Journal of Nonlinear Mathematical Physics Volume 11, Number 4 (2004), 499–507 Wave Motion

Inverse Spectral Problem for the Periodic Camassa-Holm Equation

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This article is part of the Proceedings of the meeting at the Mathematical Research Institute at Oberwolfach (Germany) titled "Wave Motion", which took place during January 25-31, 2004

Abstract

We consider the direct/inverse spectral problem for the periodic Camassa-Holm equation. In fact, we survey the direct/inverse spectral problem for the periodic weighted operator $Ly = m^{-1}(-y''+\frac{1}{4}y)$ acting in the space $L^2(\mathbb{R}, m(x)dx)$, where $m = u_{xx}-u > 0$ is a 1-periodic positive function and u is the solution of the Camassa-Holm equation $u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$. For the operator L we describe the complete solution of the inverse spectral problem: i) uniqueness, prove that the spectral data uniquely determines the potential, ii) characterization, give conditions for some data to be the spectral data of some potential, iii) reconstruction, give an algorithm for recovering the potential from the spectral data, iv) a priori estimates, obtain two-sided a priori estimates of u, m in terms of gap lengths.

Consider the well known Camassa-Holm equation [2]:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1}$$

This equation describes the motion of solitary waves on shallow water, $u(x,t)$ being the fluid velocity in the x-direction. We add some backgraund of the Camassa-Holm equation. Firstly, note that the paper [17] presents an alternative derivation of the equation, different from the original approach devised by Camassa-Holm. Secondly, we remark that the Camassa-Holm equation was derived as a model in elasticity in the paper [13]. Finally, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group on the circle (see [37] for a formal derivation and see [11, 12] for a detailed discussion of the geometric viewpoint and its physical implications). Thus, there is a purely geometric interpretation of the Camassa-Holm equation as a geodesic flow on the diffeomorphism group of the circle [37, 11, 12].

It is known (see [7]) that $m \equiv u_{xx} - u$ retains its signature under the shallow water flow if it is of one sign to start with. In looking for spatially periodic solutions of Eq.(1) a key point is to understand the associated spectral problem:

$$-f'' + \frac{1}{4}f = \lambda m f, \quad m = u_{xx} - u, \quad \lambda \in \mathbb{C}. \tag{2}$$

Throughout this paper we assume that a 1-periodic function $m(x) > 0, x \in \mathbb{R}$ and $m, m' \in L^2(\mathbb{R}), \mathbb{T} = \mathbb{R}/\mathbb{Z}$. It is clear from the results in [36], [9] that this restriction in the spectral problem which we consider in this paper corresponds precisely to the physically relevant case of periodic waves that do not break. Define the periodic weighted operator $Lf = m^{-1}(-f'' + \frac{1}{4}f)$ in $L^2(\mathbb{R}, m(x)dx)$. Let $\vartheta(x,\lambda), \varphi(x,\lambda)$ be fundamental solutions of the Eq. $-f'' + \frac{1}{4}f = \lambda m f$, with the condition $\vartheta'(0,\lambda) = \varphi(0,\lambda) = 0$ and $\vartheta(0,\lambda) = \varphi'(0,\lambda) = 1$. Define the Lyapunov function $\Delta(\lambda) = \frac{1}{2}(\varphi'(1,\lambda) + \vartheta(1,\lambda))$. Let $\mu_n, n \geqslant 1$ be be the Dirichlet spectrum of the equation $-f'' + \frac{1}{4}f = \lambda m f$. It follows from standard arguments (see [33], [31]) that the spectrum of L is absolutely continuous and consists of intervals $\sigma_n = [\lambda_{n-1}^+, \lambda_n^-]$, where $0 < \lambda_{n-1}^+ < \lambda_n^- \leqslant \lambda_n^+, n \geqslant 1$. These intervals are separated by gaps $\gamma_n = (\lambda_n^+, \lambda_n^+)$ of length $|\gamma_n| \geqslant 0$. If a gap γ_n is degenerate, i.e. $|\gamma_n| = 0$, then the corresponding segments σ_n, σ_{n+1} merge. Note that $\Delta(\lambda_n^\pm) = (-1)^n, n \geqslant 1$ and recall that $\mu_n \in [\lambda_n^-, \lambda_n^+]$. It is well known that the sequence $0 < \lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \ldots$ is the spectrum of the operator with 2-periodic boundary conditions, i.e. $f(x+2) = f(x), x \in \mathbb{R}$. Here equality means that $\lambda_n^- = \lambda_n^+$ is a double eigenvalue. The lowest eigenvalue λ_0^+ is simple and the corresponding eigenfunction is 1-periodic. The eigenfunctions corresponding to $\lambda_n^\pm$ have period 1 when n is even and they are antiperiodic, $f(x+1) = -f(x),\ x \in \mathbb{R}$, when n is odd.

