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The Heun equation and the Calogero-Moser-Sutherland system V: generalized Darboux transformations

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Abstract

We obtain isomonodromic transformations for Heun's equation by generalizing the Darboux transformation, and we find pairs and triplets of Heun's equation which have the same monodromy structure. By composing generalized Darboux transformations, we establish a new construction of the commuting operator which ensures that the finite-gap property is satisfied. As an application, we prove some previous conjectures in part III.

1 Introduction

It was shown in [10] that some pairs of Schrödinger operators are isomonodromic. Set

$$H_1 = -\frac{d^2}{dx^2} + 6\wp(x), \tag{1.1}$$

$$H_2 = -\frac{d^2}{dx^2} + 2\wp(x) + 2\wp(x+\omega_1) + 2\wp(x+\omega_2), \tag{1.2}$$

where $\wp(x)$ is the Weierstrass $\wp$-function with periods $(2\omega_1, 2\omega_3)$ and $\omega_2 = -\omega_1 - \omega_3$. For the operators H_1 and H_2, we have integral representations of eigenfunctions for any eigenvalues, and the monodromy of eigenfunctions is calculated in terms of hyperelliptic integrals (see [7, 9, 10]). It is shown that there exist eigenfunctions $\Lambda_1(\pm x, E)$ (resp. $\Lambda_2(\pm x, E)$) of H_1 (resp. H_2) corresponding to the eigenvalue E such that

$$\Lambda_1(\pm(x+2\omega_i), E) = \Lambda_1(\pm x, E) \exp\left(\mp\frac{1}{2}\int_{\sqrt{3g_2}}^{E} \frac{(-6\tilde{E}\eta_i + (2\tilde{E}^2 - 3g_2)\omega_i)d\tilde{E}}{\sqrt{-(\tilde{E}^2-3g_2)(\tilde{E}^3 - 9g_2\tilde{E}/4 - 27g_3/4)}}\right), \tag{1.3}$$

$$\Lambda_2(\pm(x+2\omega_i), E) = \Lambda_2(\pm x, E) \exp\left(\mp\frac{1}{2}\int_{\sqrt{3g_2}}^{E} \frac{(-6\tilde{E}\eta_i + (2\tilde{E}^2 - 3g_2)\omega_i)d\tilde{E}}{\sqrt{-(\tilde{E}^2-3g_2)(\tilde{E}^3 - 9g_2\tilde{E}/4 - 27g_3/4)}}\right), \tag{1.4}$$

for $i = 1, 2, 3$, where $e_i = \wp(\omega_i)$, $g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$, $g_3 = 4e_1e_2e_3$, $\eta_i = \zeta(\omega_i)$ and $\zeta(x)$ is the Weierstrass zeta function. Hence the monodromy of eigenfunctions of H_1 corresponding to the eigenvalue E coincides with that of the eigenfunctions of H_2, although the expression of the functions $\Lambda_1(x, E)$ and $\Lambda_2(x, E)$ is quite different (see section 7).

In this paper, we investigate these phenomena by means of the Darboux transformation and generalized Darboux transformation. Let $\phi_0(x)$ be an eigenfunction of the operator $H = -d^2/dx^2 + q(x)$ corresponding to an eigenvalue E_0, i.e.

$$\left(-\frac{d^2}{dx^2} + q(x)\right)\phi_0(x) = E_0\phi_0(x).$$

For this case, the potential $q(x)$ is given by $q(x) = (\phi_0'(x)/\phi_0(x))' + (\phi_0'(x)/\phi_0(x))^2 + E_0$. Set $L = d/dx - \phi_0'(x)/\phi_0(x)$ and $\tilde{H} = -d^2/dx^2 + q(x) - 2(\phi_0'(x)/\phi_0(x))'$. Then we have

$$\tilde{H}L = LH.$$

Hence, if $\phi(x)$ is an eigenfunction of the operator H corresponding to the eigenvalue E, then $L\phi(x)$ is an eigenfunction of the operator $\tilde{H}$ corresponding to the eigenvalue E. This transformation is called the Darboux transformation. We generalize the operator L to be the differential operator of higher order, and will call this operator the generalized Darboux transformation.

In the present study we consider the Schrödinger operator given by the Hamiltonian of the BC_1 Inozemtsev model, which may be written in the form

$$H^{(l_0,l_1,l_2,l_3)} = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i), \tag{1.5}$$

where $\omega_0 = 0$. Following the results of Treibich and Verdier [14], who showed that if $l_i \in \mathbb{Z}_{\geq 0}$ for all $i \in \{0, 1, 2, 3\}$, the potential of operator (1.5) is an algebro-geometric finite-gap potential, this potential is now referred to as the Treibich-Verdier potential. The interested reader may refer to [3, 6, 7, 9, 13] for further results in this area. The algebro-geometric finite-gap property in turn provides a possible means of determining the eigenfunctions and monodromy of the operator $H^{(l_0,l_1,l_2,l_3)}$.

Let $f(x)$ be an eigenfunction of the operator $H^{(l_0,l_1,l_2,l_3)}$ corresponding to the eigenvalue E, that is we have

$$\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)\right) f(x) = Ef(x). \tag{1.6}$$

This equation is an elliptic representation of Heun's equation, where Heun's equation is the standard canonical form of a Fuchsian equation with four singularities (see [5]). Thus, solving Heun's equation is equivalent to studying eigenvalues and eigenfunctions of the Hamiltonian of the BC_1 Inozemtsev model.

We now describe the main result of this paper. Let α_i be a number such that $\alpha_i = -l_i$ or $\alpha_i = l_i + 1$ for all $i \in \{0, 1, 2, 3\}$, and set $d = -\sum_{i=0}^{3} \alpha_i/2$. If $d \in \mathbb{Z}_{\geq 0}$, then there exists a differential operator L of order $d + 1$ which satisfies

$$H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}L = LH^{(l_0,l_1,l_2,l_3)}.$$

Note that Khare and Sukhatme [4] essentially established this result for the case $d = 0$, that is for the case of the original Darboux transformation. It follows immediately that, if $\phi(x)$ is an eigenfunction of the operator $H^{(l_0,l_1,l_2,l_3)}$ corresponding to an eigenvalue E, then $L\phi(x)$ is an eigenfunction of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ corresponding to the eigenvalue E. Since all coefficients of the operator L with respect to the differential $(d/dx)^k$ $(k = 0,\dots,d+1)$ are shown to be doubly-periodic, the operator L preserves the data of monodromy. Hence the operators $H^{(l_0,l_1,l_2,l_3)}$ and $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ are isomonodromic, and isospectral, because boundary conditions for spectral problems are characterized by monodromy. Note that the condition $d \in \mathbb{Z}_{\geq 0}$ corresponds to the quasi-solvability of the operator $H^{(l_0,l_1,l_2,l_3)}$.

For the case when l_0, l_1, l_2, l_3 are all integers, there exists an operator $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$ such that $H^{(l_0,l_1,l_2,l_3)}$ and $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$ are connected by an isomonodromic transformation. In some cases, these operators are also self-dual. For example, the operator $H_1(= H^{(2,0,0,0)})$ in Eq.(1.1) is connected to the operator $H_2(= H^{(1,1,1,0)})$ in Eq.(1.2) by the transformation $L = d/dx - \wp'(x)/(2(\wp(x)-e_1)) - \wp'(x)/(2(\wp(x)-e_2))$, i.e. we have

$$H^{(1,1,1,0)}L = LH^{(2,0,0,0)}.$$

For the case when $l_0+1/2, l_1+1/2, l_2+1/2, l_3+1/2$ are all integers, there exist two operators $H^{(l_0^{(1)},l_1^{(1)},l_2^{(1)},l_3^{(1)})}$ and $H^{(l_0^{(2)},l_1^{(2)},l_2^{(2)},l_3^{(2)})}$ such that $H^{(l_0,l_1,l_2,l_3)}$, $H^{(l_0^{(1)},l_1^{(1)},l_2^{(1)},l_3^{(1)})}$ and $H^{(l_0^{(2)},l_1^{(2)},l_2^{(2)},l_3^{(2)})}$ are connected by isomonodromic transformations.

