Periodic Traveling Water Waves with Isobaric Streamlines

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Abstract

It is shown that in water of finite depth, there are no periodic traveling waves with the property that the pressure in the underlying fluid flow is constant along streamlines. In the case of infinite depth, there is only one such solution, which is due to Gerstner.

1 Introduction

Consideration is given to periodic traveling waves on the surface of a perfect fluid without the assumption that the flow is irrotational. While a great deal is known about traveling waves associated with irrotational flows, traveling waves with rotational underlying flow have not been studied much so far. There has been some controversy over the existence of such traveling waves [1], but it has recently been established that periodic traveling waves with non-zero vorticity do exist for the two-dimensional Euler equations [4, 5]. Another recent article deals with symmetry properties of these solutions [3]. The present paper is concerned with a particular issue regarding these traveling-waves solutions, namely with the behavior of the pressure functional. The main result states that for a traveling wave, the pressure cannot be constant along streamlines unless the depth is infinite. If the depth is infinite, then the only traveling-wave solution that has the above property is given by an explicit formula due to Gerstner [7]. Since it can be shown that this flow has non-zero vorticity, it follows in particular that if the flow is assumed irrotational, then a traveling-wave solution which has constant pressure along streamlines is not at all possible, whether or not the depth is finite.

The remainder of the introduction is devoted to the description of the physical setting and the governing equations. The main result of the paper is stated in Section 2. In Section 3, the problem is reformulated in variables convenient for the proof which is given in Section 4.

In order to explain the problem in more detail, let us first recall the physical setup. Let x denote the horizontal, and y the vertical variable. We assume that there is no variation of the flow in the transverse direction, so that the flow is two-dimensional. Let $u(x, y, t)$ and $v(x, y, t)$ be the horizontal and vertical velocity, respectively. It is assumed that the free surface can be described by a function $\eta(x,t)$, and that the domain contains an infinite expanse of fluid in the x-direction. In the y-direction, the fluid is either confined between the flat bottom at depth $-d$ and the free surface η, or in the case of infinite depth it is bounded only above by the free surface at $y=\eta$. The governing equations are the Euler equations

$$\left.\begin{array}{rcl} u_t + uu_x + vu_y &=& -P_x, \\ v_t + uv_x + vv_y &=& -P_y - g, \end{array}\right\}$$

where $P(x,y,t)$ is the pressure, and g is the gravitational acceleration. For convenience, the density of the fluid is assumed to be unity. The incompressibility is expressed by the condition that

$$u_x + v_y = 0. \tag{1.1}$$

The dynamic boundary condition at the free surface is

$$P = P_0 \quad \text{on} \quad y = \eta(x,t), \tag{1.2}$$

where P_0 is the atmospheric pressure which is assumed to be constant. The kinematic condition that the normal velocity of the fluid at the free boundary is equal to the normal velocity of the free boundary (cf. [14]) is given by the equation

$$v = \eta_t + u\eta_x \quad \text{on} \quad y = \eta(x,t).$$

At the bottom, it is assumed that the fluid cannot flow through the boundary, and this is expressed by the boundary condition

$$v = 0 \quad \text{on} \quad y = -d. \tag{1.3}$$

In the case of infinite depth, this condition is replaced by the requirement that there be no motion at great depths, viz.

$$\lim_{y\to-\infty} \left(u^2 + v^2\right) = 0. \tag{1.4}$$

In general, for traveling waves moving to the left at a speed $c > 0$, it is assumed that the free surface η, the velocity (u,v), and the pressure P depend only on the quantity $x + ct$, rather than on x and t separately. The Euler equations are then

$$\left.\begin{array}{rcl} cu_x + uu_x + vu_y &=& -P_x, \\ cv_x + uv_x + vv_y &=& -P_y - g, \end{array}\right\} \tag{1.5}$$

while the free-surface condition is

$$v = c\eta_x + u\eta_x \quad \text{on} \quad y = \eta(x,t). \tag{1.6}$$

Now since we are looking for traveling waves which are periodic in the x-variable, it appears that solutions of (1.5) may be rescaled so that they are periodic with wavelength 2π. If this is done, the gravitational constant will be given by a new value $\tilde{g}$. If the depth is finite, then it will also be given by a new value $\tilde{d}$. In the following, it is assumed that this rescaling has been carried out, and the original variables are used.