First results about the periodic weighted operator were obtained by Lyapunov [33]. He proved that the spectrum of the periodic weighted operator has band structure. Later Krein [31] extended this result to a more general setting including 2×2 systems. These results cover L (and T, T_0 see below). Some spectral problems for L were considered in [3, 4], [7]. The general weighted Sturm-Liouville (direct) problem was studied by Constantin [5], i.e., in this case m changes sign. Unfortunately, for the last case there is no a deep result from inverse spectral theory. In the case $m > 0$ Korotyaev [22] obtained the following results about "the direct problem" for periodic weighted operators:

i) sharp asymptotics of various parameters,

ii) two-sided estimates,

iii) the global quasimomentum.

Now we consider the inverse spectral problem for the operator L.

The inverse spectral problem consist of the following parts:

i) Uniqueness. Prove that the spectral data uniquely determines the potential.

ii) Characterization. Give conditions for some data to be the spectral data of some potential.

iii) Reconstruction. Give an algorithm for recovering the potential from the spectral data.

iv) A priori Estimates: Obtain two-sided a priori estimates of q in terms of gap lenghts.

First results (the reconstruction problem) about the inverse spectral problem for the operator L was obtained by Constantin [3]. Using the Liouville substitution he transformed L to the Hill operator $\mathcal{L} = -\frac{d^2}{dy^2} + Q(y)$ acting in $L^2(\mathbb{R})$, where the potential Q is given by

$$Q(y) = \frac{1}{4m} + \frac{m''}{4m^2} - \frac{5(m')^2}{16m^3}, \quad y(x) = \int_0^x \sqrt{m(s)}ds, \tag{3}$$

here $m = m(x(y))$ and $x = x(y)$ is the inverse function of $y(x)$. Using methods from [14, 38], Constantin [3] proved that $m \in C^5(\mathbb{T})$ can be recovered from the anti/periodic and the Dirichlet spectra. The necessary exact asymptotics were obtained by a transformation of Eq.(2) to the Hill equation. In fact Constantin [3] derived the trace formula (a new one) and solved the so-called Dubrovin equation. We formulate the result of Constantin [3].

Theorem 1. *Assume that* $m \in C^5(\mathbb{T})$. *Then*

$$m(x) = -\frac{1}{4\lambda_0^+} + \frac{1}{2}\sum_{n\geqslant 1}\left(\frac{1}{\mu_n(x)} - \left(\frac{1}{\mu_n(x)}\right)'' - \frac{1}{2\lambda_n^-} - \frac{1}{2\lambda_n^+}\right), \quad x \in [0,1].$$

Here $\mu_n(t), t \in [0,1], n \geqslant 1$ *is a unique periodic solution of the system*

$$\frac{d\mu_n(t)}{dt} = \frac{\mu_n(t)\sqrt{\Delta^2(\mu_n(t)) - 1}}{\sinh\frac{1}{2}\prod_{m\neq n}(1 - (\mu_n(t)/\mu_m(t)))}, \quad \mu_n(0) = \mu_n, \tag{4}$$

where the signature of the radical $\sqrt{\Delta^2(\mu_n) - 1}$ *is given by*

$$\sqrt{\Delta^2(\mu_n) - 1} = -\Delta(\mu_n) - \frac{1}{\frac{\partial\varphi(1,\mu_n)}{\partial\lambda}}\int_0^1 m(x)\varphi^2(x,\mu_n)dx, \quad n \geqslant 1.$$