In the paper [9], the finite-gap property of the operator $H^{(l_0,l_1,l_2,l_3)}$ for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ is studied (see also [14, 13, 3, 6]). In particular, a differential operator A of odd order which commutes with $H^{(l_0,l_1,l_2,l_3)}$ is constructed. In the present study, we propose a new method for the construction of the commuting operator by composing four generalized Darboux transformations. Note that each generalized Darboux transformation is written explicitly. In section 6 we will show that the commuting operator coincides with the one defined in [9]. As an application, we prove the previous conjectures in [9]. That is, we establish that the polynomial defined by quasi-solvability coincides with the one defined by using the doubly-periodic function $\Xi(x,E)$, which is expressed as a product of two eigenfunctions. We also prove that the commuting operator is characterized by annihilating the spaces of quasi-solvability. Furthermore, for the isomonodromic pair $H^{(l_0,l_1,l_2,l_3)}$ and $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$, it is shown that several functions related to the monodromy of $H^{(l_0,l_1,l_2,l_3)}$ coincide with those of $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$. For the case $l_1 = l_2 = l_3 = 0$ the operator $H^{(l_0,0,0,0)}$ is called the Lamé operator. An explicit expression for the commuting operator of the Lamé operator is obtained.

This paper is organized as follows: In section 2, we review the connection between quasi-solvability and the generalized Darboux transformation, which was essentially established by Aoyama, Sato and Tanaka [1]. In section 3, we consider generalized Darboux transformations for the case of Heun's equation. In section 4 (resp. section 5), we investigate isomonodromic transformations for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ (resp. $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0} - 1/2$). In section 6, we construct a differential operator of odd order which commutes with $H^{(l_0,l_1,l_2,l_3)}$ and investigate its properties. In section 7, we consider isomonodromic pairs for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, while in section 8 we obtain an explicit expression for the commuting operator of the Lamé operator.

2 Quasi-solvability and the generalized Darboux transformation

We review the relationship between quasi-solvability and the generalized Darboux transformation, which was essentially established by Aoyama, Sato and Tanaka [1].

We set $H = -d^2/dx^2 + q(x)$. Let n be a positive integer. If the operator H preserves a n-dimensional space U of functions, then the operator H is called quasi-solvable, and there exists a basis $\langle f_1(x), \dots, f_n(x)\rangle$ of the invariant space such that $Hf_j(x) = \sum_i a_{i,j} f_i(x)$ for some constants $a_{i,j}$ $(1 \le i, j \le n)$. Let $P_{H,U}(t)$ be the characteristic polynomial of the operator H on the space U. Then the set $\{E|P_{H,U}(E) = 0\}$ coincides with the set of eigenvalues of the operator H on the space U. The model is thus partially solved, and this is the origin of the "quasi-solvability" description. For the space U, there exists a monic differential operator of order n,

$$L = \left(\frac{d}{dx}\right)^n + \sum_{i=1}^{n} c_i(x) \left(\frac{d}{dx}\right)^{n-i}, \tag{2.1}$$

such that $Lf(x) = 0$ for all $f(x) \in U$. It is uniquely determined and written as

$$L = \left| \begin{pmatrix} f_1(x) & \frac{d}{dx} f_1(x) & \dots & \left(\frac{d}{dx}\right)^{n-1} f_1(x) \\ f_2(x) & \frac{d}{dx} f_2(x) & \dots & \left(\frac{d}{dx}\right)^{n-1} f_2(x) \\ \vdots & & \vdots & \vdots \\ f_n(x) & \frac{d}{dx} f_n(x) & \dots & \left(\frac{d}{dx}\right)^{n-1} f_n(x) \end{pmatrix} \right|^{-1} \left| \begin{pmatrix} f_1(x) & \frac{d}{dx} f_1(x) & \dots & \left(\frac{d}{dx}\right)^{n} f_1(x) \\ f_2(x) & \frac{d}{dx} f_2(x) & \dots & \left(\frac{d}{dx}\right)^{n} f_2(x) \\ \vdots & & \vdots & \vdots \\ f_n(x) & \frac{d}{dx} f_n(x) & \dots & \left(\frac{d}{dx}\right)^{n} f_n(x) \\ 1 & \frac{d}{dx} & \dots & \left(\frac{d}{dx}\right)^{n} \end{pmatrix} \right|.$$

Proposition 2.1. (c.f. [1]) Suppose that the operator $H = -d^2/dx^2 + q(x)$ preserves an n-dimensional space U of functions. Let L be the differential operator expressed by Eq.(2.1) which annihilates the functions in U, and set $\tilde{H} = -d^2/dx^2 + q(x) + 2c_1'(x)$. Then we have

$$\tilde{H}L = LH.$$

Proof. By a direct calculation, it follows that the order of the differential operator $\tilde{H}L - LH$ is at most $n-1$. Assume that $\tilde{H}L - LH \neq 0$, and denote the order by k. Then the dimension of solutions to the differential equation $(\tilde{H}L - LH)f(x) = 0$ is k. Let $g(x) \in U$. Since the operator H preserves the space U, we have $Hg(x) \in U$. By definition of the space U, we have $Lg(x) = 0$ and $LHg(x) = 0$. Hence we have $(\tilde{H}L - LH)g(x) = 0$ for $g(x) \in U$, but this contradicts the premise that the dimension of solutions is $k (\le n-1)$. We therefore obtain $\tilde{H}L = LH$. ■

We consider the case $n = 1$, Let $\phi_0(x)$ be a non-zero function in U. Then $U = \mathbb{C}\phi_0(x)$, and the operator which annihilates $\phi_0(x)$ is given by $L = d/dx - \phi_0'(x)/\phi_0(x)$. The operator $\tilde{H}$ is given by $\tilde{H} = H - 2(\phi_0'(x)/\phi_0(x))'$. Hence the proposition reproduces the Darboux transformation. In this sense, the transformation in the proposition may be called the generalized Darboux transformation.

Note that, in Proposition 2.1, if the operator H is diagonalizable on the space U, then the proposition is related to Crum's theorem [2] by choosing $f_1(x), \dots, f_n(x)$ as eigenfunctions. However, the analysis of the next section is undertaken for the more general case when $f_1(x), \dots, f_n(x)$ are not necessarily eigenfuntions.

3 Generalized Darboux transformation for Heun's equation

In this section, we apply Proposition 2.1 to Heun's equation. For this purpose, we recall the quasi-solvability of Heun's equation.

Proposition 3.1. [8, Proposition 5.1] Let α_i be a number such that $\alpha_i = -l_i$ or $\alpha_i = l_i+1$ for all $i \in \{0,1,2,3\}$, and set $d = -\sum_{i=0}^{3}\alpha_i/2$. Suppose that $d \in \mathbb{Z}_{\geq 0}$, and let $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ be the $d+1$-dimensional space spanned by

$$\left\{\widehat{\Phi}(\wp(x))\wp(x)^n\right\}_{n=0,\dots,d}, \tag{3.1}$$

where $\widehat{\Phi}(z) = (z-e_1)^{\alpha_1/2}(z-e_2)^{\alpha_2/2}(z-e_3)^{\alpha_3/2}$. Then the operator $H^{(l_0,l_1,l_2,l_3)}$ (see Eq.(1.5)) preserves the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$.

Set $z = \wp(x)$ and $\widehat{H}^{(l_0,l_1,l_2,l_3)} = \widehat{\Phi}(z)^{-1} \circ H^{(l_0,l_1,l_2,l_3)} \circ \widehat{\Phi}(z)$. Proposition 3.1 is then proved by showing that the operator $\widehat{H}^{(l_0,l_1,l_2,l_3)}$ preserves the space spanned by $(z-e_2)^r$ $(r = 0,\dots,d)$. For details, see the appendix or [8, Proposition 5.1].

We now proceed to calculate the differential operator which annihilates the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$.