In the case of infinite depth, there is an explicit traveling-wave solution due to Gerstner [7], which has become known as Gerstner's wave. It is most conveniently expressed in Lagrangian variables. The particle paths in Gerstner's wave are circles given by

$$\left.\begin{array}{rcl} x(t) & = & a + e^b \sin(a + ct), \\ y(t) & = & b - e^b \cos(a + ct), \end{array}\right\} \tag{1.7}$$

where $c > 0$ is the speed of the surface wave, and the position of the particle at time $t = 0$ is given by (a, b), with $b \leq b_0 \leq 0$. Note that particles move counterclockwise, and the surface wave is propagating to the left.

One may consider a moving frame of reference by making the change of variables $(x, y) \to (x + ct, y)$. The particle paths then coincide with the streamlines, and are given in parametric form by

$$\left.\begin{array}{rcl} x(s) & = & s + e^b \sin(s), \\ y(s) & = & b - e^b \cos(s). \end{array}\right\} \tag{1.8}$$

These are the equations for a trochoid. They can be written in functional form as

$$\pm x = \arcsin\left(\frac{b - y}{e^b}\right) - \sqrt{(e^b)^2 - (b - y)^2}, \tag{1.9}$$

up to an arbitrary shift in the horizontal direction. The free surface is described by the trochoid with $b = b_0$. In the limiting case $b_0 = 0$, the surface is given by a cycloid. For a more thorough review of Gerstner's solution, the reader is referred to [2, 7, 9, 11].

Inspecting the flow corresponding to Gerstner's solution, it appears that the pressure P is constant along streamlines. This can be seen as follows. As stated in [2], the pressure associated to Gerstner's solution is given by

$$P = C - gb + \frac{g}{2}e^{2b} = C - g\left(b - \frac{1}{2}e^{2b}\right),$$

for some constant C, while the stream function, as defined in the next section, is given by $\psi = b - \frac{1}{2}e^{2b}$. Therefore, the pressure can be written as $P = C - g\psi$. This raises the question whether this is a special property of only Gerstner's solution, or if there are other solutions with the same property. In this paper, it is shown that if the depth is finite, then the only traveling-wave solution that has this property is the trivial shear flow with a flat surface and zero vertical velocity everywhere. If the depth is infinite, then there is a unique non-trivial traveling-wave solution with this property, and this solution is Gerstner's wave.

2 Main results

This section contains the statement of the main results of this article. In formulating these results, it is convenient to use the stream function ψ which is defined by requiring that

$$\psi_x = -v, \quad \text{and} \quad \psi_y = u - c.$$

This definition shows that

$$-\triangle\,\psi = \omega,$$

where $\omega = v_x - u_y$ is the vorticity of the flow. It can be shown that if a C^2- flow has no stagnation points, i.e. points where $v = 0$ and $u = c$, the vorticity ω is given by a function $\omega = \gamma(\psi)$ [5]. Experimental evidence suggests that the speed of the surface wave is considerably larger than the speed of the individual water particles [10, 13]. Therefore it is reasonable to assume that u is always less than the speed c of the progressive wave, and consequently that there are no stagnation points. The equation for ψ is then

$$-\triangle\,\psi = \gamma(\psi).$$

We define the primitive of γ by

$$\Gamma(p) = \int_0^p \gamma(-s)\,ds + C, \tag{2.1}$$

for some constant C to be determined later.

The stream function can also be defined using the mass flux across a vertical line in the fluid. The relative flux across a vertical line $x = x_0$ is defined by

$$p_0 = \int_{-d}^{\eta(x_0)} \big[u(x,y) - c\big]\,dy, \tag{2.2}$$

which is independent of x_0. Note that $p_0 < 0$, since $u < c$. The stream function ψ is determined uniquely up to a constant, and it is constant on the free surface and on the bottom. If we take the stream function to be zero at the free surface, then $\psi = p_0$ at the bottom.