We consider the other inverse problems (uniqueness, characterization and a priori estimates) for the case $m, m' \in L^2(\mathbb{T})$ and $m > 0$. Let G be the unitary transformation

$$G : L^2(\mathbb{R}, m(x)dx) \to L^2(\mathbb{R}, \rho^2(y)dy), \quad Gf(y) \equiv f(x(y)), \tag{5}$$

and $x(y)$ is the inverse function of $y(x) = \int_0^x \sqrt{m(x)}ds$. Then without loss of generality we may furthermore assume that $\int_0^1 \sqrt{m(x)}dx = 1$, $m(0) = m(1) = 1$. We obtain the operator T given by

$$T \equiv GLG^{-1} = T_0 + r, \quad T_0 f = -\rho^{-2}(\rho^2(y)f')' = -f'' - 2qf', \quad r \equiv \frac{1}{4\rho^4}, \tag{6}$$

where

$$\rho^2(y) = \sqrt{m(x(y))} = e^{2\int_0^y q(s)ds}, \qquad q = \frac{\rho'}{\rho} \in H \equiv \left\{q \in L^2_{\mathbb{R}}(\mathbb{T}) : \int_0^1 q(y)dy = 0\right\}.$$

We identify q with its periodic extension to the real line.

The inverse spectral theory for the periodic weighted operator T_0 has been studied in [21, 22, 18]. The operator T with respect to $q = \rho'/\rho$ is *non-linear* perturbation of T_0, and this produces an additional complication for the inherently non-linear inverse spectral theory which cannot be resolved by simple appeal to some trivial general principles.

Let $\mu_n, n \geqslant 1$, be the Dirichlet spectrum of the equation $-f'' - 2qf' + rf = \lambda f$ with boundary condition $f(0) = f(1) = 0$. For technical reasons (in order to apply the results of [25, 26]), it is convenient to consider a shift in the spectral parameter by looking at $T - \lambda_0^+$. Thus we introduce the fundamental solutions $y_1(x,\lambda,q), y_2(x,\lambda,q)$ of the equation

$$-f'' - 2qf' + rf = (\lambda + \lambda_0^+)f, \quad \lambda \in \mathbb{C}, \tag{7}$$

satisfying the conditions: $y_2(0,\lambda,q) = y_1'(0,\lambda,q) = 0, \quad y_2'(0,\lambda,q) = y_1(0,\lambda,q) = 1$. Here and below we use the notation $(') = \partial/\partial x, (\dot{}) = \partial/\partial\lambda$. Next we introduce the Lyapunov function $\Delta(\lambda,q) = \frac{1}{2}(y_2'(1,\lambda,q) + y_1(1,\lambda,q))$ and note that $\Delta(\lambda_n^\pm, q) = (-1)^n$, $n \geqslant$

1, $\Delta(0,q) = 1$. For each $n \geqslant 1$ there exists a unique point $\lambda_n \in [\lambda_n^-, \lambda_n^+]$ such that $\dot\Delta(\lambda_n, q) = 0$. It follows from the arguments in [22] that $\mu_n \in [\lambda_n^-, \lambda_n^+]$ for any $n \geqslant 1$.

By analogy with the Marchenko-Ostrovski mapping, introduced in [21] for the periodic weighted operator, we construct the mapping $h : q \to h(q) = \{h_n\}_1^\infty$ from H into $\ell^2 \oplus \ell^2$ by the rule: $h_n = (h_{cn}, h_{sn}) \in \mathbb{R}^2$, where the components have the form

$$h_{cn} = -\log[(-1)^n y_2'(1, \mu_n, q)], \qquad h_{sn} = ||h_n|^2 - h_{cn}^2|^{1/2} \mathrm{sign}(\lambda_n - \mu_n), \tag{8}$$

