Journal of Nonlinear Mathematical Physics Volume 11, Number 4 (2004), 534–548 WAVE MOTION

$L_p - L_q$ Decay Estimates for Wave Equations with Time-Dependent Coefficients

Michael REISSIG

Faculty of Mathematics and Computer Science, TU Bergakademie Freiberg, Agricolastr.1, 09595 Freiberg, Germany
E-mail: reissig@math.tu-freiberg.de

This article is part of the Proceedings of the meeting at the Mathematical Research Institute at Oberwolfach (Germany) titled "Wave Motion", which took place during January 25-31, 2004

Abstract

The goal of this survey article is to explain the up-to-date state of the theory of $L_p - L_q$ decay estimates for wave equations with time-dependent coefficients. We explain the influence of oscillations in the coefficients by using a precise classification. Moreover, we will see how mass and dissipation terms take influence on the $L_p - L_q$ decay estimates.

1 Introduction

If one is interested in the Cauchy problem for non-linear wave equations like

$$u_{tt} - \triangle u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \ \ u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x), \tag{1.1}$$

then one should be prepared to meet several phenomena. Under reasonable regularity assumptions one can expect a local (in time) well-posedness result for solutions valued in Sobolev spaces. On the other hand the nonlinearity may cause a blow-up behavior of these solutions. Thus it seems to be reasonable to ask if under the assumption that the zero solution is a steady state solution there exist global (in time) small data solutions. This question applied to the above model (1.1) means to prove that for all $\varepsilon \in (0, \varepsilon_0(\varphi,\psi)]$ the Cauchy problem

$$u_{tt} - \triangle u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \ \ u(0,x) = \varepsilon\varphi(x), \ u_t(0,x) = \varepsilon\psi(x) \tag{1.2}$$

has a global in time solution. A positive answer to this question yields a *stability result* for the solution (here the trivial one) generating an equilibrium.

One of the key tools to prove such a global existence result is the so-called *Strichartz' decay estimate* (see [17]) for the energy $E(u)(t) := (\nabla u(t,\cdot), u_t(t,\cdot))|_{L_q}$ basing on the $L_q(\mathbb{R}^n)$-norm

$$E(u)(t)|_{L_q} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})} E(u)(0)|_{W_p^{N_p}} \tag{1.3}$$

on the conjugate line $2 \le q \le \infty$, $1/p + 1/q = 1$ for solutions of the Cauchy problem for classical wave equations, where $N_p > n\Big(\frac{1}{p} - \frac{1}{q}\Big)$.

Generalizing such type of estimates (with $-\frac{n}{2}$ instead of $-\frac{n-1}{2}$ in the decay rate) to Klein-Gordon equations or damped wave equations (with an additional term $-\frac{1}{2}$ in the decay rate of the latter case coming from the dissipation itself) one can show the global (in time) existence of small data solutions for

$$u_{tt} - \triangle u + m^2 u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u),\ \ u(0,x) = \varepsilon\varphi(x),\ u_t(0,x) = \varepsilon\psi(x),\ m > 0;$$
$$u_{tt} - \triangle u + u_t = f(u_t, \nabla u, \nabla u_t, \nabla^2 u),\ \ u(0,x) = \varepsilon\varphi(x),\ u_t(0,x) = \varepsilon\psi(x).$$

In all these applications the right-hand side is supposed to possess a structure related to the energy and to satisfy a special asymptotic behavior around 0, thus the *mass* or *dissipation* term is not allowed to include into the right-hand side.

Those *Strichartz' decay estimates* are proved in the literature for special partial differential equations or systems of partial differential equations with constant coefficients in connection with the study of models from e.g. elasticity, thermoelasticity, thermoviscoelasticity or electrodynamics (see [7] and references therein).

Several years ago the author formulated the following question:
Are we able to prove such $L_p - L_q$ decay estimates for wave equations with variable coefficients and with mass and dissipation but without drift term like

$$u_{tt} - a(t,x) \triangle u + m(t,x)u + b(t,x)u_t = 0.$$

Today we have a good understanding of the influence of time-dependent coefficients on such $L_p - L_q$ decay estimates. In general, the dependence on spatial and time variables causes difficulties. We have only a few results about energy decay (see [6], [3] and references therein). But we have to emphasize that in special situations where the coefficients depend only on the spatial variables a self-adjoint structure of the "elliptic part" allows to apply methods from spectral theory. An additional dependence on the time variable excludes the application of such methods up to now.

In Section 2 we will describe the up-to-date knowledge about $L_p - L_q$ decay estimates for wave equations with variable time-dependent coefficients containing mass and dissipation terms, too. The coefficients are allowed to be subjected to altering processes. Moreover, our precise knowledge helps to classify new non-linear models for which one can show the global existence of small data solutions. Some concluding remarks are given in Section 3.

2 Wave models with time-dependent coefficients

2.1 Wave equations with weak dissipation

The results of this subsection are taken from [11] and [19].

2.1.1 A model case

We begin our considerations for the Cauchy problem

$$u_{tt} - \triangle u + b(t)u_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \tag{2.1}$$

where as usually $b(t) \geq 0$. In this case the energy decays although it is not clear if it decays to 0. In the paper [11] the authors supposed the case of a *weak dissipation*, that is, $b' < 0, \lim_{t\to\infty} b(t) = 0$. The goal is to get a complete picture from the free wave ($b(t) \equiv 0$) to the damped wave case ($b(t) \equiv 1$, which does not represent a weak dissipation).

What kind of expectations does a specialist have?

• If b has a very weak influence, then there should be a relation to the free wave equation. Such relations are described by so-called *scattering results* and the corresponding *scattering operator*.

• If b has a weak influence, then the classical energy should decay to 0 and the corresponding $L_p - L_q$ decay estimate is the classical Strichartz decay estimate with an additional term as a time-dependent coefficient coming from the energy decay to 0 itself. Such weak dissipations will be called *non-effective*.

• There exists a critical case which brings a change of the influence of the weak dissipation. This critical case is discussed in [19].