Proposition 3.2. Under the assumption of Proposition 3.1, the monic differential operator of order $d+1$ which annihilates the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is given by

$$L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = \wp'(x)^{d+1}\widehat{\Phi}(\wp(x)) \circ \left(\frac{1}{\wp'(x)}\frac{d}{dx}\right)^{d+1} \circ \widehat{\Phi}(\wp(x))^{-1}. \tag{3.2}$$

Writing the operator $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ in the form

$$L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = \left(\frac{d}{dx}\right)^{d+1} + \sum_{i=1}^{d+1} c_i(x)\left(\frac{d}{dx}\right)^{d+1-i}, \tag{3.3}$$

then

$$c_1(x) = -\frac{d+1}{4}\left(\sum_{i=1}^{3}\frac{2\alpha_i+d}{\wp(x)-e_i}\right)\wp'(x). \tag{3.4}$$

If i is even (resp. odd), then $c_i(x)$ is expressed as $c_i(x) = R_i(\wp(x))$ (resp. $c_i(x) = R_i(\wp(x))\wp'(x)$), where $R_i(z)$ is a rational function in z.

Proof. It is trivial that the operator $(d/dz)^{d+1}$ annihilates the space spanned by z^r $(r = 0,\dots,d)$, and the operator $\widehat{\Phi}(z) \circ (d/dz)^{d+1} \circ \widehat{\Phi}(z)^{-1}$ annihilates the space spanned by $\widehat{\Phi}(z)z^r$ $(r = 0,\dots,d)$. Upon writing

$$\widehat{\Phi}(z) \circ \left(\frac{d}{dz}\right)^{d+1} \circ \widehat{\Phi}(z)^{-1} = \left(\frac{d}{dz}\right)^{d+1} + \sum_{i=1}^{d+1}\hat{c}_i(z)\left(\frac{d}{dz}\right)^{d+1-i},$$

then $\hat{c}_1(z) = -\sum_{i=1}^{3}((d+1)\alpha_i)/(2(z-e_i))$, and all coefficients $\hat{c}_i(z)$ are rational functions in z. By the transformation $z=\wp(x)$, the monic operator

$$\begin{aligned} L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} &= \wp'(x)^{d+1}\widehat{\Phi}(\wp(x))\circ\left(\frac{1}{\wp'(x)}\frac{d}{dx}\right)^{d+1}\circ\widehat{\Phi}(\wp(x))^{-1} \qquad (3.5)\\ &= \wp'(x)^{d+1}\left(\frac{1}{\wp'(x)}\frac{d}{dx}\right)^{d+1} + \sum_{i=1}^{d+1}\hat{c}_i(\wp(x))\wp'(x)^{d+1}\left(\frac{1}{\wp'(x)}\frac{d}{dx}\right)^{d+1-i} \end{aligned}$$

annihilates the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$. Upon writing $((1/\wp'(x))(d/dx))^i = \sum_{j=0}^{i}\hat{b}_j(x)(d/dx)^j$, then if j is even (resp. odd), then $\hat{b}_j(x)$ is expressed as $r_j(\wp(x))$ (resp. $r_j(\wp(x))\wp'(x)$), where $r_j(z)$ is a rational function in z. Combining the above with the relation $\wp'(x)^2 = 4(\wp(x)-e_1)(\wp(x)-e_2)(\wp(x)-e_3)$, it follows that if i is even (resp. odd), then $c_i(x)$ is expressed as $c_i(x)=R_i(\wp(x))$ (resp. $c_i(x)=R_i(\wp(x))\wp'(x)$), where $R_i(z)$ is a rational function in z. We now calculate $c_1(x)$. By Eqs.(3.5, A.2), we have

$$c_1(x) = -\frac{(d+1)d}{2}\frac{\wp''(x)}{\wp'(x)} + \hat{c}_1(\wp(x))\wp'(x) = -\frac{d+1}{4}\left(\sum_{i=1}^{3}\frac{2\alpha_i+d}{\wp(x)-e_i}\right)\wp'(x).$$

■

Theorem 3.3. Let α_i be a number such that $\alpha_i=-l_i$ or $\alpha_i=l_i+1$ for all $i\in\{0,1,2,3\}$, and set $d=-\sum_{i=0}^{3}\alpha_i/2$. Suppose that $d\in\mathbb{Z}_{\geq 0}$, and let $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ be the operator defined in Proposition 3.2. Then we have

$$H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}H^{(l_0,l_1,l_2,l_3)}. \qquad (3.6)$$

Proof. It follows from Propositions 2.1 and 3.2 that

$$(H^{(l_0,l_1,l_2,l_3)} + 2c_1'(x))L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}H^{(l_0,l_1,l_2,l_3)},$$

where $c_1(x)$ is defined in Eq.(3.4). By Eq.(A.2), we have

$$\begin{aligned} 2c_1'(x) &= \frac{d+1}{2}\left(\sum_{i=1}^{3}\frac{2\alpha_i+d}{(\wp(x)-e_i)^2}\right)\wp'(x)^2 - \frac{d+1}{2}\left(\sum_{i=1}^{3}\frac{2\alpha_i+d}{\wp(x)-e_i}\right)\wp''(x)\\ &= -(d+1)(2(\alpha_1+\alpha_2+\alpha_3)+3d)\wp(x) + \sum_{i=1}^{3}(d+1)(2\alpha_1+d)\wp(x+\omega_i). \end{aligned}$$

Since $\alpha_i=-l_i$ or $\alpha_i=l_i+1$, we have $l_i(l_i+1) = -\alpha_i(-\alpha_i+1)$. Hence we obtain $H^{(l_0,l_1,l_2,l_3)}+2c_1'(x) = H^{(\alpha_0-1,\alpha_1-1,\alpha_2-1,\alpha_3-1)}+2c_1'(x) = H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ and Eq.(3.6). ■

We consider the converse relation to Eq.(3.6). The operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ preserves the $d+1$-dimensional space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$, because $-\alpha_i-d\in\{-(\alpha_i+d),\alpha_i+d+1\}$ and $-\sum_{i=0}^{3}(-\alpha_i-d)/2=d$.

Proposition 3.4. (i) We have

$$H^{(l_0,l_1,l_2,l_3)}L_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d} = \\ L_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}. \tag{3.7}$$

(ii) The characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ coincides with that of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$.

Proof. Set $\tilde{l}_i = \alpha_i + d$ $(i = 0, 1, 2, 3)$. Then $-\sum_{i=0}^{3} \tilde{l}_i/2 = d$. By Theorem 3.3, we have

$$H^{(-\tilde{l}_0+d,-\tilde{l}_1+d,-\tilde{l}_2+d,-\tilde{l}_3+d)}L_{-\tilde{l}_0,-\tilde{l}_1,-\tilde{l}_2,-\tilde{l}_3} = L_{-\tilde{l}_0,-\tilde{l}_1,-\tilde{l}_2,-\tilde{l}_3}H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)},$$

and hence we obtain Eq.(3.7). We will prove (ii) in the appendix. ■

Let $f_1(x, E)$ and $f_2(x, E)$ be a basis of solutions to the differential equation $(H^{(l_0,l_1,l_2,l_3)} - E)f(x) = 0$. Since the operator $H^{(l_0,l_1,l_2,l_3)}$ is doubly-periodic, the functions $f_1(x+2\omega_k, E)$ and $f_2(x+2\omega_k, E)$ $(k = 1, 3)$ are also solutions to $(H^{(l_0,l_1,l_2,l_3)} - E)f(x) = 0$, and they can be written as linear combinations of $f_1(x, E)$ and $f_2(x, E)$. Hence we have the monodromy matrices

$$(f_1(x + 2\omega_k, E)\ f_2(x + 2\omega_k, E)) = (f_1(x, E)\ f_2(x, E)) \begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{pmatrix}. \tag{3.8}$$

Now assume that $\alpha_0, \dots, \alpha_3, d$ satisfy the assumption of Theorem 3.3. Set $\tilde{f}_i(x, E) = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} f_i(x, E)$ $(i = 1, 2)$. It follows from Eq.(3.6) that $\tilde{f}_1(x, E)$ and $\tilde{f}_2(x, E)$ are eigenfunctions of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ with the eigenvalue E. If E is not an eigenvalue of $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$, then $\{\tilde{f}_1(x, E), \tilde{f}_2(x, E)\}$ is a basis of solutions to the differential equation $(H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)} - E)f(x) = 0$. It is shown in Proposition 3.2 that operator $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is doubly-periodic, and it follows from Eq.(3.8) that

$$(\tilde{f}_1(x + 2\omega_k, E)\ \tilde{f}_2(x + 2\omega_k, E)) = (\tilde{f}_1(x, E)\ \tilde{f}_2(x, E)) \begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{pmatrix}.$$

Hence the monodromy structure of $H^{(l_0,l_1,l_2,l_3)}$ coincides with that of $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ for the case that E is not an eigenvalue of $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$. If E is an eigenvalue of $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$, then by Proposition 3.4 (ii) we have that it is also the eigenvalue of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$. Thus the operator $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ defines an isomonodromic transformation from $H^{(l_0,l_1,l_2,l_3)}$ to $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$. Similarly $L_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$ defines an isomonodromic transformation from $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ to $H^{(l_0,l_1,l_2,l_3)}$.