and the function $|h_n|^2 = h_{cn}^2 + h_{sn}^2$ is defined by the equation

$$\cosh |h_n| = (-1)^n \Delta(\lambda_n, q). \tag{9}$$

It is necessary to check that h is actually well defined by these formulae (which amounts to showing that $h_n^2 - h_{cn}^2 \geqslant 0$), and we shortly recall the argument from [21]. Note that the Wronskian identity $\rho^2(y_2' y_1 - y_1' y_2) = 1$ at $x = 1$ implies $(-1)^n y_2'(1, \mu_n(q), q) > 0$, since $(-1)^n y_1(1, \mu_n(q), q) > 0$. Using the relation

$$y_2'(1, \mu_n(q), q) = (-1)^n e^{-h_{cn}(q)},$$

we deduce that $y_1(1, \mu_n(q), q) = (-1)^n e^{h_{cn}(q)}$ and then

$$(-1)^n \Delta(\mu_n(q), q) = \cosh h_{cn}(q), \quad n \geqslant 1. \tag{10}$$

Thus, by (9) and (10), we get $h_n^2 - h_{cn}^2 \geqslant 0$, since $(-1)^n \Delta(\lambda, q)$ has a maximum at λ_n on the segment $[\lambda_n^-, \lambda_n^+]$. In particular, h_{sn} is well defined.

Using the periodic spectrum $\{\lambda_n^\pm\}$ and the Dirichlet spectrum $\{\mu_n\}$, we now construct the gap length mapping $q \to g(q) = \{g_n\}_1^\infty$, introduced in [24] for the Hill operator. The vector $g_n \equiv (g_{cn}, g_{sn}) \in \mathbb{R}^2$ has the components:

$$g_{cn} = \frac{\lambda_n^+ + \lambda_n^-}{2} - \mu_n, \qquad g_{sn} = \left| \frac{|\gamma_n|^2}{4} - g_{cn}^2 \right|^{1/2} \mathrm{sign} h_{sn}. \tag{11}$$

Note that from the vector g we can compute the gaps lengths $|\gamma_n|, \mathrm{sign} h_{sn}$, and g_{cn} for any $n \geqslant 1$. However, we do not know the position of the gaps and the Dirichlet eigenvalues.

By $\ell_p^2, p \geqslant 1$, we denote the space of real sequences $f = \{f_n\}_1^\infty$ equipped with the norm $\|f\|_p^2 = \sum (2\pi n)^{2p} |f_n|^2$. In the case $p = 0$ we write $\|\cdot\|_0 = \|\cdot\|$.

To describe the geometric picture in the complex plane $\mathbb{C}$, we finally need the relation between the quasimomentum domain K and the spectral domain $\mathcal{Z}$. The quasimomentum domain is given by $K = \mathbb{C} \setminus \cup \Gamma_n$, where $\Gamma_n = (\pi n - i|h_n|, \pi n + i|h_n|\ )$ is an excised slit of height $|h_n| = |h_{-n}| \geqslant 0$, $n \geqslant 1$. The spectral domain is given by $\mathcal{Z} = \mathbb{C} \setminus \cup \widetilde{g}_n$ where $\widetilde{g}_n = (z_n^-, z_n^+) = -\widetilde{g}_{-n}, n \geqslant 1$ are horizontal slits of length $|\widetilde{g}_n| \geqslant 0$ and $z_n^\pm = \sqrt{\lambda_n^\pm - \lambda_0^+} > 0, n \geqslant 1$. Recall that a map $f : H \to H_1$ is a real analytic isomorphism between the Hilbert spaces H and H_1 if f is bijective and both f and f^{-1} are real analytic maps between H, H_1. We formulate the results from [26] and [1].