• If b has a stronger influence, then the $L_p - L_q$ decay estimate is similar to that one for the damped wave equation but with another decay function related with the dissipation itself. Such weak dissipations will be called *effective*.

The critical case will be described by the family of Cauchy problems

$$u_{tt} - \triangle u + \frac{\mu}{1+t} u_t = 0,\ \mu > 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{2.2}$$

Using the theory of special functions (Bessel functions) a $L_p - L_q$ decay estimate was proved in [19] for the *energy operator* $\mathbb{E}(t) : (\langle D\rangle\varphi, \psi) \in L_{p,r} \to (u_t(t,\cdot), |D|u(t,\cdot)) \in L_q$, where $L_{p,r}$ denotes the Bessel potential space of order r basing on $L_p(\mathbb{R}^n)$.

Theorem 1. *The energy operator $\mathbb{E}(t)$ satisfies the $L_p - L_q$ decay estimate*

$$\|\mathbb{E}(t)\|_{L_{p,r}\to L_q} \leq C(1+t)^{\max\{-\frac{(n-1)}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}, -n(\frac{1}{p}-\frac{1}{q})-1\}} \tag{2.3}$$

on the conjugate line with $p \in (1,2]$ and $r = n(\frac{1}{p} - \frac{1}{q})$.

Remarks

• If $p = q = 2$, then the estimate (2.3) yields the $L_2 - L_2$ decay estimate

$$\|\mathbb{E}(t)\|_{L_2\to L_2} \leq C(1+t)^{\max\{-\frac{\mu}{2}, -1\}}.$$

Thus the norm of the energy operator depends on μ if $\mu \leq 2$, otherwise it is independent. Thus in the critical case $\mu = 2$ we obtain a maximal energy decay.

• If $\mu \leq 2$, then the $L_p - L_q$ decay estimate generalizes the Strichartz decay estimate (1.3). In this case the dissipation is non-effective.

• If $\mu \geq n+3$, then the $L_p - L_q$ decay estimate (2.3) yields

$$\|\mathbb{E}(t)\|_{L_{p,r}\to L_q} \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{q})-1}.$$

This decay estimate hints that the family of Cauchy problems (2.2) is something intermediate between Cauchy problems with non-effective and effective weak dissipations (cf. with the remarks after Theorem 3).

2.1.2 Non-effective weak dissipations

Let us devote to (2.1) with non-effective dissipations. Due to [11] such dissipations are characterized by the assumptions

(A1) $\int_0^t b(\tau)d\tau = \infty$;
(A2) $b(t) \geq 0$, $b'(t) < 0$, $b(t) \to 0$ if $t \to \infty$;
(A3) $(1+t)b(t) \leq C$ and $\lambda(t) \leq C(1+t)$ for $t \geq 0$, where $\lambda(t) = \exp(\frac{1}{2}\int_0^t b(\tau)d\tau)$;
(A4) $|D_t^k b(t)| \leq C_k b(t)(\frac{1}{1+t})^k$ for $t \geq 0$ and all positive integers k;
(A5) $\limsup_{t\to\infty} tb(t) < 1$.

Theorem 2. *Under the assumptions (A1) to (A5) the energy operator $E(t)$ satisfies the $L_p - L_q$ decay estimate*

$$\|\mathbb{E}(t)\|_{L_{p,r}\to L_q} \leq C\lambda(t)^{-1}(1+t)^{-\frac{(n-1)}{2}(\frac{1}{p}-\frac{1}{q})} \tag{2.4}$$

on the conjugate line with $p \in (1, 2]$ and $r = n(\frac{1}{p} - \frac{1}{q})$.

Remarks

- We conclude from (2.4) the decay estimate (2.3) for $\mu < 1$ (take account of (A5)), although it holds formally for $\mu \leq 2$, too.
- The classical energy decay is described by $\lambda(t)^{-1}$. This function may tend arbitrary slow to 0 as the next example shows:

$b(t) = \frac{\mu}{(e^{[n]}+t)\log(e^{[n]}+t)\cdots\log^{[n]}(e^{[n]}+t)}$ implies $\lambda(t) = (\log^{[n]}(e^{[n]}+t))^{\mu/2}$.

Here $\log^{[n]}$ denotes the n-times iterated logarithm and $e^{[n]}$ denotes the n-times application of e to the power.

Proof. We will only present the main steps of the proof. The x-independence of coefficients allows to apply the partial Fourier transformation. Thus we have to study a Cauchy problem for an ordinary differential equation depending on the parameter $|\xi|^2$. The goal is to find a WKB-representation of its solution. As usually these representations contain terms like

$$e^{i\phi(t,\xi)}a(t,\xi)w(\xi), \tag{2.5}$$

where $\phi = \phi(t,\xi)$ is the so-called *phase function* and $a = a(t,\xi)$ is the so-called *amplitude function*. One has to show that a finite number of derivatives of a with respect to ξ satisfies symbol like estimates. For this reason we divide the extended phase space $\mathbb{R}^n \times (0,\infty)$ (it is essential to include the t-variable into the phase space) into *zones*, into the *dissipative zone* $Z_{diss}(N) = \{(t,\xi) : (1+t)|\xi| \leq N\}$ and the *hyperbolic zone* $Z_{hyp}(N) = \{(t,\xi) : (1+t)|\xi| \geq N\}$. In both zones one has to estimate the fundamental solution. In $Z_{diss}(N)$ this can be done in a more or less straightforward way. More preparations needs the consideration in the hyperbolic zone. Here we need classes of symbols

$$S_N\{m_1,m_2,m_3\} =: \Big\{a = a(t,\xi) : |D_t^k D_\xi^\alpha a(t,\xi)| \leq C_{k,\alpha}|\xi|^{m_1-|\alpha|}b(t)^{m_2}\Big(\frac{1}{1+t}\Big)^{m_3+k}$$
$$\text{in } Z_{hyp}(N) \text{ for all } k \text{ and for all } \alpha\Big\}.$$

Then one can start a diagonalization procedure related to these symbol classes to the partial Fourier transformed Cauchy problem (2.1). A sufficiently large number of steps guarantees that the amplitudes behave like symbols for a finite number of derivatives with respect to ξ. Some additional techniques lead to the desired WKB-representation consisting of terms like (2.5). Consequently, we arrive at a representation of the solution to (2.1) by the aid of Fourier multipliers.