On the multiplicity of eigenvalues of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$, we have the following proposition:

Proposition 3.5. (c.f. [12]) Let α_i be a number such that $\alpha_i = -l_i$ or $\alpha_i = l_i + 1$ for all $i \in \{0, 1, 2, 3\}$, and set $d = -\sum_{i=0}^{3} \alpha_i/2$. If $d \in \mathbb{Z}_{\geq 0}$ and $\alpha_i \neq \alpha_j$ for some $i, j \in \{0, 1, 2, 3\}$, then zeros of the characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ are distinct for generic periods $(2\omega_1, 2\omega_3)$.

The proof of this proposition is given in Appendix B, but we note here the meaning of "generic" in the above. In Appendix B, it is shown that there exist periods $(2\omega_1, 2\omega_3)$ such that zeros of the characteristic polynomial are distinct. Hence the discriminant of the characteristic polynomial is not identically zero, and the set of periods such that the discriminant of the characteristic polynomial is not equal to zero is open-dense. We describe a set of such periods to be generic periods.

4 The case of integers

We investigate quasi-solvability and generalized Darboux transformation for the case $l_i \in \mathbb{Z}_{\geq 0}$ $(i = 0, 1, 2, 3)$. Throughout this section, assume $l_i \in \mathbb{Z}_{\geq 0}$ for $i = 0, 1, 2, 3$.

Let $\mathcal{F}$ be the space spanned by meromorphic doubly periodic functions up to signs, that is

$$\mathcal{F} = \bigoplus_{\epsilon_1,\epsilon_3=\pm 1} \mathcal{F}_{\epsilon_1,\epsilon_3},$$
$$\mathcal{F}_{\epsilon_1,\epsilon_3} = \{f(x)\text{: meromorphic } | f(x + 2\omega_1) = \epsilon_1 f(x),\ f(x + 2\omega_3) = \epsilon_3 f(x)\},$$

If $\alpha_i \in \mathbb{Z}$ $(i = 0, 1, 2, 3)$ and $-\sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\geq 0}$, then $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is a subspace of $\mathcal{F}_{\epsilon_1,\epsilon_3}$ for suitable $\epsilon_1, \epsilon_3 \in \{\pm 1\}$, because $(\wp(x) - e_i)^{1/2} = \wp_i(x)$ $(i = 1, 2, 3)$, where $\wp_i(x)$ is the co-$\wp$ function which is single-valued and contained in $\mathcal{F}_{\epsilon_1,\epsilon_3}$ $(\epsilon_1, \epsilon_3 \in \{\pm 1\})$. Invariant subspaces of $\mathcal{F}$ with respect to the operator $H^{(l_0,l_1,l_2,l_3)}$ are studied in [7] (see also [3, 8, 9]).

Let $\alpha_i \in \{-l_i, l_i + 1\}$ $(i = 0, 1, 2, 3)$,

$$U_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = \begin{cases} V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}, & \sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\leq 0}; \\ V_{1-\alpha_0,1-\alpha_1,1-\alpha_2,1-\alpha_3}, & \sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\geq 2}; \\ \{0\}, & \text{otherwise,} \end{cases} \tag{4.1}$$

and

$$\tilde{L}_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = \begin{cases} L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}, & \sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\leq 0}; \\ L_{1-\alpha_0,1-\alpha_1,1-\alpha_2,1-\alpha_3}, & \sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\geq 2}; \\ 1, & \text{otherwise.} \end{cases} \tag{4.2}$$

If $l_0 + l_1 + l_2 + l_3$ is even, then the operator $H^{(l_0,l_1,l_2,l_3)}$ (see Eq.(1.5)) preserves the spaces

$$U_{-l_0,-l_1,-l_2,-l_3},\ U_{-l_0,-l_1,l_2+1,l_3+1},\ U_{-l_0,l_1+1,-l_2,l_3+1},\ U_{-l_0,l_1+1,l_2+1,-l_3}. \tag{4.3}$$

Each space is contained in $\mathcal{F}_{\epsilon_1,\epsilon_3}$ for some $\epsilon_1, \epsilon_3 \in \{\pm 1\}$, and the correspondence between the spaces and the signs of (ϵ_1, ϵ_3) is one-to-one. Let V be the sum of these spaces, then V is given by the direct sum of these spaces, i.e,

$$V = U_{-l_0,-l_1,-l_2,-l_3} \oplus U_{-l_0,-l_1,l_2+1,l_3+1} \oplus U_{-l_0,l_1+1,-l_2,l_3+1} \oplus U_{-l_0,l_1+1,l_2+1,-l_3}, \tag{4.4}$$

and is the maximal finite-dimensional invariant subspace in $\mathcal{F}$ with respect to the action of the operator $H^{(l_0,l_1,l_2,l_3)}$. Let k_i be the rearrangement of l_i such that $k_0 \geq k_1 \geq k_2 \geq k_3 (\geq 0)$. Set

$$g = \begin{cases} k_0, & k_0 + k_3 \geq k_1 + k_2; \\ (k_0 + k_1 + k_2 - k_3)/2, & k_0 + k_3 < k_1 + k_2. \end{cases} \tag{4.5}$$

Then $g \in \mathbb{Z}_{\geq 0}$ and $\dim V = 2g+1$. We set

$$l_0^e = (-l_0 + l_1 + l_2 + l_3)/2, \quad l_1^e = (l_0 - l_1 + l_2 + l_3)/2, \\ l_2^e = (l_0 + l_1 - l_2 + l_3)/2, \quad l_3^e = (l_0 + l_1 + l_2 - l_3)/2. \tag{4.6}$$

Note that $l_0^e + l_1^e + l_2^e + l_3^e = l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}$. It follows directly from Proposition 3.6 that

$$H^{(l_0^e,l_1^e,l_2^e,l_3^e)}\tilde{L}_{-l_0,-l_1,-l_2,-l_3} = \tilde{L}_{-l_0,-l_1,-l_2,-l_3}H^{(l_0,l_1,l_2,l_3)},$$
$$H^{(l_1^e,l_0^e,l_3^e,l_2^e)}\tilde{L}_{-l_0,-l_1,l_2+1,l_3+1} = \tilde{L}_{-l_0,-l_1,l_2+1,l_3+1}H^{(l_0,l_1,l_2,l_3)},$$
$$H^{(l_2^e,l_3^e,l_0^e,l_1^e)}\tilde{L}_{-l_0,l_1+1,-l_2,l_3+1} = \tilde{L}_{-l_0,l_1+1,-l_2,l_3+1}H^{(l_0,l_1,l_2,l_3)},$$
$$H^{(l_3^e,l_2^e,l_1^e,l_0^e)}\tilde{L}_{-l_0,l_1+1,l_2+1,-l_3} = \tilde{L}_{-l_0,l_1+1,l_2+1,-l_3}H^{(l_0,l_1,l_2,l_3)}.$$

Thus the operators which are linked by generalized Darboux transformations from $H^{(l_0,l_1,l_2,l_3)}$ are

$$H^{(l_0^e,l_1^e,l_2^e,l_3^e)},\ H^{(l_1^e,l_0^e,l_3^e,l_2^e)},\ H^{(l_2^e,l_3^e,l_0^e,l_1^e)},\ H^{(l_3^e,l_2^e,l_1^e,l_0^e)}, \\ H^{(l_0,l_1,l_2,l_3)},\ H^{(l_1,l_0,l_3,l_2)},\ H^{(l_2,l_3,l_0,l_1)},\ H^{(l_3,l_2,l_1,l_0)}. \tag{4.7}$$

As is discussed in section 3, these eight operators are isomonodromic. Note that the operators $H^{(l_1,l_0,l_3,l_2)}$, $H^{(l_2,l_3,l_0,l_1)}$, $H^{(l_3,l_2,l_1,l_0)}$ are obtained from the operator $H^{(l_0,l_1,l_2,l_3)}$ by the shift transformation $x \to x + \omega_i$ $(i = 1, 2, 3)$.

