0,x} - v_{0,x}|(x)\, dx \tag{5.2}$$

where

$$M = \sup_{(0,T)} \|u - u_{xx}(t,\cdot)\|_{\mathcal{M}} + \|v - v_{xx}(t,\cdot)\|_{\mathcal{M}} \quad .$$

The uniqueness then follows directly from Gronwall Lemma.
Existence. We will decompose the proof into three steps.
Step 1. This first step consists in constructing a sequence of smooth solutions of (C-H) with initial data that approximate u_0. Let ρ_n be the classical Friedrichs mollifier sequence. For u_0 as in the statement of the theorem, we define $u_0^n = \rho_n * u_0 \in H^\infty(I\!R)$ and we call u_n the associated smooth solution of (C-H). Setting $y_n = u_n - u_{n,xx}$, from (2.1) we infer that

$$\partial_t y_n + \partial_x(u_n y_n) = -y_n \partial_x u_n$$

and thus

$$\partial_t |y_n| + \partial_x(u_n |y_n|) = -|y_n| \partial_x u_n \quad .$$

Integrating in space, it follows that

$$\frac{d}{dt}\|y(t)\|_{L^1} \le \inf_{(0,t)} (\partial_x u(t)) \|y(t)\|_{L^1}$$

and since

$$\|\partial_x u_n\|_{L^\infty} = \|\partial_x p * y_n\|_{L^\infty} \le \frac{1}{2}\|y_n\|_{L^1} \quad , \tag{5.3}$$

one deduces that

$$\|y_n(t)\|_{L^1} \le \frac{2\|y_n(0)\|_{L^1}}{2 - t\|y_n(0)\|_{L^1}} = \frac{2\|y_n(0)\|_{\mathcal{M}}}{2 - t\|y_n(0)\|_{\mathcal{M}}} \quad . \tag{5.4}$$

[4] $W^{1,1}(I\!R)$ is the space of $L^1(I\!R)$ functions with derivatives in $L^1(I\!R)$ and $BV(I\!R)$ is the space of function with bounded variation

(5.3)-(5.4) and the blow-up criterion (3.1) show that $u_n \in C([0,T];H^\infty(I\!R))$ with $T = T(\|y_0\|_{\mathcal{M}}) > 0$. Furthermore,

$$\|u_{n,x}\|_{L^\infty_{T,x}} \le C(T,\|y_0\|_{\mathcal{M}}) \quad , \quad \|u_n(t)\|_{H^1} = \|u_0^n\|_{H^1} \quad \text{on } [0,T] \quad , \tag{5.5}$$

$$\|u_n\|_{L^\infty_T W^{1,1}} = \|p * y_n\|_{L^\infty_T W^{1,1}} \le C(T,\|y_0\|_{\mathcal{M}}), \tag{5.6}$$

$$\text{and} \qquad \|u_{n,xx}\|_{L^\infty_T L^1_x} \le \|u_n\|_{L^\infty_T L^1_x} + \|y_n\|_{L^\infty_T L^1_x} \le C(T,\|y_0\|_{\mathcal{M}}) \ . \tag{5.7}$$

Moreover, from Section 4.1 we see that if y_0 has a definite sign then $T = +\infty$ and

$$\|u_{n,x}\|_{L^\infty_{t,x}} \le \|u_n\|_{L^\infty_{t,x}} \le \|u_n\|_{L^\infty_t H^1(I\!R)} = \|u_0^n\|_{H^1} \quad . \tag{5.8}$$

Actually, one can prove (see [7]) that if y_0 has a definite sign then

$$\|y_n(t)\|_{L^1} = \|y_0^n\|_{L^1}, \quad \forall t \in I\!R \quad . \tag{5.9}$$

Step 2. In this step we pass to the limit to get the existence of a weak solution. Indeed, from (1.2) and (5.4)-(5.5), $\{u_n\}$ is bounded in $H^1(I\!R)$ and $\{u_{n,x}\}$ is bounded in BV on $(0,T)$. Hence, from classical compactness theorems in Sobolev spaces together with Helly's theorem there exists a subsequence $\{u_{n_k}\}$ and an element u of $H^1(I\!R)$ with $u_x \in BV$ such that

$$u_{n_k} \rightharpoonup u \text{ weakly in } H^1((0,T)\times I\!R) \tag{5.10}$$

$$u_{n_k} \to u \text{ a.e. on } (0,T)\times I\!R \tag{5.11}$$

$$\partial_x u_{n_k} \to \partial_x u \text{ a.e. on } (0,T)\times I\!R \tag{5.12}$$

The main difficulty for passing to the limit in (1.2) comes from the nonlinear term $\partial_x p * [u_{n_k}^2 + \frac{1}{2}u_{n_k,x}^2]$. But we note that $\{u_{n_k}^2(t) + \frac{1}{2}u_{n_k,x}^2(t)\}$ is uniformly bounded in $L^1(I\!R)$ and therefore (5.10)-(5.12) imply that for a.e. $t \in (0,T)$, its weak limit is $u^2(t) + \frac{1}{2}u_x^2(t)$. As $\partial_x p \in L^2(I\!R)$, a.e. in $(0,T)\times I\!R$ we thus have

$$\partial_x p * [u_{n_k}^2 + \frac{1}{2}u_{n_k,x}^2] \underset{n_k\to\infty}{\to} \partial_x p * [u^2 + \frac{1}{2}u_x^2]$$

which proves that u satisfies (1.2) at least in the distributional sense. Moreover by the same argument as in Section 2.1, $u \in C_w([0,T];H^1(I\!R))$.