Having defined the stream function, we are now ready to state the main results of this paper. In the following, the term "2π-periodic" always refers to the x-variable. Note also that we assume that γ is a C^1 function.

Theorem 1. *(Finite Depth) If u, v, η and P are twice continuously differentiable, 2π-periodic functions that solve (1.1, 1.2, 1.3, 1.5, 1.6), and the pressure P is constant along the streamlines $\psi = p$, then $\eta \equiv 0$.*

Thus it appears that the only possible traveling-wave solution in this case is the trivial shear flow. If the depth is infinite, there is a non-trivial solution given by Gerstner's wave.

Theorem 2. *(Infinite Depth) If u, v, η and P are twice continuously differentiable, 2π-periodic functions that solve (1.1, 1.2, 1.4, 1.5, 1.6), and the pressure P is constant along the streamlines $\psi = p$, then either $\eta \equiv 0$, or the particle paths are given by (1.7).*

3 Equivalent formulation

For the proof of these theorems, it will be convenient to reformulate the problem in different variables. First note that in terms of the stream function, the Euler equations become

$$\left.\begin{array}{rcl} \psi_y\psi_{xy} - \psi_x\psi_{yy} &=& -P_x, \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} &=& -P_y - g. \end{array}\right\} \tag{3.1}$$

Since the stream function is defined in such a way that the resulting problem is stationary, it is possible to integrate the equations (3.1) to yield the Bernoulli condition

$$\frac{(u-c)^2+v^2}{2}+P+gy-\Gamma(\psi)=E, \tag{3.2}$$

for some constant E. By virtue of the assumption that $\omega=\gamma(\psi)$, this equation holds throughout the fluid [8]. We can normalize Γ to absorb the constant E, and the equation can then be used to eliminate the pressure P from the boundary conditions. The problem for the stream function ψ is then

$$\triangle\psi \quad = \quad -\gamma(\psi) \quad \text{on} \quad -d<y<\eta(x), \tag{3.3}$$

with the boundary conditions

$$\left.\begin{array}{rcll} |\nabla\psi|^2+gy & = & -P_0 & \text{on} \quad y=\eta(x), \\ \psi & = & 0 & \text{on} \quad y=\eta(x), \end{array}\right\} \tag{3.4}$$

and

$$\psi \quad = \quad p_0 \quad \text{on} \quad y=-d. \tag{3.5}$$

Besides the formulation in terms of the stream function, we will need another formulation in terms of the height h above the flat bottom. As was mentioned before, ψ is constant along the surface and the flat bottom. Moreover, by construction ψ is strictly decreasing as a function of y. Hence for every fixed x, the height h above the flat bottom is a well defined function of ψ. Therefore, we may make the transformation $q=x$, $p=-\psi$, and regard ψ as an independent variable. Thereby, the domain of the fluid is mapped to a rectangle R, defined by $0\le q\le 2\pi$, and $p_0\le p\le 0$. This change of variables was introduced by Dubreil-Jacotin [6], and recently used in [5]. Now defining

$$h(q,p)=y+d, \tag{3.6}$$

it can be shown that the following relations hold.

$$h_q=\frac{v}{u-c}, \qquad h_p=\frac{-1}{u-c}. \tag{3.7}$$

By assumption, h is periodic in q, and it is plain that the free surface is given by $\eta(x)=h(x,0)$. An explicit calculation shows that h satisfies the quasi-linear elliptic equation

$$(1+h_q^2)h_{pp}-2h_ph_qh_{pq}+h_p^2h_{qq} \quad = \quad -\gamma(-p)h_p^3 \quad \text{on} \quad p_0<p<0, \tag{3.8}$$

with boundary conditions

$$\left.\begin{array}{rcll} 1+h_q^2+ghh_p^2 & = & 0 & \text{on} \quad p=0, \\ h & = & 0 & \text{on} \quad p=p_0. \end{array}\right\} \tag{3.9}$$

This observation can be formalized, and a convenient statement is given by the next proposition.