Assume that $l_0,\ l_1,\ l_2,\ l_3 \in \mathbb{Z}$, $l_0 \geq l_1 \geq l_2 \geq l_3 \geq 0$ and $l_0 + l_1 + l_2 + l_3$ is even. Set $\tilde{l}_0 = l_3^e$, $\tilde{l}_1 = l_2^e$, $\tilde{l}_2 = l_1^e$ and $\tilde{l}_3 = \max(l_0^e, -l_0^e - 1)$. Then $\tilde{l}_0 \geq \tilde{l}_1 \geq \tilde{l}_2 \geq \tilde{l}_3 \geq 0$, and the operator $H^{(l_0,l_1,l_2,l_3)}$ is isomonodromic to $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$. Note that, if $l_0 + l_3 \neq l_1 + l_2$ (resp. $l_0 + l_3 = l_1 + l_2$), then we have $(l_0, l_1, l_2, l_3) \neq (\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$ (resp. $(l_0, l_1, l_2, l_3) = (\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$).

We consider the case $l_0 + l_1 + l_2 + l_3$: odd. If $l_0 + l_1 + l_2 + l_3$ is odd, then the operator $H^{(l_0,l_1,l_2,l_3)}$ preserves the spaces

$$U_{-l_0,-l_1,-l_2,l_3+1},\ U_{-l_0,-l_1,l_2+1,-l_3},\ U_{-l_0,l_1+1,-l_2,-l_3},\ U_{l_0+1,-l_1,-l_2,-l_3}. \tag{4.8}$$

Each space is contained in $\mathcal{F}_{\epsilon_1,\epsilon_3}$ for some $\epsilon_1, \epsilon_3 \in \{\pm 1\}$, and the correspondence between the spaces and the signs of (ϵ_1, ϵ_3) is one-to-one. Let V be the sum of these spaces, then V is given by

$$V = U_{-l_0,-l_1,-l_2,l_3+1} \oplus U_{-l_0,-l_1,l_2+1,-l_3} \oplus U_{-l_0,l_1+1,-l_2,-l_3} \oplus U_{l_0+1,-l_1,-l_2,-l_3}, \tag{4.9}$$

and it is the maximal finite-dimensional invariant subspace in $\mathcal{F}$ with respect to the action of the operator $H^{(l_0,l_1,l_2,l_3)}$. Let k_i be the rearrangement of l_i such that $k_0 \geq k_1 \geq k_2 \geq k_3 (\geq 0)$ and set

$$g = \begin{cases} k_0, & k_0 \geq k_1 + k_2 + k_3 + 1; \\ (k_0 + k_1 + k_2 + k_3 + 1)/2, & k_0 < k_1 + k_2 + k_3 + 1. \end{cases} \tag{4.10}$$

Then $g \in \mathbb{Z}_{\geq 0}$ and $\dim V = 2g + 1$. Upon setting

$$l_0^o = (l_0 + l_1 + l_2 + l_3 + 1)/2, \quad l_1^o = (l_0 + l_1 - l_2 - l_3 - 1)/2, \\ l_2^o = (l_0 - l_1 + l_2 - l_3 - 1)/2, \quad l_3^o = (l_0 - l_1 - l_2 + l_3 - 1)/2. \tag{4.11}$$

it follows from Proposition 3.6 that

$$H^{(l^o_0,l^o_1,l^o_2,l^o_3)}\tilde{L}_{l_0+1,-l_1,-l_2,-l_3} = \tilde{L}_{l_0+1,-l_1,-l_2,-l_3}H^{(l_0,l_1,l_2,l_3)},$$
$$H^{(l^o_1,l^o_0,l^o_3,l^o_2)}\tilde{L}_{-l_0,l_1+1,-l_2,-l_3} = \tilde{L}_{-l_0,l_1+1,-l_2,-l_3}H^{(l_0,l_1,l_2,l_3)},$$
$$H^{(l^o_2,l^o_3,l^o_0,l^o_1)}\tilde{L}_{-l_0,-l_1,l_2+1,-l_3} = \tilde{L}_{-l_0,-l_1,l_2+1,-l_3}H^{(l_0,l_1,l_2,l_3)},$$
$$H^{(l^o_3,l^o_2,l^o_1,l^o_0)}\tilde{L}_{-l_0,-l_1,-l_2,l_3+1} = \tilde{L}_{-l_0,-l_1,-l_2,l_3+1}H^{(l_0,l_1,l_2,l_3)}.$$

Thus the operators which are linked by generalized Darboux transformations from $H^{(l_0,l_1,l_2,l_3)}$ are

$$H^{(l^o_0,l^o_1,l^o_2,l^o_3)},\ H^{(l^o_1,l^o_0,l^o_3,l^o_2)},\ H^{(l^o_2,l^o_3,l^o_0,l^o_1)},\ H^{(l^o_3,l^o_2,l^o_1,l^o_0)}, \quad (4.12)$$
$$H^{(l_0,l_1,l_2,l_3)},\ H^{(l_1,l_0,l_3,l_2)},\ H^{(l_2,l_3,l_0,l_1)},\ H^{(l_3,l_2,l_1,l_0)},$$

and these eight operators are isomonodromic.

Assume that $l_0,\ l_1,\ l_2,\ l_3 \in \mathbb{Z}$, $l_0 \geq l_1 \geq l_2 \geq l_3 \geq 0$ and $l_0+l_1+l_2+l_3$ is odd. Set $\tilde{l}_0 = l^o_0$, $\tilde{l}_1 = l^o_1$, $\tilde{l}_2 = l^o_2$ and $\tilde{l}_3 = \max(l^o_3, -l^o_3-1)$. Then $\tilde{l}_0 \geq \tilde{l}_1 \geq \tilde{l}_2 \geq \tilde{l}_3 \geq 0$, and the operator $H^{(l_0,l_1,l_2,l_3)}$ is isomonodromic to $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$. Note that, if $l_0 \neq l_1+l_2+l_3+1$ (resp. $l_0 = l_1+l_2+l_3+1$), then we have $(l_0,l_1,l_2,l_3) \neq (\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)$ (resp. $(l_0,l_1,l_2,l_3) = (\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)$).

We now reproduce results in this section for the Lamé equation and the associated Lamé equation. For the special case when three (resp. two) of l_i $(i=0,1,2,3)$ are zero, Eq.(1.6) is called the Lamé equation (resp. the associated Lamé equation). For simplicity, we consider the case $l_2 = l_3 = 0$, $l_0, l_1 \in \mathbb{Z}$ and $l_0 \geq l_1 \geq 0$.

If l_0+l_1 is even and $l_0 > l_1$, then the operator $H^{(l_0,l_1,0,0)}$ is isomonodromic to $H^{((l_0+l_1)/2,(l_0+l_1)/2,(l_0-l_1)/2,(l_0-l_1)/2-1)}$ by the transformation $L_{-l_0,l_1+1,1,0}$. In particular, if l_0 is even and $l_1 = 0$ (the case of Lamé equation), then $H^{(l_0,0,0,0)}$ is isomonodromic to $H^{(l_0/2,l_0/2,l_0/2,l_0/2-1)}$. Note that if l_0+l_1 is even and $l_0 = l_1$, then the operator $H^{(l_0,l_1,0,0)}$ is self-dual.