Theorem 2. *i) The mappings $h : H \to \ell^2 \oplus \ell^2, g : H \to \ell_{-1}^2 \oplus \ell_{-1}^2$ are real analytic isomorphisms and the following two-sided estimates are fulfilled:*

$$\|q\| \leqslant 2\sqrt{2} \|h(q)\| (1 + 4\|h(q)\|)^3, \quad \|h(q)\| \leqslant 3\|q\|(1 + 4\|q\| + 2e^{4\|q\|})^3, \tag{12}$$

$$\|q\| \leqslant (9\pi)^4 \|g(q)\|_{-1}(1+\|g(q)\|_{-1})^7, \quad \|g(q)\|_{-1} \leqslant 2\|q\|(1+4\|q\|+2e^{4\|q\|})^7. \tag{13}$$

ii) For each $h^* \in \ell^2 \oplus \ell^2$ there exists a unique function $q \in H$ such that $h(q) = h^*$ and a unique conformal mapping $k : \mathcal{Z} \to K$ such that $\cos k(z) = \Delta(z^2, q), z \in \mathcal{Z}$, and the following identities and asymptotics are fulfilled:

$$k(z) = z - \frac{\|q\|^2 + \|r\|^2 - \lambda_0^+ + o(1)}{2z}, \qquad \text{as} \qquad z \to i\infty, \tag{14}$$

$$k(z_n \pm i0) = \pi n \pm i|h_n|, \quad k(m_n \pm i0) = \pi n \pm ih_{cn}, \qquad n \geqslant 1, \tag{15}$$

where $m_n = \sqrt{\mu_n - \lambda_0^+} > 0$ and $z_n = \sqrt{\lambda_n - \lambda_0^+} > 0$

Remark. 1) The mappings corresponding to h and g for the unperturbed operator T_0 were shown to be real analytic isomorphisms in [21], [22], [18]. For the perturbed operator T, assuming $q, q' \in H$, it was shown in [26] that h, g are real analytic isomorphisms also. If $q \in H$, it was shown in [26] that h, g are real analytic local isomorphisms and injections. In [1] a surjection for the case $q \in H$ was proved.

2) Assertion i) gives a full characterization of the spectrum in terms of $\{h_n\}$ or $\{g_n\}$ for the case $q \in H$. Assertion ii) explains the geometric sense of the mapping h, which is analogous to the Marchenko-Ostrovski mapping for the Hill operator (see [34]). We emphasize that both mappings h and $k(\cdot)$ give deep geometric insight into the global spectral structure of our operator. Using h, we may construct all of the spectrum. In fact, h determines (uniquely) the conformal mapping $k(\cdot)$ from the spectral domain $\mathcal{Z}$ with slits $\widetilde{g}_n$ onto the quasimomentum domain K with slits Γ_n (specified by h), and the two banks of these slits are identified as follows: The horizontal slit $\widetilde{g}_n$ is transformed to the vertical slit Γ_n. The points $\pi n \pm 0$ on the right and left sides of Γ_n are images of the edge points $z_n^\pm$ of the slit $\widetilde{g}_n$. The points $z_n \pm i0$ on the upper and lower sides of $\widetilde{g}_n$ are transformed to the edge points $\pi n \pm i|h_n|$ of the slit Γ_n (see (15)). The points $m_n \pm i0$ on the upper and lower sides of $\widetilde{g}_n$ are transformed to the points $\pi n \pm ih_{cn}$ on one of the sides of the slit Γ_n (see (15)). Thus, the mapping k (and, therefore, h) allows to recover the sequences $\{z_n^\pm\}, \{m_n\}$ from $\{h_n\}$ and inversely $\{h_n\}$ from $\{z_n^\pm\}, \{m_n\}$. In addition, the quasimomentum k is crucial to derive the double-sided estimates in assertion i) (see [22], [27], [29]).