Now one has to discuss these Fourier multipliers. This will be done after localizing the amplitudes by using of a dyadic decomposition related to the extended phase space. The part of the dyadic decomposition which belongs to the dissipative zone generates Fourier multipliers which can be studied by the *Hardy-Littlewood inequality*. The part which belongs to the hyperbolic zone needs some *Littman-type lemmas* (see [4]). These are results for oscillating integrals with localized amplitude away from the origin. The *stationary phase method* together with usual properties of the Fourier transformation leads to a $L_1 - L_\infty$ estimate for Fourier multipliers with localized amplitude. After deriving a $L_2 - L_2$ estimate (here we can use directly the structure of the terms (2.5)) some interpolation gives $L_p - L_q$ estimates on the conjugate line. Gluing all these estimates for Fourier multipliers with localized amplitudes together leads to the desired $L_p - L_q$ decay estimate for the Fourier multiplier, the energy operator itself on the conjugate line for $p \in (1, 2]$. The supposed regularity we need to avoid constants depending on the parameter of the dyadic decomposition in the $L_p - L_q$ decay estimates for the Fourier multipliers with localized amplitudes. ■

Comments

• The study of wave equations with non-effective dissipation bases on two completely different ideas: the application of Hardy-Littlewood inequality on the one hand and the proof of Littman-type lemmas for oscillating integrals on the other hand.

• The energy decay in the $L_2 - L_2$ estimate appears from the behavior of the amplitudes in the hyperbolic zone.

2.1.3 Effective weak dissipations

Effective dissipations are characterized in [11] by the assumptions (A1),(A2),(A4) and (A6) $tb(t) \to \infty$ if $t \to \infty$.

Theorem 3. *Under the assumptions (A1),(A2),(A4) and (A6) the energy operator* $\mathbb{E}(t)$ *satisfies the* $L_p - L_q$ *decay estimate*

$$\|\mathbb{E}(t)\|_{L_{p,r} \to L_q} \leq C\Big(1 + \int_0^t b(\tau)^{-1} d\tau\Big)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \tag{2.6}$$

on the conjugate line with $p \in (1, 2]$ *and* $r = n(\frac{1}{p} - \frac{1}{q})$.

Remarks

• If we choose $\mu \geq n+3$ in (2.3) and $b(t) = \frac{\mu}{1+t}$ in (2.6), then we conclude (2.3). Although this family of dissipations does not satisfy (A6) the last observation hints that this family is something intermediate between families of non-effective and effective dissipations.

• The classical damped wave equation has no weak dissipation but nevertheless an effective dissipation as the corresponding $L_p - L_q$ decay estimate shows.

• Typical examples of effective weak dissipations are the following ones:
$b(t) = \frac{\log^{[n]}(e^{[n]}+t)}{1+t}$; $b(t) = (1+t)^{-\kappa}, \kappa \in (0, 1)$; $b(t) = \frac{1}{\log^{[n]}(e^{[n]}+t)}$.

Proof. We will only discuss some changes to the proof of Theorem 2. $L_p - L_q$ decay estimates for the solutions of the damped wave equation were studied in [5] after the construction of explicit representation of the solution. This construction bases on hyperbolic as well as elliptic WKB-constructions. This results from the the fact that $b(t) \equiv 1$ represents an effective dissipation (in a more general sense as considered in this subsection).

Consequently, our approach needs more zones, the most essential ones are the dissipative zone (see the previous proof), the *elliptic zone* $Z_{ell}(t_0, \varepsilon, N) = \{(t,\xi) : \frac{N}{1+t} \le |\xi| \le (1-\varepsilon)\frac{b(t)}{2}\} \cap \{t \ge t_0\}$ and the hyperbolic zone $Z_{hyp}(N') = \{(t,\xi) : |\xi| \ge N' b(t)\}$. Of importance is the *separating line* $\Gamma_{sep} = \{(t,\xi) : |\xi| = \frac{b(t)}{2}\}$. In the elliptic and hyperbolic zone we define classes of symbols

$$S_{ell}\{m_1, m_2\} =: \Big\{ a = a(t,\xi) : |D_t^k D_\xi^\alpha a(t,\xi)| \le C_{k,\alpha} b(t)^{m_1 - |\alpha|} \Big(\frac{1}{1+t}\Big)^{m_2+k} \text{ in } Z_{ell}(t_0,\varepsilon,N) \text{ for all } k \text{ and for all } \alpha \Big\};$$

$$S_{hyp}\{m_1, m_2, m_3\} =: \Big\{ a = a(t,\xi) : |D_t^k D_\xi^\alpha a(t,\xi)| \le C_{k,\alpha} \langle\xi\rangle_t^{m_1 - |\alpha|} b(t)^{m_2} \Big(\frac{1}{1+t}\Big)^{m_3+k} \Big\} \text{ in } Z_{hyp}(N') \text{ for all } k \text{ and for all } \alpha \Big\},$$

where $\langle\xi\rangle_t = \sqrt{\big||\xi|^2 - \frac{1}{4}b^2(t)\big|}$. In both zones we carry out a WKB-construction of elliptic or of hyperbolic type. In the case of effective dissipations the $L_p - L_q$ decay estimates follow from standard properties for Fourier multipliers without using the stationary phase method and from estimates of the fundamental solution by a Gauss function. In the phase function from (2.5) there appears an integral over $\langle\xi\rangle_t$ which would cause new difficulties to prove a Littman-type result if it would be necessary. ■

Comment

• The energy decay in the $L_2 - L_2$ estimate appears from the behavior of the amplitudes in a neighborhood of the separating line Γ_{sep}.

2.1.4 Scattering result

If we do not suppose in Theorem 2 the assumption (A1), then the corresponding $L_2 - L_2$ estimate gives that the classical energy does not tend to 0 if t tends to infinity. This proposes that there is a connection between a solution to (2.1) and a solution to the free wave equation.