If l_0+l_1 is odd and $l_0 > l_1-1$, then the operator $H^{(l_0,l_1,0,0)}$ is isomonodromic to $H^{((l_0+l_1+1)/2,(l_0+l_1-1)/2,(l_0-l_1-1)/2,(l_0-l_1-1)/2)}$ by the transformation $L_{-l_0,l_1+1,1,1}$. In particular, if l_0 is odd and $l_1 = 0$ (the case of Lamé equation), then $H^{(l_0,0,0,0)}$ is isomonodromic to $H^{((l_0+1)/2,(l_0-1)/2,(l_0-1)/2,(l_0-1)/2)}$. Note that if l_0+l_1 is odd and $l_0 = l_1-1$, then the operator $H^{(l_0,l_1,0,0)}$ is self-dual.

Thus we have confirmed the conjecture of Khare and Sukhatme [4] regarding the isospectral partner of the Lamé and associated Lamé equations.

5 The case of half-integers

We investigate quasi-solvability and generalized Darboux transformation for the case $l_i + 1/2 \in \mathbb{Z}$ $(i=0,1,2,3)$. Introduce $l_i = n_i - 1/2$ and suppose that $n_i \in \mathbb{Z}_{\geq 0}$ for all $i \in \{0,1,2,3\}$.

Candidates for the space related to quasi-solvability written in the form of Eq.(3.1) are described as $V_{\tilde{n}_0+\frac{1}{2},\tilde{n}_1+\frac{1}{2},\tilde{n}_2+\frac{1}{2},\tilde{n}_3+\frac{1}{2}}$, where $\tilde{n}_i \in \{\pm n_i\}$ $(i=0,1,2,3)$. The condition $\sum_{i=0}^{3}\tilde{n}_i \in 2\mathbb{Z}$ is necessary for the existence of non-zero space $V_{\tilde{n}_0+\frac{1}{2},\tilde{n}_1+\frac{1}{2},\tilde{n}_2+\frac{1}{2},\tilde{n}_3+\frac{1}{2}}$. In other words, if $\sum_{i=0}^{3} n_i$ is odd, then the space related to quasi-solvability given in the form of Eq.(3.1) does not exist.

Henceforth in this section, we assume that $\sum_{i=0}^{3} n_i$ is even. Then the operator $H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})}$ preserves the spaces

$$
\begin{aligned}
& U_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}},\ U_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},n_2+\frac{1}{2},n_3+\frac{1}{2}},\\
& U_{-n_0+\frac{1}{2},n_1+\frac{1}{2},-n_2+\frac{1}{2},n_3+\frac{1}{2}},\ U_{-n_0+\frac{1}{2},n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}},\\
& U_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},n_3+\frac{1}{2}},\ U_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}},\\
& U_{-n_0+\frac{1}{2},n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}},\ U_{n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}},
\end{aligned}
$$

where $U_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is defined in Eq.(4.1). Unlike the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}$, these eight spaces are not disjoint. In fact, these eight spaces are subspaces of the space $U_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}$. Upon setting

$$
\begin{aligned}
& n_0^{(1)} = (n_0+n_1+n_2+n_3)/2, \quad n_1^{(1)} = (n_0+n_1-n_2-n_3)/2,\\
& n_2^{(1)} = (n_0-n_1+n_2-n_3)/2, \quad n_3^{(1)} = (n_0-n_1-n_2+n_3)/2,\\
& n_0^{(2)} = (-n_0+n_1+n_2+n_3)/2, \quad n_1^{(2)} = (n_0-n_1+n_2+n_3)/2,\\
& n_2^{(2)} = (n_0+n_1-n_2+n_3)/2, \quad n_3^{(2)} = (n_0+n_1+n_2-n_3)/2,
\end{aligned}
$$

it follows from Proposition 3.6 that

$$
\begin{aligned}
& H^{(n_0^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2})}\tilde{L}_{n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}\\
& \quad = \tilde{L}_{n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},\\
& H^{(n_1^{(1)}-\frac{1}{2},n_0^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2})}\tilde{L}_{-n_0+\frac{1}{2},n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}\\
& \quad = \tilde{L}_{-n_0+\frac{1}{2},n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},\\
& H^{(n_2^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2},n_0^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2})}\tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}}\\
& \quad = \tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},\\
& H^{(n_3^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_0^{(1)}-\frac{1}{2})}\tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},n_3+\frac{1}{2}}\\
& \quad = \tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},
\end{aligned}
$$

where $\tilde{L}_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is defined in Eq.(4.2). We also obtain

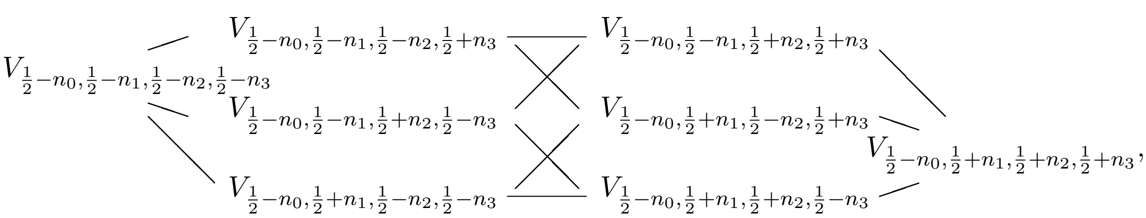
$$
\begin{aligned}
&H^{(n_0^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2},n_3^{(2)}-\frac{1}{2})}\tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}\\
&\quad=\tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},\\
&H^{(n_1^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2},n_3^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2})}\tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},n_2+\frac{1}{2},n_3+\frac{1}{2}}\\
&\quad=\tilde{L}_{-n_0+\frac{1}{2},-n_1+\frac{1}{2},n_2+\frac{1}{2},n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},\\
&H^{(n_2^{(2)}-\frac{1}{2},n_3^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2})}\tilde{L}_{-n_0+\frac{1}{2},n_1+\frac{1}{2},-n_2+\frac{1}{2},n_3+\frac{1}{2}}\\
&\quad=\tilde{L}_{-n_0+\frac{1}{2},n_1+\frac{1}{2},-n_2+\frac{1}{2},n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},\\
&H^{(n_3^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2})}\tilde{L}_{-n_0+\frac{1}{2},n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}}\\
&\quad=\tilde{L}_{-n_0+\frac{1}{2},n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}}H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})}.
\end{aligned}
$$

Thus the operators which are linked by generalized Darboux transformations from $H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})}$ are

$$
\begin{aligned}
&H^{(n_0^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2})},\ H^{(n_1^{(1)}-\frac{1}{2},n_0^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2})},\\
&H^{(n_2^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_0^{(1)}-\frac{1}{2})},\ H^{(n_3^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_0^{(1)}-\frac{1}{2})},\\
&H^{(n_0^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2},n_3^{(2)}-\frac{1}{2})},\ H^{(n_1^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2},n_3^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2})},\\
&H^{(n_2^{(2)}-\frac{1}{2},n_3^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2})},\ H^{(n_3^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2})},\\
&H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})},\ H^{(n_1-\frac{1}{2},n_0-\frac{1}{2},n_3-\frac{1}{2},n_2-\frac{1}{2})},\\
&H^{(n_2-\frac{1}{2},n_3-\frac{1}{2},n_0-\frac{1}{2},n_1-\frac{1}{2})},\ H^{(n_3-\frac{1}{2},n_2-\frac{1}{2},n_1-\frac{1}{2},n_0-\frac{1}{2})}.
\end{aligned}
$$

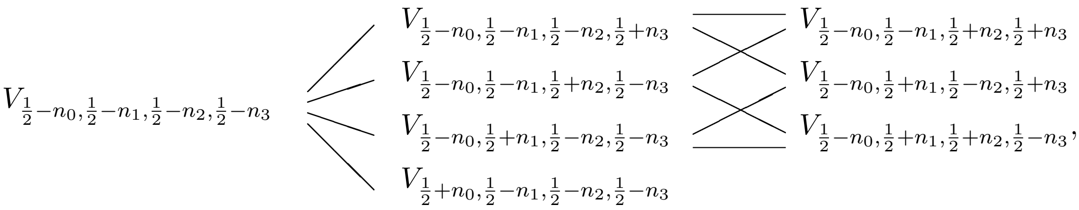
We now investigate the case $n_0,\ n_1,\ n_2,\ n_3\in\mathbb{Z}_{\geq 0}$, $n_0+n_1+n_2+n_3\in 2\mathbb{Z}_{\geq 0}$ and $n_0\geq n_1\geq n_2\geq n_3$. Recall that $l_i=n_i-1/2$ $(i=0,1,2,3)$. We will subdivide this case into three subcases, $n_0\geq n_1+n_2+n_3$, $n_1+n_2-n_3\leq n_0<n_1+n_2+n_3$ and the case $n_0<n_1+n_2-n_3$. If $\alpha_0+\alpha_1+\alpha_2+\alpha_3=2$, then we set $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}=\{0\}$.