In the proof of Theorem 2 useful factorizations $h(q) = h^0(V(q))$ and $g(q) = g^0(V(q))$ were used, where $h^0 : H \to \ell^2 \oplus \ell^2$ is a Marchenko-Ostrovski mapping (i.e. it is similar to the mapping h) and $g^0 : H \to \ell^2 \oplus \ell^2$ is the gap-length mapping for the Hill operator $-y'' + p'y$ in $L^2(\mathbb{R})$ with a periodic potential p', where $p = V(q)$ and $V : H \to H$ is the nonlinear mapping defined by

$$p = V(q), \quad p'(x) = q'(x) + q^2(x) + r(x) - \int_0^1 (q^2(s) + r(s))ds, \quad q \in H. \tag{16}$$

All derivatives here are in the distributional sense. In order to use the factorization $h = h^0 \circ V$ and $g = g^0 \circ V$ we need precise control on h^0, g^0 and V. Concerning h^0, g^0 it is sufficient for our purpose to use the following result from [25]:
The mappings $h^0 : H \to \ell^2 \oplus \ell^2$, $g^0 : H \to \ell^2 \oplus \ell^2$ are real analytic isomorphisms. Moreover, the following estimates are fulfilled:

$$\|q\| \leqslant 2\|h^0(q)\|(1 + 4\|h^0(q)\|), \quad \|h^0(q)\| \leqslant 3\|q\|(1 + 2\|q\|), \quad q \in H, \tag{17}$$

$$\|q\| \leqslant 48\pi^2\|\gamma\|_{-1}(1+\|\gamma\|_{-1})^3, \qquad \|\gamma\|_{-1} \leqslant 2\|q\|(1+2\|q\|)^3, \quad \gamma = \{|\gamma_n|\}_1^{\infty}. \tag{18}$$

Concerning V, the analysis in [26] is not yet sufficient for our purpose. As a preliminary step, we shall however use the following result from [26]:
The mapping $V : H \to H$ is real analytic, one-to-one and locally invertible, and an estimate of $\|V(q)\|$ in terms of $\|q\|$ holds.
The analytic problem is to obtain an additional one-sided estimate on V and to show that V is a surjection. We formulate the needed result from [1].

Theorem 3. *The mapping $V : H \to H$ is a real analytic isomorphism and the following two-sided estimates are fulfilled:*

$$\|q\| \leqslant \sqrt{2}\|p\|(1+\sqrt{2}\|p\|), \quad \|p\| \leqslant \|q\|(1+4\|q\|+2e^{4\|q\|}), \quad p = V(q). \tag{19}$$

From Theorem 2 a priori estimates of the solution $u(x,t)$ of the Camassa-Holm equation (1.2) at fixed time t are follow. We formulate these results by estimating the function $m(x,t) = u_{xx}(x,t) - u(x,t)$ in terms of the spectral parameters.

Theorem 4. *The following identities are fulfilled:*

$$\|m(\cdot,t)\|^2 = \|u(\cdot,t)\|^2 + 2\|u'(\cdot,t)\|^2 + \|u''(\cdot,t)\|^2, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{T})},$$

$$\|m'(\cdot,t)\|^2 = \|u'(\cdot,t)\|^2 + 2\|u''(\cdot,t)\|^2 + \|u'''(\cdot,t)\|^2. \tag{20}$$

Let $m, m' \in L^2(\mathbb{T}), m > 0$ and $h = h(q), \gamma = g(q)$, where $q = \frac{\rho'}{\rho}$, and ρ is connected with m by the transformation (5). Then the following estimates are fulfilled:

$$\|m\| \leqslant e^{B_h}, \quad \|m'\| \leqslant B_h e^{3B_h/2}, \quad B_h = 8\sqrt{2}\|h\|(1+4\|h\|)^3, \tag{21}$$

$$\|m\| \leqslant e^{C_\gamma}, \quad \|m'\| \leqslant C_\gamma e^{3C_\gamma/2}, \quad C_\gamma = (18\pi)^4\|\gamma\|_{-1}(1+\|\gamma\|_{-1})^7. \tag{22}$$

As our final result in this paper, we describe the trace formula for the operator T. Let $\mu_n(\tau)$ be the Dirichlet eigenvalues of the equation

$$-f'' + \frac{1}{4}f = \lambda m(x+\tau)f, \quad \lambda \in \mathbb{C}, \quad \tau \in [0,1]. \tag{23}$$

Renark that $\lambda_n^- \leqslant \mu_n(\tau) \leqslant \lambda_n^+$ (see [K1]). We formulate the following result from [1].