We denote by $E := |D|^{-1}L_2 \times L_2$ the energy space. Our goal is to compare in the energy space the solution $(u(t,\cdot), u_t(t,\cdot))$ to (2.1) with the solution $(v(t,\cdot), v_t(t,\cdot))$ to the Cauchy problem

$$v_{tt} - \triangle v = 0,\ v(0,x) = \tilde{\varphi}(x),\ u_t(0,x) = \tilde{\psi}(x). \tag{2.7}$$

For this reason we define the operators $S(t,s) := (u(s), u_t(s))^T \to (u(t), u_t(t))^T$ and $S_0(t,s) := (v(s), v_t(s))^T \to (v(t), v_t(t))^T$ describing the evolution of the solutions to (2.1), (2.7), respectively. To get a scattering result one defines the operator $W_+ := \lim_{t\to\infty} S_0(0,t)S(t,0)$. In [11] it is shown the norm-convergence

$\|W_+ - S_0(0,t)S(t,0)\|_{L_2\to L_2} \to 0$ if $t\to\infty$. By the aid of this operator we can describe the above mentioned connection.

Theorem 4. *We assume for the weak dissipation $b=b(t)$ the condition $\int_0^\infty b(t)dt<\infty$. Then there exists an isomorphism $W_+ : E \to E$ of the energy space such that for the solution u of (2.1) to data (φ,ψ) and the solution v to (2.7) to data $(\tilde\varphi,\tilde\psi) := W_+(\varphi,\psi)$ the estimate*

$$\|(u(t,\cdot),u_t(t,\cdot)) - (v(t,\cdot),v_t(t,\cdot))\|_E \le C\|(\varphi(\cdot),\psi(\cdot))\|_E \int_t^\infty b(\tau)d\tau \tag{2.8}$$

holds, where the constant C depends only on $\|b\|_{L_1}$.

Remarks

- We restricted ourselves to the forward Cauchy problem. Without new difficulties one can consider the backward Cauchy problem, too. We have only to suppose $b\in L_1(\mathbb{R})$. Then we get W_-. Using W_+ and W_- one can define the *scattering operator* which is of special interest.
- Besides the above mentioned norm-convergence we can describe by (2.8) how the convergence rate can be estimated. This rate can be arbitrary small how the following examples show:

$b(t) = \frac{1}{(e^{[n]}+t)\log(e^{[n]}+t)\cdots(\log^{[n]}(e^{[n]}+t))^\gamma}$, $\gamma>1$, $b(t)=(1+t)^{-\kappa}$, $\kappa>1$.

2.2 Wave equations with time dependent coefficients containing mass and dissipation

2.2.1 Wave equations with variable speed of propagation

To get a first impression what may happen in the case of variable speed of propagation let us consider

$$u_{tt} - (2+\sin t)\triangle u = 0,\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x). \tag{2.9}$$

Then in [12] the following result is proved.

Theorem 5. *There are no constants p,q,r and C and a nonnegative function f defined on $\mathbb{N}$ such that the estimate*

$$\|\mathbb{E}(m)\|_{L_{p,r}\to L_q} \le Cf(m) \tag{2.10}$$

is fulfilled for all $m\in\mathbb{N}$ while $\log f(m)=o(m)$ as $m\to\infty$.

Remarks

- The conditions for f are very near to optimal ones. Indeed, due to Gronwall's inequality one can prove the L_2-L_2 estimate (which gives in general a very weak energy estimate)

$$\|\mathbb{E}(t)\|_{L_2\to L_2} \le C\exp(C_0 t)$$

with suitable nonnegative constants C and C_0. Choosing $t=m$, $m\in\mathbb{N}$, $p=q=2$, $r=0$, we get an inequality like (2.10) with $\log f(m)=O(m)$ as $m\to\infty$.
- The above example shows the deteriorating influence of oscillating behavior of time-dependent coefficients on L_p-L_q decay estimates.

Proof. Let us sketch how the oscillating behavior of the coefficient implies the *instability of the zero solution*. For this reason we study a family of solutions $\{u_M\}$ to (2.9) with data $\{\varphi_M, \psi_M\}$, $M \in \mathbb{N}$. With a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = 1$ when $|x| \leq 1$, and $\chi(x) = 0$ when $|x| \geq 2$ let us choose the initial data (known from *geometric optics*)

$$\varphi_M(x) = e^{ix\cdot y}\chi(\tfrac{x}{M^2}), \quad \psi_M = e^{ix\cdot y}\chi(\tfrac{x}{M^2})C_0,$$

with a suitable constant C_0. Using the finite propagation speed of solutions one can construct a dependence domain with size depending on $O(M)$ in such a way that the solution u_M has to possess in this domain the structure $u_M(t,x) = \exp(ix \cdot y)w_M(t)$, where w_M is a solution to $w_{tt} + |y|^2(2 + \sin t)w = 0$. In this way the relation to Floquet's theory should be clear. If we choose $|y|^2$ from an *interval of instability* for Hill's equation with periodic coefficient $2 + \sin t$, then the fundamental matrix has an eigenvalue with modul larger than one. Thus the effect of instability comes in. ■

The next question is to understand if a "slower" oscillating behavior may lead to some kind of instability behavior. For this reason let us consider the Cauchy problem

$$u_{tt} - (2 + \sin((\log(t + 30))^\alpha)) \triangle u = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \tag{2.11}$$

with $\alpha > 0$. Then the following statement can be concluded from the results of [10].