If $n_0\geq n_1+n_2+n_3$, then the operator $H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})}$ preserves the spaces



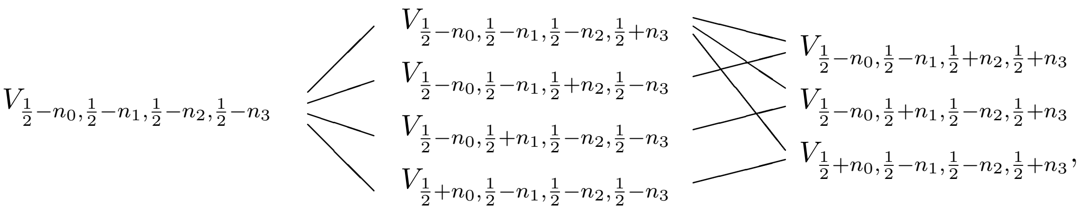
where the notation $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$—$V_{\tilde{\alpha}_0,\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3}$ means that $V_{\tilde{\alpha}_0,\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3}$ is a subspace of $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$. For this case, the operator $H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})}$ is isomonodromic to $H^{(n_0^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2})}$ and $H^{(n_3^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},-n_0^{(2)}-\frac{1}{2})}$ by the transformations $L_{-n_0+\frac{1}{2},n_1+\frac{1}{2},n_2+\frac{1}{2},n_3+\frac{1}{2}}$ and $L_{-n_0+\frac{1}{2},n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}}$ respectively.

If $n_1+n_2-n_3 \le n_0 < n_1+n_2+n_3$, then the operator $H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})}$ preserves the spaces



and it is isomonodromic to $H^{(n_0^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2},n_3^{(1)}-\frac{1}{2})}$ and $H^{(n_3^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2})}$ by the transformations $L_{n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}$ and $L_{-n_0+\frac{1}{2},n_1+\frac{1}{2},n_2+\frac{1}{2},-n_3+\frac{1}{2}}$ respectively.

If $n_0 < n_1+n_2-n_3$, then the operator $H^{(n_0-\frac{1}{2},n_1-\frac{1}{2},n_2-\frac{1}{2},n_3-\frac{1}{2})}$ preserves the spaces



and it is isomonodromic to $H^{(n_0^{(1)}-\frac{1}{2},n_1^{(1)}-\frac{1}{2},n_2^{(1)}-\frac{1}{2},-n_3^{(1)}-\frac{1}{2})}$ and $H^{(n_3^{(2)}-\frac{1}{2},n_2^{(2)}-\frac{1}{2},n_1^{(2)}-\frac{1}{2},n_0^{(2)}-\frac{1}{2})}$ by the transformations $L_{n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},-n_3+\frac{1}{2}}$ and $L_{n_0+\frac{1}{2},-n_1+\frac{1}{2},-n_2+\frac{1}{2},n_3+\frac{1}{2}}$ respectively.

6 Finite-gap potential

In this section, we construct an odd-order differential operator $\tilde{A}$ (see below) which commutes with the operator $H^{(l_0,l_1,l_2,l_3)}$ by composing generalized Darboux transformations for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\ge 0}$. We also show that the commuting operator coincides with the one constructed in [9]. We adopt the notation of section 4.

Proposition 6.1. If $l_0+l_1+l_2+l_3$ is even, we set

$$\tilde{A} = \tilde{L}_{-l_3^e,l_2^e+1,l_1^e+1,-l_0^e}\tilde{L}_{-l_1,l_0+1,-l_3,l_2+1}\tilde{L}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}\tilde{L}_{-l_0,-l_1,-l_2,-l_3}, \tag{6.1}$$

while if $l_0+l_1+l_2+l_3$ is odd, we set

$$\tilde{A} = \tilde{L}_{l_2^o+1,-l_3^o,-l_0^o,-l_1^o}\tilde{L}_{-l_1,-l_0,l_3+1,-l_2}\tilde{L}_{-l_0^o,l_1^o+1,-l_2^o,-l_3^o}\tilde{L}_{l_0+1,-l_1,-l_2,-l_3}. \tag{6.2}$$

We then have that the operator $\tilde{A}$ commutes with $H^{(l_0,l_1,l_2,l_3)}$, i.e.,

$$\tilde{A}H^{(l_0,l_1,l_2,l_3)} = H^{(l_0,l_1,l_2,l_3)}\tilde{A}. \tag{6.3}$$

Proof. For the case when $l_0+l_1+l_2+l_3$ is even, Eq.(6.3) may be proven by applying the following relations,

$$\begin{aligned} &H^{(l_0^e,l_1^e,l_2^e,l_3^e)}\tilde{L}_{-l_0,-l_1,-l_2,-l_3}=\tilde{L}_{-l_0,-l_1,-l_2,-l_3}H^{(l_0,l_1,l_2,l_3)}, \\ &H^{(l_1,l_0,l_3,l_2)}\tilde{L}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}=\tilde{L}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}H^{(l_0^e,l_1^e,l_2^e,l_3^e)}, \\ &H^{(l_3^e,l_2^e,l_1^e,l_0^e)}\tilde{L}_{-l_1,l_0+1,-l_3,l_2+1}=\tilde{L}_{-l_1,l_0+1,-l_3,l_2+1}H^{(l_1,l_0,l_3,l_2)}, \\ &H^{(l_0,l_1,l_2,l_3)}\tilde{L}_{-l_3^e,l_2^e+1,l_1^e+1,-l_0^e}=\tilde{L}_{-l_3^e,l_2^e+1,l_1^e+1,-l_0^e}H^{(l_3^e,l_2^e,l_1^e,l_0^e)}. \end{aligned} \tag{6.4}$$

For the case when $l_0+l_1+l_2+l_3$ is odd, Eq.(6.3) may be proven by the relations

$$\begin{aligned} &H^{(l_0^o,l_1^o,l_2^o,l_3^o)}\tilde{L}_{l_0+1,-l_1,-l_2,-l_3}=\tilde{L}_{l_0+1,-l_1,-l_2,-l_3}H^{(l_0,l_1,l_2,l_3)}, \\ &H^{(l_1,l_0,l_3,l_2)}\tilde{L}_{-l_0^o,l_1^o+1,-l_2^o,-l_3^o}=\tilde{L}_{-l_0^o,l_1^o+1,-l_2^o,-l_3^o}H^{(l_0^o,l_1^o,l_2^o,l_3^o)}, \\ &H^{(l_2^o,l_3^o,l_0^o,l_1^o)}\tilde{L}_{-l_1,-l_0,l_3+1,-l_2}=\tilde{L}_{-l_1,-l_0,l_3+1,-l_2}H^{(l_1,l_0,l_3,l_2)}, \\ &H^{(l_0,l_1,l_2,l_3)}\tilde{L}_{l_2^o+1,-l_3^o,-l_0^o,-l_1^o}=\tilde{L}_{l_2^o+1,-l_3^o,-l_0^o,-l_1^o}H^{(l_2^o,l_3^o,l_0^o,l_1^o)}. \end{aligned} \tag{6.5}$$

■

Note that the order of the differential operator $\tilde{A}$ is equal to $2g+1$, where g is defined in Eq.(4.5) or Eq.(4.10). Recall that the space V can be expressed in the form of Eq.(4.4) or Eq.(4.9), with $\dim V=2g+1$, and it is the maximal finite-dimensional invariant subspace in $\mathcal{F}$ with respect to the action of the operator $H^{(l_0,l_1,l_2,l_3)}$.

Proposition 6.2. The operator $\tilde{A}$ is the monic differential operator of minimum order which annihilates all elements in V.

Proof. We prove this proposition for the case when $l_0+l_1+l_2+l_3$ is even. The case when $l_0+l_1+l_2+l_3$ is odd may be proven similarly.