Proposition 1. *Assume that* $\gamma \in C^1([0,|p_0|])$. *Let* (u,v,η,P) *be a* 2π*-periodic* C^2 *solution of (1.1, 1.2, 1.3, 1.5, 1.6), and let* h *be given by (3.6). Then* h *is a* 2π*-periodic* C^2 *solution of (3.8, 3.9).*

The proof is given in [5]. In the proof, use is also made of the functions F and G, defined by

$$F(q,p) = \frac{1}{h_p} \qquad \text{and} \qquad G(q,p) = -\frac{h_q}{h_p}. \tag{3.10}$$

From $h_{pq} = h_{qp}$ and (3.8), it follows that F and G satisfy the following equations on the rectangle R.

$$F_q + F_p G - G_p F = 0,$$

$$G_q + G_p G + F F_p = \gamma(-p).$$

These relations will be of value in the proof of Lemma 1 appearing in the next section.

4 Proof of Theorem 1

The proof of Theorem 1 is achieved by establishing a number of auxiliary results. If it is assumed that the pressure is constant along streamlines, then we look for flows with the property that

$$\frac{F^2 + G^2}{2} + gh + P(p) - \Gamma(p) = 0, \tag{4.1}$$

where Γ is the primitive of γ, normalized to yield $\Gamma(0) = -E$.

Lemma 1. *Assume that* $\gamma \in C^1([0,|p_0|])$. *Let* (u,v,η,P) *be a* 2π*-periodic* C^2 *solution of (1.1, 1.2, 1.3, 1.5, 1.6), and let* h *be given by (3.6). If the pressure* P *is constant along the streamlines* $\psi = p$, *then* h *satisfies the equation*

$$\frac{1}{h_p(p,q)} = h(p,q)P_p(p) + \beta(p), \tag{4.2}$$

for some function $\beta(p)$.

Proof. Recall that F and G satisfy the equations

$$F_q + GF_p - FG_p = 0, \tag{4.3}$$

and

$$G_q + GG_p + FF_p = \gamma. \tag{4.4}$$

Differentiating (4.1) with respect to p and using (4.4), we find

$$P_p = G_q - gh_p.$$

Multiplying by G, and using (3.10), we see that

$$P_pG = GG_q - gh_pG = GG_q + gh_q.$$

Combining this with the relation obtained by differentiating (4.1) with respect to q, we get

$$P_pG + FF_q = 0.$$

If we divide the previous relation by F and use (3.10), we have

$$-P_ph_q + F_q = 0$$

or

$$\partial_q\Big(P_ph - \frac{1}{h_p}\Big) = 0.$$

Integrating the last equation yields (4.2). ■

Because of the unknown character of the functions P_p and β, the integration of (4.2) is problematic. However, as it turns out, it is possible to circumvent this difficulty.

Lemma 2. *Assume that $\gamma \in C^1([0, |p_0|])$. Let (u, v, η, P) be a 2π-periodic C^2 solution of (1.1, 1.2, 1.3, 1.5, 1.6), such that the pressure P is constant along the streamlines $\psi = p$. Let h be given by (3.6), Let Γ be given by (2.1), and let β be given by (4.2). Consider the functions*

$$a_0 = -2P_p\beta + 2P\beta_p - g - 2\Gamma\beta_p + \gamma\beta, \tag{4.5}$$

$$a_1(p) = 2g\beta_p - 2P_p^2 + \gamma P_p - 2\Gamma P_{pp} + 2PP_{pp}, \tag{4.6}$$

and

$$a_2(p) = 2gP_{pp}. \tag{4.7}$$

If $h_q(p, q) \neq 0$ for some $q \in [0, 2\pi]$ and $p \in (p_0, 0)$, then $a_0(p) = 0$, $a_1(p) = 0$, and $a_2(p) = 0$.