Theorem 6. *Consider the Cauchy problem (2.11). Then there exists a constant C such that the following $L_p - L_q$ estimate holds:*

$$\|\mathbb{E}(t)\|_{L_{p,r}\to L_q} \leq C(1+t)^{s_0 - \frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})} \tag{2.12}$$

on the conjugate line with $p \in (1,2]$, $r = n(\frac{1}{p} - \frac{1}{q})$ and
- *$s_0 = 0$ if $\alpha \leq 1$;*
- *$s_0 = \varepsilon$ if $\alpha \in (1,2)$ for all sufficiently small positive ε;*
- *s_0 is a fixed positive constant if $\alpha = 2$;*
- *there does not exist a positive constant s_0 such that (2.12) is satisfied if $\alpha > 2$.*

Comment

- The results for the above family of Cauchy problems (2.11) explain the sensitivity of oscillating behavior of coefficients on $L_p - L_q$ decay estimates. The constant s_0 describes how the decay rate differs from the classical Strichartz' decay rate for the wave operator.
- From the last statement we understand the change of possible $L_p - L_q$ decay estimates from the case $\alpha \leq 2$ to the case $\alpha > 2$. But we can give a more precise description of the new quality which comes in for $\alpha > 2$. For this reason we assume that the statement for $\alpha = 2$ remains valid for $\alpha > 2$. Then one expects for the family of forward Cauchy problems

$$u_{tt} - (2 + \sin((\log(t + 30))^\alpha)) \triangle u = 0,\ u(t_0,x) = \varphi(x),\ u_t(t_0,x) = \psi(x), \tag{2.13}$$

a $L_p - L_q$ estimate of the energy operator like

$$\|\mathbb{E}(t,t_0)\|_{L_{p,r}\to L_q} \leq C(1+t)^{s_0}$$

with a finite s_0 uniformly for all $t \geq t_0 \geq 0$. But this estimate is a contradiction to the following result from [10] (cf. with Theorem 5):

Theorem 7. *Let us consider (2.13) with $\alpha > 2$. There are no constants p, q, r, s and C_1, C_2 such that for all initial times t_0 and for all initial data $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, the following $L_p - L_q$ estimate holds for all $t \geq t_0$:*

$$E(u)(t)\big|_{L^q} \leq C_1 \exp(C_2(\log(t+e))^s) E(u)(t_0)\big|_{Lp,r},$$

where $s < \alpha - 1$. Here the (non-standard) energy $E(u)(t)\big|_{L_{p,r}}$ is defined by

$$E(u)(t)\big|_{L_{p,r}} := \Big\|\sigma(t)\nabla_x u(t,\cdot)\Big\|_{L_{p,r}} + \Big\|\frac{1}{\sigma(t)^2}\partial_t(u(t,\cdot)\sigma(t))\Big\|_{L_{p,r}}$$

with $\sigma(t) := \sqrt{\frac{\alpha(\log(t+e))^{\alpha-1}}{t+e}}$.

The result from Theorem 6 proposes the following classification of oscillating coefficients for the strictly hyperbolic Cauchy problem with bounded coefficient,

$$u_{tt} - a(t) \triangle u = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{2.14}$$

Definition 1. Let $a = a(t)$ be a smooth function satisfying

$$|D_t^k a(t)| \leq C_k \Big(\frac{1}{t}\Big(\log t\Big)^\gamma\Big)^k, \quad k \in \mathbb{N},\ \text{for large}\, t. \tag{2.15}$$

We say that the oscillations of a are *very slow, slow* or *fast* if $\gamma = 0$, $0 < \gamma < 1$ or $\gamma = 1$ respectively. If (2.15) is not satisfied for $\gamma = 1$, then we say that a has *very fast oscillations*.

If we apply this definition to the coefficient from (2.11), then we see that the oscillations are very slow, slow, fast, very fast if $\alpha \leq 1$, $\alpha \in (1,2)$, $\alpha = 2$, $\alpha > 2$.

One can generalize the statement from Theorem 6 to Cauchy problems (2.14) satisfying (2.15). To derive decay estimates of the energy operator one can use the methods to study $L_p - L_q$ decay estimates for wave equations with non-effective dissipation in the sense to couple representation of solutions by Fourier multipliers with Hardy-Littlewood inequality and stationary phase method. We define suitable zones of the phase space, symbol classes in these zones, e.g. such classes of amplitudes satisfying in the hyperbolic zone estimates like

$$|D_t^k D_\xi^\alpha a(t,\xi)| \leq C_{k,\alpha}|\xi|^{m_1-|\alpha|}(\tfrac{1}{t+e^3}(\log(t+e^3))^\gamma)^{m_2+k} \tag{2.16}$$

for all k and α with a real m_1 and a nonnegative m_2. Here one term takes account of the oscillating behavior of a. In the WKB-representations there appear phase functions (see (2.5)) containing terms like $|\xi|\int_0^t \sqrt{a(s)}ds$. The proof of Littman-type lemmas using the method of stationary phase is more or less standard. The other steps are technical but well understood.

The proof of the statement of Theorem 6 for $\alpha > 2$ bases on the application of Floquet's theory, too. The coefficient is in opposite to that one from (2.9) not pure periodic. If one tries to transform it to the periodic case, then additional terms appear. For this reason the method how to prove such *instability results* for non-periodic coefficients should be developed (cf. with [16], where for the first time such an approach was presented).

The specialists of wave equations may ask the following question:
Is is possible to generalize the results for (2.14) to strictly hyperbolic Cauchy problems with bounded coefficients like

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)?$$

This seems to be a difficult problem, but not from the point of view of the construction of WKB-solutions. This approach can be generalized without new difficulties. The main problem seems to be to prove a Littman-type result. In the oscillating integrals with localized amplitudes there appear the phase functions $\phi_\pm = \phi_\pm(t,\xi) = i(x\cdot\xi \pm \int_0^t \sum_{k,l=1}^n a_{kl}(s)\xi_k\xi_l ds)$. One has to study the rank of the *Hesse matrix* in *stationary points* of these phase functions. The main difficulty is to get an estimate uniformly for all t. Here the integral together with oscillations in a_{kl} bring difficulties. In [15] a so-called *stabilization condition* for the coefficients a_{kl} and b_k was introduced to derive $L_p - L_q$ decay estimates for the solutions of the strictly hyperbolic Cauchy problem with *increasing in time coefficients* (see the following considerations)

$$u_{tt} + \sum_{k=1}^{n} b_k(t) u_{x_k t} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Such a stabilization condition restricts the admissible oscillating behavior of coefficients.