Theorem 5. *Let $m, m' \in L^2(\mathbb{T})$ and $m > 0$. Then each function $\mu_n(\cdot) \in C^2(\mathbb{T}), n \geqslant 1$ and the following trace formula is fulfilled:*

$$-\log m(\tau) = \log(4\lambda_0^+) + \sum_1^\infty \Big(\log\lambda_n^+ + \log\lambda_n^- - 2\log\mu_n(\tau)\Big), \quad \tau \in \mathbb{T}. \tag{24}$$

The series converges absolutely and uniformly on $[0,1]$. It is differentiable with respect to τ and its derivative belongs to $L^2(\mathbb{T})$.

Remark. i) The identity (24) is called the trace formula for T. It is the analogue of the well known trace formula

$$q(\tau) = \lambda_0 + \sum_1^\infty \Big(\lambda_n^+ + \lambda_n^- - 2\mu_n(\tau) \Big)$$

for the Hill operator $-\partial_x^2+q$, where $\mu_n(\tau)$ denotes the Dirichlet eigenvalue for the $\tau-$shifted eigenvalue. This formula was proved firstly in the case of more regular potentials (see e.g. [33]), and then extended to potentials in L^2 in [24], the method from [24] was used in the proof.

ii) Note that in view of Theorem 2, the function h determines $\lambda_n^\pm$ and $\mu_n(0)$. Furthermore, $h \in H$ implies $b, b' \in L^2(\mathbb{T})$. Thus the trace formula is valid for $h \in H$ and allows - at least formally - to reconstruct q from the spectral data of the shifted potential.

Assuming, as above, that $m(x) \geqslant \varepsilon > 0$, the trace formula for the Camassa Holm equation was proved in [7] for the finite gap case assuming, of course, that periodic finite gap potentials q do exist for the Camassa-Holm equation (the invariance principle in [26] implies that they actually do). In Theorem 5 the result for the finite gap case [7] is extened to the case $m' \in L^2(\mathbb{T})$.

The proof of these results is by a (nontrivial) generalization of methods developed for the Hill operator. The corresponding inverse spectral problem for the Schrödinger operator with periodic potential $V(x)$ was considered in a great number of papers. Marchenko and Ostrovski constructed in [34, 35] the mapping $h : V \to h(V)$ and proved that this mapping is a continuous isomorphism. Garnett and Trubowitz [16] proved that h and g are real analytic isomorphisms in the case of even potentials. Kargaev and Korotyaev [19] reproved the result of [16] (in the case of even potentials) by a direct method. We remind the reader that the proofs by the direct method are short and efficient, but that this approach needs some a priori estimates of potentials in terms of spectral data. For the general case (including non-even potentials) Korotyaev [23] proved that h is a real analytic isomorphism by the direct method. The gap length mapping g for the Hill operator was constructed by Korotyaev [24]. Moreover, it was proved in [24] that g is a real analytic isomorphism. Two-sided estimates for various parameters of the Hill operator (the norm of a periodic potential, effective masses, gap lengths, height of slits $|h_n|$ etc.) were obtained in [27, 28, 29]. The results for the Hill operator [34, 35], [23, 24] were extended in [25] to the case of singular potentials V' with $V \in L^2(\mathbb{T})$. This paper proves that the mappings h, g are real analytic isomorphisms in this class of potentials.

The inverse problem in terms of the vertical slits mapping h for the periodic weighted operator T_0 (or, in other words, for the equation $-f'' = \lambda m f$ with positive m) was solved in [21, 22], and the corresponding problem for the gap length mapping was solved in [21, 18]. As we already mentioned above, the mappings h, g were constructed for a certain class of perturbations of T_0 and used to solve the inverse problem in [25, 26].

We mention that inverse scattering for the Camassa-Holm equation is not developed (see [6]). Here there is an open interesting problem.

Finally we want to mention that a nonstationary problem for the Camassa-Holm equation was investigated in a series of papers of Constantin et al. (see [8, 9, 11, 10] and the references given in these papers).

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