Proof. We know from (4.1) and (3.10) that

$$\frac{h_q^2 + 1}{2h_p^2} + gh - \Gamma + P = 0. \tag{4.8}$$

In combination with (4.2), we find

$$h_q^2 = -\frac{2(gh - \Gamma + P)}{[\beta + P_ph]^2} - 1. \tag{4.9}$$

Differentiating (4.9) with respect to p, and using (4.2) and (4.8), we obtain two expressions for $2h_qh_{qp}$ in terms of h and derivatives of γ, β, and P. We find the polynomial equation

$$a_2(p)h^2 + a_1(p)h + a_0(p) = 0, \tag{4.10}$$

where a_0, a_1 and a_2 are given by (4.5), (4.6) and (4.7), respectively. Differentiation of (4.10) with respect to q yields

$$2a_2(p)hh_q + a_1(p)h_q = 0.$$

If $h_q(p,q) \neq 0$, we can divide by h_q, and differentiate again to obtain

$$2a_2(p)h_q = 0.$$

Since $h_q \neq 0$, this shows that $a_2(p) = 0$. Consequently, (4.10) reduces to

$$a_1(p)h + a_0(p) = 0.$$

Differentiating this with respect to q, we find that

$$a_1(p)h_q = 0.$$

Now since $h_q \neq 0$, it follows that $a_1(p) = 0$. Finally equation (4.10) yields that $a_0(p) = 0$. ■

Lemma 3. *Assume that $\gamma \in C^1([0,|p_0|])$. Let (u, v, η, P) be a 2π-periodic C^2 solution of (1.1, 1.2, 1.3, 1.5, 1.6), such that the pressure P is constant along the streamlines $\psi = p$. Let h be given by (3.6). If $h_q(p,q) \neq 0$ for some p and q, then h is locally given by a trochoid.*

Proof. Assuming that P only depends on p yields equation (4.1). Now assume that $h_q(q,p) \neq 0$ at some point (p,q), and by continuity in an open neighborhood U around (p,q). Lemma 2 shows that then $a_2 \equiv 0$, $a_1 \equiv 0$ and $a_0 \equiv 0$ in a neighborhood U_p of p. From $a_2 \equiv 0$ we obtain

$$P = Ap + B, \tag{4.11}$$

for some constants A and B. Inserting $P = Ap + B$, into (4.6), and using $a_1 \equiv 0$, we obtain

$$2g\beta_p - 2A^2 + \gamma A = 0. \tag{4.12}$$

We will first prove that A cannot be zero. If $A = 0$, equation (4.12) shows that β is a constant, so that (4.5) and the fact that $a_0(p) \equiv 0$ imply that $\gamma(p) = \frac{g}{\beta}$. Equation (4.2) becomes $h_p = \dfrac{1}{\beta}$, so that

$$h(p,q) = \frac{1}{\beta}\,p + k(q),$$

with k periodic. But according to (3.8), the equation for k is

$$\frac{1}{\beta^2}\,k_{qq} = -\frac{g}{\beta}\,\frac{1}{\beta^3},$$

so that the function k cannot be periodic. It follows that $A \neq 0$.

Note that it follows from (4.12) that β is given by the expression

$$\beta(p) = -\frac{A}{2g}\,\Gamma(p) + \frac{A^2}{g}\,p + C_0, \tag{4.13}$$

for some constant of integration C_0. Using now (4.2), equation (4.8) becomes

$$h_q^2\,(P_p h + \beta)^2 + 2(gh - \Gamma + P) + (P_p h + \beta)^2 = 0.$$

As $A \neq 0$, this can be rewritten as

$$h_q^2\Big(h + \frac{\beta}{A}\Big)^2 + \Big(h + \frac{\beta}{A} + \frac{g}{A^2}\Big)^2 = \frac{g^2}{A^4} + \frac{2}{A^2}\Gamma + \frac{2g}{A^3}\beta - \frac{2}{A}p - \frac{2}{A^2}B. \tag{4.14}$$

Since it was assumed that $h_q \neq 0$, the previous relation forces the existence of $K(p) \neq 0$ with

$$K(p)^2 = \frac{g^2}{A^4} + \frac{1}{A^2}\Gamma(p) + \frac{2g}{A^3}C_0 - \frac{2}{A^2}B,$$

where (4.13) was used. Letting $H = h + \frac{\beta}{A} + \frac{g}{A^2}$, and suppressing the dependence of H and K on p for the moment, the differential equation (4.14) becomes

$$\frac{H_q(H - g/A^2)}{\sqrt{K^2 - H^2}} \overset{+}{\underset{-}{=}} 1. \tag{4.15}$$