The next step is to study $L_p - L_q$ decay estimates for

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x). \tag{2.17}$$

The coefficient consists of two parts, one part $\lambda(t)^2$ describes the *increasing behavior* of the coefficient, the other part $b(t)^2$ describes the *oscillating behavior*. In opposite to the *deteriorating influence of oscillations* an increasing behavior has an *improving* influence. The latter was shown for special examples in [9], [11], and [1]. Thus we expect an interplay between the influence of both parts. That it is really so show the next examples. The related hyperbolic energy for the solutions to (2.17) is $E(u)(t)\big|_{L_q} := \|(\lambda(t)\nabla u(t,\cdot), u_t(t,\cdot))\|_{L_q}$.

Examples The following examples allow $L_p - L_q$ decay estimates for (2.17):
- *Logarithmic growth*: $\lambda(t) = \log(t+e^3)$, $b(t) = 2 + \sin((\log(t+e^3))^\alpha)$ for $\alpha \le 2$ (cf. with (2.11) and Theorem 6);
- *Potential growth*: $\lambda(t) = (1+t)^\beta$, $\beta > 0$, $b(t)$ as in the previous example;
- *Exponential growth*: $\lambda(t) = \exp(t^\beta), \beta \ge 1/2$, $b(t) = 2 + \sin(t + e^3)$;
- *Superexponential growth*: $\lambda(t) = \exp(\exp(t^\beta)), \beta > 0$, $b(t)$ as in the previous example.

If we apply the main result from [16] to special situations then we see that the following examples do not allow $L_p - L_q$ decay estimates for (2.17):
- *Potential growth*: $\lambda(t) = (1+t)^\beta$, $\beta \ge 0$, $b(t)$ is an arbitrary periodic, non-constant, smooth and positive function;
- *Exponential growth*: $\lambda(t) = \exp(t^\beta), \beta < 1/2, b(t)$ is as in the previous example.

The interplay between increasing and oscillating behavior of the coefficients will be described by a classification of oscillations generalizing the proposed classification from Definition 1.

Definition 2. Let the oscillating part $b = b(t)$ satisfy with $\Lambda(t) := \int_0^t \lambda(s)ds$ the conditions

$$|D_t^k b(t)| \le C_k \Big(\frac{\lambda(t)}{\Lambda(t)} \Big(\log \Lambda(t) \Big)^{\gamma} \Big)^k, \quad k \in \mathbb{N}, \ \text{for large } \ t. \tag{2.18}$$

We say that the oscillations of b are *very slow, slow* or *fast* if $\gamma = 0, \ 0 < \gamma < 1$ or $\gamma = 1$ respectively. If (2.18) is not satisfied for $\gamma = 1$, then we say that b has *very fast oscillations.*

Remarks

- If $\lambda(t) \equiv 1$, then we obtain the condition (2.15).
- If $\lambda(t) = \exp(t^{\beta}), \beta > 0$, and $b(t)$ is an arbitrary periodic, non-constant, smooth and positive function, then the oscillations are very slow, slow, fast, very fast if $\beta \ge 1$, $\beta \in (1/2, 1)$, $\beta = 1/2$, $\beta < 1/2$, respectively.
- If the oscillations are at least fast, then one can expect $L_p - L_q$ decay estimates. If the oscillations are very fast, then one cannot expect such decay estimates. One has to expect statements like those from Theorems 5 and 7 (see [16]).

The $L_p - L_q$ decay estimates are similar to (2.12). If we assume $\lambda(0) > 0$, then the following estimate holds for the energy operator $\mathbb{E}(t)$ which bases on the hyperbolic energy $E(u)(t)|_{L_q} := \|(\lambda(t)\nabla u(t,\cdot), u_t(t,\cdot))\|_{L_q}$:

$$\|\mathbb{E}(t)\|_{L_{p,r} \to L_q} \le C(1 + \Lambda(t))^{s_0 - \frac{n-1}{2}(\frac{1}{p} - \frac{1}{q})} \tag{2.19}$$

on the conjugate line with $p \in (1, 2]$ and with a suitable r. The necessary regularity is related to $n(\frac{1}{p} - \frac{1}{q})$. The improving influence of the increasing part is reflected to $\Lambda(t)$ in the decay function. The decay rate is unchanged.

Comment

- Up to now we have not studied the exact relation between the type of oscillations and the constant s_0 appearing in the decay rate (cf. with Theorem 6). In [14] the case of fast oscillations is studied. The case of slow oscillations is studied in [12]. Nevertheless a sharp relation between γ in (2.18) and s_0 in (2.19) is open up to now. We have the following conjectures:

Conjecture 1 At most potential growth of λ

$$\|\mathbb{E}(t)\|_{L_{p,r} \to L_q} \le C\sqrt{\lambda(t)}(1 + \Lambda(t))^{s_0 - \frac{n-1}{2}(\frac{1}{p} - \frac{1}{q})}$$

with $s_0 = 0$, $s_0 = \varepsilon$, s_0 is a finite positive constant, and there does not exist a finite s_0 if $\gamma = 0, \gamma \in (0, 1), \gamma = 1$, (2.18) is not satisfied for $\gamma = 1$, respectively. The regularity r is related to $n(\frac{1}{p} - \frac{1}{q})$.

Conjecture 2 At least exponential growth of λ

$$\|\mathbb{E}(t)\|_{L_{p,r} \to L_q} \le C\sqrt{\lambda(t)}(1 + \Lambda(t))^{s_0 - \frac{n-1}{2}(\frac{1}{p} - \frac{1}{q})}$$

with $s_0 = \varepsilon$, s_0 is a finite positive constant, and there does not exist a finite s_0 if $\gamma \in [0, 1), \gamma = 1$, (2.18) is not satisfied for $\gamma = 1$, respectively. The regularity r is related to $n(\frac{1}{p} - \frac{1}{q})$.