Step 3. In this last step we prove the continuity in $H^1(I\!R)$ and the conservation laws. Since for almost every $t \in [0,T]$, $u_{n_k}(t) \rightharpoonup u(t)$ weakly in $H^1(I\!R)$, one has

$$\|u(t)\|_{H^1} \le \liminf_{n_k\to\infty} \|u_{n_k}\|_{H^1} = \liminf_{n_k\to\infty} \|u_0^{n_k}\|_{H^1} = \|u_0\|_{H^1} \quad . \tag{5.13}$$

But u belonging to $C_w([0,T];H^1(I\!R))$, (5.13) is actually true for all $t \in [0,T]$. Now, it is worth noticing that one can reverse time for the Camassa-Holm equation. Hence, taking $u(t)$ with $t > 0$ as initial data and reversing time, one obtains by uniqueness for all $t \ge 0$,

$$\|u_0\|_{H^1} \le \|u(t)\|_{H^1} \quad .$$

The H^1-norm of u is thus constant in time. This combined with the weak continuity of u in $H^1(I\!R)$ implies the strong continuity of u in $H^1(I\!R)$. Note that for a.e. $t \in [0,T]$, $u_{n_k}(t) \rightharpoonup u(t)$ weakly in $H^1(I\!R)$ and that

$$\lim_{n_k\to\infty} \|u_{n_k}(t)\|_{H^1} = \lim_{n_k\to\infty} \|u_0^{n_k}\|_{H^1} = \|u_0\|_{H^1} = \|u(t)\|_{H^1} \quad .$$

This implies that $u_{n_k}(t)$ converges in fact strongly to $u(t)$ in $H^1(\mathbb{R})$ for a.e. $t \in (0,T)$. Hence, using that $F(\cdot)$ is continuous from $H^1(\mathbb{R})$ to $\mathbb{R}$, we deduce that for all $t \in (0,T)$,

$$F(u(t)) = \lim_{n_k \to \infty} F(u_{n_k}(t)) = \lim_{n_k \to \infty} F(u_0^{n_k}) = F(u_0) \quad .$$

5.2 "Weak" weak solutions

In [28] Xin and Zhang proved the existence of global weak solutions of (1.2) for initial data in $H^1(\mathbb{R})$. They are able to pass to the limit in viscous approximations by using Young measures associated to $\partial_x u_\varepsilon$ where u_ε is the viscous approximation. The key elements are the uniform a priori estimates (5.14) and (5.15) below.

Theorem 5. *[28] Assume that $u_0 \in H^1(\mathbb{R})$. Then (1.2) has a global weak solution*

$$u \in C(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})) \quad ,$$

with u_0 as initial data, such that

$$\|u(t,\cdot)\|_{H^1} \le \|u_0\|_{H^1}\,, \quad \forall t > 0 \quad .$$

Moreover,

$$p * (u^2 + \frac{1}{2}u_x^2) \in L^\infty(R, W^{1,\infty}(\mathbb{R})), \quad \partial_x u \in L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}),\ p < 3, \tag{5.14}$$

and there exists $C = C(\|u_0\|_{H^1})$ such that

$$\|\partial_x u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le (C + \frac{1}{t}) \quad \forall t > 0 \quad . \tag{5.15}$$

Unfortunately there is a priori no uniqueness result for these solutions and their energy is only known to be non increasing. However, in [29], Xin and Zhang proved that if one has a better temporal integrability on $\|\partial_x u\|_{L^\infty}$ that the one given by (5.15), uniqueness holds. More precisely,

Theorem 6. *[29] If further there exists $b \in L^2_{loc}(\mathbb{R}_+)$ such that*

$$\|\partial_x u(t,\cdot)\|_{L^\infty} \le b(t) \tag{5.16}$$

then the weak solution given by Theorem 5 is unique in some class.

Note that Theorem 5 insures only the existence of a function $b \in L^p(\mathbb{R})$, with $p < 1$, such that (5.16) holds. On the other hand, from (5.8), if $y_0 = u_0 - u_{0,xx}$ is a positive bounded measure then $\|\partial_x u(t)\|_{L^\infty}$ is uniformly bounded on $\mathbb{R}_+$ and thus satifies the hypotheses of Theorem 6. But then u is actually a "strong" weak solution " which can be handled by Theorem 4.

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