Integrating, we obtain

$$\int_{H_0}^{H(q)} \frac{s - g/A^2}{\sqrt{K^2 - s^2}}\, ds \overset{+}{\underset{-}{=}} q$$

or

$$\sqrt{K^2 - H_0^2} - \sqrt{K^2 - H^2(q)} + \frac{g}{A^2}\left[\arcsin\left(\frac{H(q)}{K}\right) - \arcsin\left(\frac{H_0}{K}\right)\right] \overset{+}{\underset{-}{=}} q. \tag{4.16}$$

It can be shown by an elementary argument that for a fixed p, the solution of the ordinary differential equation (4.15) is unique as long as $H - \frac{g}{A^2} = h + \frac{1}{A}\beta$ is nonzero. Therefore at least locally, the streamlines are given by (4.16), which is the equation for a trochoid. ■

Proof of Theorem 1. It follows from Lemma 3 that whenever the height h depends on $x = q$, then the streamlines are given locally by a trochoid which can be written in parametric form as

$$\begin{aligned} q(s) &= \frac{g}{A^2}s + K(p)\sin(s), \\ h(s) &= -\frac{g}{A^2} - \frac{1}{A}\beta(p) - K(p)\cos(s). \end{aligned}$$

To show that this is not possible if the depth of the fluid is finite, we will show that the equation of continuity is not satisfied. The evolution of a particle is given by the composition $\mathcal{T}\mathcal{T}_0^{-1}$, where the two mappings are given by

$$\mathcal{T}_0\begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} \frac{g}{A^2}s + K(p)\sin(s) \\ -\frac{g}{A^2} - \frac{1}{A}\beta(p) - K(p)\cos(s) \end{pmatrix},$$

and

$$\mathcal{T}\begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} \frac{g}{A^2}(s+t) + K(p)\sin(s+t) \\ -\frac{g}{A^2} - \frac{1}{A}\beta(p) - K(p)\cos(s+t) \end{pmatrix}.$$

The incompressibility of the fluid can be expressed by the condition that the map $\mathcal{T}\mathcal{T}_0^{-1}$ is area-preserving. But the Jacobians are given by

$$J_0 = -\frac{g}{A^3}\beta' - KK' - \left(\frac{g}{A^2}K' + \frac{1}{A}\beta' K\right) A\cos(s),$$

and

$$J = -\frac{g}{A^3}\beta' - KK' - \left(\frac{g}{A^2}K' + \frac{1}{A}\beta' K\right) A\cos(s+t).$$

It is plain the JJ_0^{-1} will depend on t unless

$$K' + \frac{A}{g}\beta' K = 0.$$

Solving this equation for K gives

$$K(p) = K_0\, e^{-\frac{A}{g}\beta(p)},$$

for some constant K_0. Now remember that

$$\beta(p) = -\frac{A}{2g}\,\Gamma(p) + \frac{A^2}{g}\,p + C_0.$$

Since it was assumed that $\gamma = \Gamma_p$ is continuous up to the boundary, Γ cannot be logarithmic. Thus the dependence of K on p is exponential, so that all streamlines are given by nontrivial trochoids. In particular, this contradicts the requirement that the flat bottom at finite depth is a streamline. We conclude that for finite depth, the only possible flow is a shear flow with a flat surface. ■

The proof of Theorem 2 is exactly the same, except that the definition of h has to be altered slightly. In this case, h is defined as the height above the line $y = 0$. Since (4.16) is precisely the equation for a trochoid, (1.7) is in fact the only non-trivial solution of (1.1, 1.2, 1.4, 1.5, 1.6), which has constant pressure along streamlines.

Remark. Since the proof does not depend on a special form of the vorticity function γ, the special case $\gamma = 0$ is also allowed. Thus it follows that for irrotational flow, there is no non-trivial periodic traveling wave with the property that the pressure is constant along streamlines, even if the depth is infinite.

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