2.2.2 Variable Mass

In the introduction we recalled Strichartz' decay estimate (1.3) for the solutions to the wave equation. If we are interested in the Cauchy problem for the Klein-Gordon equation

$$u_{tt} - \triangle u + m^2 u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; m > 0,$$

then $L_p - L_q$ decay estimates for its solutions are proven in [18] or [7]. The method of proof bases on a transformation of the Klein-Gordon equation to a wave equation in $n+1$ variables. If we study the corresponding representation of the solution by Fourier multipliers, then we expect the following $L_p - L_q$ decay estimate for the energy operator $\mathbb{E}(t)$ basing on the energy $E(u)(t)\big|_{L_q} := \|(u(t,\cdot), \nabla u(t,\cdot), u_t(t,\cdot))\|_{L_q}$:

$$\|\mathbb{E}(t)\|_{L_{p,r} \to L_q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \tag{2.20}$$

on the conjugate line with $p \in (1,2]$ and $r = n(\frac{1}{p} - \frac{1}{q})$.
Comment
• Using Fourier multipliers the term $\langle \xi \rangle_m := \sqrt{|\xi|^2 + m^2}$ appears instead of $|\xi|$ in the phase functions (the mass term is included into the phase). Then the stationary phase method gives the better Littman-type estimate with $n/2$ instead of $(n-1)/2$ (cf. (2.20) with (1.3)) because the Hesse matrix has full rank n instead of $n-1$ in stationary points.

The paper [13] is devoted to the Klein-Gordon type model

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u + m^2 \lambda(t)^2 b(t)^2 u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; m > 0. \tag{2.21}$$

Here λ and b describe the same (increasing or oscillating) behavior as in (2.17). Then the mass term has an essential influence on the interplay between $\lambda(t)$ and $b(t)$ as the following examples show:
Examples
• If $\lambda(t) \equiv 1$ and $b(t)$ is a periodic, non-constant, smooth and positive function, then we can prove a similar result to Theorem 5.
• If $\lambda(t) = (1+t)^l$ and b is as before, then we have a $L_p - L_q$ decay estimate if $l \ge 1$. If we compare this observation with the examples from the previous subsection we see, that now less growth of λ is necessary to compensate the periodic part to get $L_p - L_q$ decay estimates.

Open problem Up to now we have only a conjecture what happens in the case $\lambda(t) = (1+t)^l$, $l \in (0,1)$ and $b(t)$ supposed as a periodic, non-constant, smooth and positive function.

The above examples propose a change of condition (2.18) describing the interplay between increasing and oscillating part.

Definition 3. Let the oscillating part $b = b(t)$ satisfy with $\Lambda(t) := \int_0^t \lambda(s)ds$ the conditions

$$|D_t^k b(t)| \le C_k \Big(\frac{\lambda(t)}{\Lambda(t)^\gamma} \Big)^k, \quad k \in \mathbb{N}, \text{ for large } t. \tag{2.22}$$

We say that the oscillations of b are *slow* or *fast* if $\gamma \in (1/2, 1], \gamma = 1/2$ respectively. If (2.22) is not satisfied for $\gamma = 1/2$, then we say that b has *very fast oscillations*.

Then the results from [13] yield the following statement.

Theorem 8. *Consider the Cauchy problem (2.21) under the assumption (2.22). Then there exists a constant C such that the following $L_p - L_q$ estimate holds:*

$$\|\mathbb{E}(t)\|_{L_{p,r}\to L_q} \le C\sqrt{\lambda(t)}(1+\Lambda(t))^{s_0-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \tag{2.23}$$

on the conjugate line with $p \in (1,2]$, $r = n(\frac{1}{p}-\frac{1}{q})$ and
- $s_0 = 0$ *if* $\gamma \in (1/2, 1]$*;*
- s_0 *is a fixed positive constant if* $\gamma = 1/2$.

Here we are interested only in estimates of the hyperbolic energy
$E(u)(t)\big|_{L_q} := \|(\lambda(t)\nabla u(t,\cdot), u_t(t,\cdot))\|_{L_q}$.

There appears the term $n/2$ in the decay rate of the estimate (2.23). We call the *mass term effective* if such an improvement of the decay rate in the sense that $n/2$ instead of $(n-1)/2$ appears. The paper [3] is devoted to a family of Klein-Gordon models which allow to study the change from $(n-1)/2$ to $n/2$, that is, the change from models with *non-effective mass term* to models with *effective mass term*.

The family of Cauchy problems is

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u + \frac{\lambda(t)^2 b(t)^2}{(e^3+\Lambda(t))^{2\gamma}(\log(e^3+\Lambda(t)))^{2\delta}} u = 0,$$
$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

under the condition

$$|D_t^k b(t)| \le C_k\Big(\frac{\lambda(t)}{\Lambda(t)^\beta}\Big(\log \Lambda(t)\Big)^{-\omega}\Big)^k \qquad \text{for large } t,$$

for $\beta \in [0,1]$, $\gamma \ge 0$ and $\delta \ne 0$ only for $\gamma = 1$, where $\omega = 0$ if $\beta \in [0,1)$ and $\omega \in (-\infty, 0]$ if $\beta = 1$. For $\gamma = 0$ we have (2.23), for $\gamma = \infty$, due to our arrangement this means no mass, we have (2.19) if $\omega \in (-1,0]$. It seems to be reasonable and we will follow this strategy that the oscillating behavior in the mass term coincides with that one from the main part.

What kind of results do the readers expect?

Cases $\beta \in (\frac{1}{2},1);\ \beta = 1, \omega < -1$:
- If γ is small, this means we have a bit smaller mass than in the case $\gamma = 0$, then one should expect a Klein-Gordon type $L_p - L_q$ decay estimate.
- If γ is large, this means we have a small mass, then one cannot expect a $L_p - L_q$ decay estimate, maybe one has a Floquet-type result.

It should be interesting to describe the change-over from Klein-Gordon to Floquet, to find the critical mass $\gamma_0 = \gamma_0(\beta)$ and to study the influence of the mass on decay estimates.

Case $\beta = 1,\ \omega \in (-1,0]$:
- If γ is small, then one should expect a Klein-Gordon type decay rate.
- If γ is large, then one should expect a wave decay rate.

It should be interesting to describe the change-over from Klein-Gordon to wave decay rates, *to find the critical mass $\gamma_0 = \gamma_0(\beta)$ and to study the influence of the mass on decay estimates.*

Case $\beta \in [0, \frac{1}{2})$:
• If we have no mass ($\gamma = \infty$), then one cannot expect a $L_p - L_q$ decay estimate.
It should be interesting to study what happens during the change-over from $\gamma = \infty$ to $\gamma = 0$. Does a Floquet-type effect appear? What is the influence of the mass term?

In [3] there are answers to all of these questions and the change-over to the critical cases is described.

3 Concluding remarks

• The model with dissipation (related to Klein-Gordon model (2.21))

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u + a\lambda(t)b(t)u_t = 0, \;\; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x), \; a \neq 0,$$

is studied in [8]. There we introduce the same classification of oscillations is given in Definition 3. We are able to prove a statement like Theorem 8 for solutions to this wave model.
• Dissipative wave models with coefficients depending on time and spatial variables, too, are discussed in [2] and [6] (see further references therein). Using the method of weighted energy inequalities (this method differs from our approach) some $L_2 - L_2$ estimates for the energy are proved. Corresponding $L_p - L_q$ decay estimates are still open up to now.
• In the previous section we discussed the influence of increasing and oscillating parts of coefficients on $L_p - L_q$ decay estimates. The possible deteriorating influence of decreasing parts is studied in [20].
• Some results of this paper can be used to understand a new qualitative behavior of solutions to non-linear models like

$$u_{tt} - \lambda(t)^2 b(t)^2 \triangle u = \lambda(t)^2 b(t)^2 (\nabla u)^2 - (u_t)^2, \;\; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x).$$

If the corresponding Cauchy problem (2.17) shows a Floquet effect, that is, we can expect results like those from Theorems 5 and 7, then the non-linear Cauchy problem does not have the property of *existence of small data solutions* (see [21] and [20]). In opposite to this relation the proof of this property seems to be reasonable in cases for which $L_p - L_q$ decay estimates for solutions of (2.17) are known.

References

[1] Galstian A, $L_p - L_q$ decay estimates for the wave equations with exponentially growing speed of propagation, *Applicable Analysis* **82** (2003) 3, 197–214.

[2] Hirosawa F, Energy decay for degenerate hyperbolic equations of Klein-Gordon type with dissipative term, *Funkcialaj Ekvacioj* **43** (2000) 1, 163–191.

[3] Hirosawa F and Reissig M, From wave- to Klein-Gordon type decay rates, in: Schulze et.al, *Nonlinear Hyperbolic Equations, Spectral Theory, and Wavelet Transformations*, Advances in PDE, Operator Theory, Advances and Applications, Birkhäuser, vol. **145** (2003), 95–155.

[4] Littman W, Fourier transformations of surface carried measures and differentiability of surface averages, *Bull. Amer. Math. Soc.* **69** (1963), 766–770.

[5] Matsumura A, On the asymptotic behavior of solutions of semi-linear wave equations, *Publ. RIMS Kyoto Univ.* **12** (1976), 169–189.

[6] Mochizuki K and Nakazawa H, Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation, *Publ. RIMS Kyoto Univ.* **32** (1996), 401–414.

[7] Racke R, *Lectures on Nonlinear Evolution Equations*, Aspects of Mathematics, vol. **19**, Vieweg (1992).

[8] Reissig M, Klein-Gordon type decay rates for wave equations with a time-dependent dissipation, *Adv. Math. Sci. Appl.* **11** (2001) 2, 859–891.

[9] Reissig M, On $L_p - L_q$ estimates for solutions of a special weakly hyperbolic equation, Ed. Li Ta-Tsien, *Nonlinear Evolution Equations and Infinite-Dimensional Dynamical Systems*, 153-164, World Scientific (1997).

[10] Reissig M and Smith J, $L_p - L_q$ estimate for wave equation with bounded time-dependent coefficient, 32 A4, to appear in *Hokkaido Mathematical Journal*.

[11] Reissig M and Wirth J, *Wave equations with monotone weak dissipation*, Fakultät für Mathematik und Informatik, TU Bergakademie Freiberg, Preprint 2003-02, 71 A4, ISSN 1433-9307.

[12] Reissig M and Yagdjian K, About the influence of oscillations on Strichartz-type decay estimates, *Rend. Sem. Mat. Univ. Pol. Torino* **58** (2000) 3, 375–388.

[13] Reissig M and Yagdjian K, Klein-Gordon type decay rates for wave equations with time-dependent coefficients, *Banach Center Publications* vol.**52** (2000), 189–212.

[14] Reissig M and Yagdjian K, $L_p - L_q$ decay estimates for hyperbolic equations with oscillations in the coefficients, *Chin. Ann. of Math.*, Ser. B, **21** (2000) 2, 153–164.

[15] Reissig M and Yagdjian K, $L_p - L_q$ estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients, *Math. Nachr.* **214** (2000), 71–104.

[16] Reissig M and Yagdjian K, One application of Floquet's theory to $L_p - L_q$ estimates for hyperbolic equations with very fast oscillations, *Math. Meth. Appl. Sci.* **22** (1999), 937–951.

[17] Strichartz R, A priori estimates for the wave-equation and some applications, *J. Funct. Anal.* **5** (1970), 218–235.

[18] v.Wahl W, L^p-decay rates for homogeneous wave-equations, *Math. Zeitschrift* **120** (1971), 93–106.

[19] Wirth J, Solution representations for a wave equation with weak dissipation, *Math. Meth. Appl. Sci.* **27** (2004), 101–124.

[20] Wirth J, About the solvability behavior for special classes of non-linear hyperbolic equations, *Nonl. Anal.* **52** (2003), 421–431.

[21] Yagdjian K, Parametric resonance and nonexistence of global solution to non-linear wave equations, *J. Math. Anal. Appl.* **260** (2001) 1, 251-268.