Since V may be written in the form of Eq.(4.4), and $\deg\tilde{A}=\dim V$, it is sufficient to show that $\tilde{A}\phi(x)=0$ for $\phi(x)\in U_{-l_0,-l_1,-l_2,-l_3}$, $U_{-l_0,-l_1,l_2+1,l_3+1}$, $U_{-l_0,l_1+1,-l_2,l_3+1}$, $U_{-l_0,l_1+1,l_2+1,-l_3}$. It follows from the expression of the operator $\tilde{A}$ in the form of Eq.(6.1) that the operator $\tilde{L}_{-l_0,-l_1,-l_2,-l_3}$ annihilates any element in the space $U_{-l_0,-l_1,-l_2,-l_3}$. Hence we have $\tilde{A}\phi(x)=0$ for $\phi(x)\in U_{-l_0,-l_1,-l_2,-l_3}$. Let $\langle f_1(x),\dots,f_n(x)\rangle$ be a basis of the space $U_{-l_0,-l_1,l_2+1,l_3+1}$. Since the operator $H^{(l_0,l_1,l_2,l_3)}$ preserves the space $U_{-l_0,-l_1,l_2+1,l_3+1}$, the function $H^{(l_0,l_1,l_2,l_3)}f_j(x)$ may be written in the form $\sum_{i=1}^n a_{i,j}f_i(x)$ for some constants $a_{i,j}$. It follows from Eq.(6.4) that

$$\begin{aligned} H^{(l_0^e,l_1^e,l_2^e,l_3^e)}\tilde{L}_{-l_0,-l_1,-l_2,-l_3}f_j(x)&=\tilde{L}_{-l_0,-l_1,-l_2,-l_3}H^{(l_0,l_1,l_2,l_3)}f_j(x) \\ &=\sum_{i=1}^n a_{i,j}\tilde{L}_{-l_0,-l_1,-l_2,-l_3}f_j(x). \end{aligned} \tag{6.6}$$

Set $\tilde{U}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}=\tilde{L}_{-l_0,-l_1,-l_2,-l_3}U_{-l_0,-l_1,l_2+1,l_3+1}$. Then it follows from Eq.(6.6) that the space $\tilde{U}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}$ is invariant under the action of $H^{(l_0^e,l_1^e,l_2^e,l_3^e)}$. Let (ϵ_1,ϵ_3) ($\epsilon_1,\epsilon_3\in\{\pm1\}$) be such that $U_{-l_0,-l_1,l_2+1,l_3+1}\subset\mathcal{F}_{\epsilon_1,\epsilon_3}$. Then we have $\tilde{U}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}\subset\mathcal{F}_{\epsilon_1,\epsilon_3}$ and $U_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}\subset\mathcal{F}_{\epsilon_1,\epsilon_3}$. As is shown in [7, Theorem 3.1], the space $U_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}$ is the maximum subspace of $\mathcal{F}_{\epsilon_1,\epsilon_3}$ which is invariant under the action of $H^{(l_0^e,l_1^e,l_2^e,l_3^e)}$. Thus we have

$$\tilde{U}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1} = \tilde{L}_{-l_0,-l_1,-l_2,-l_3} U_{-l_0,-l_1,l_2+1,l_3+1} \subset U_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}.$$

Similarly it may be shown that

$$\tilde{L}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}\tilde{L}_{-l_0,-l_1,-l_2,-l_3}U_{-l_0,l_1+1,-l_2,l_3+1} \subset U_{-l_1,l_0+1,-l_3,l_2+1},$$
$$\tilde{L}_{-l_1,l_0+1,-l_3,l_2+1}\tilde{L}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}\tilde{L}_{-l_0,-l_1,-l_2,-l_3}U_{-l_0,l_1+1,l_2+1,-l_3} \subset U_{-l_3^e,l_2^e+1,l_1^e+1,-l_0^e}.$$

Since the operator $\tilde{L}_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}$ (resp. $\tilde{L}_{-l_1,l_0+1,-l_3,l_2+1}$, $\tilde{L}_{-l_3^e,l_2^e+1,l_1^e+1,-l_0^e}$) annihilates any element in the space $U_{-l_0^e,-l_1^e,l_2^e+1,l_3^e+1}$ (resp. $U_{-l_1,l_0+1,-l_3,l_2+1}$, $U_{-l_3^e,l_2^e+1,l_1^e+1,-l_0^e}$), we have $\tilde{A}\phi(x) = 0$ for $\phi(x) \in U_{-l_0,-l_1,l_2+1,l_3+1}$ (resp. $\phi(x) \in U_{-l_0,l_1+1,-l_2,l_3+1}$, $\phi(x) \in U_{-l_0,l_1+1,l_2+1,-l_3}$). ■

It follows from Proposition 6.2 that the kernel of the operator $\tilde{A}$ coincides with the space V. We denote the monic characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space V by $P(E)$. For simplicity, we set $u(x) = \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)$ and $H = H^{(l_0,l_1,l_2,l_3)} = -d^2/dx^2 + u(x)$.

Proposition 6.3. Set $\tilde{a}_0(x) = 1$ and $\tilde{a}_{g+1}(x) = 0$. The operator $\tilde{A}$ may be expressed in the form

$$\tilde{A} = (-1)^g \left[\sum_{j=0}^{g} \left\{ \tilde{a}_j(x)\frac{d}{dx} - \frac{1}{2}\left(\frac{d}{dx}\tilde{a}_j(x)\right) \right\} H^{g-j} + \sum_{j=0}^{g} c_j H^{g-j} \right], \tag{6.7}$$

for some even doubly-periodic functions $\tilde{a}_j(x)$ $(j = 1,\dots,g)$ and constants c_j $(j = 0,\dots,g)$, where the functions $\tilde{a}_j(x)$ $(j = 0,\dots,g)$ satisfy

$$\tilde{a}_j'''(x) - 4u(x)\tilde{a}_j'(x) + 4\tilde{a}_{j+1}'(x) - 2u'(x)\tilde{a}_j(x) = 0. \tag{6.8}$$

Proof. Since $\tilde{A}$ is a monic differential operator of order $2g+1$, it can be expressed in the form

$$\tilde{A} = (-1)^g \left[\sum_{j=0}^{g} \left(\tilde{a}_j(x)\frac{d}{dx} + \tilde{b}_j(x) \right) H^{g-j} \right],$$

where $\tilde{a}_0(x) = 1$. We have

$$[(-1)^g\tilde{A}, H] = \sum_{j=0}^{g} \left[\tilde{a}_j(x)\frac{d}{dx} + \tilde{b}_j(x), -\frac{d^2}{dx^2} + u(x) \right] H^{g-j}$$
$$= \sum_{j=0}^{g} \left(\tilde{a}_j(x)u'(x) + 2\tilde{a}_j'(x)\frac{d^2}{dx^2} + (\tilde{a}_j''(x) + 2\tilde{b}_j'(x))\frac{d}{dx} + \tilde{b}_j''(x) \right) H^{g-j}$$
$$= \sum_{j=0}^{g} \left(2\tilde{a}_j'(x)(-H + u(x)) + (\tilde{a}_j''(x) + 2\tilde{b}_j'(x))\frac{d}{dx} + \tilde{a}_j(x)u'(x) + \tilde{b}_j''(x) \right) H^{g-j}$$
$$= \sum_{j=0}^{g} \left((\tilde{a}_j''(x) + 2\tilde{b}_j'(x))\frac{d}{dx} - 2\tilde{a}_{j+1}'(x) + 2\tilde{a}_j'(x)u(x) + \tilde{a}_j(x)u'(x) + \tilde{b}_j''(x) \right) H^{g-j}$$
$$= 0.$$

Hence we obtain

$$\tilde{a}_j''(x) + 2\tilde{b}_j'(x) = 0, \quad -2\tilde{a}_{j+1}'(x) + 2\tilde{a}_j'(x)u(x) + \tilde{a}_j(x)u'(x) + \tilde{b}_j''(x) = 0,$$

and so

$$\tilde{b}_j(x) = -\tilde{a}_j'(x)/2 + c_j, \quad \tilde{a}_j'''(x) - 4u(x)\tilde{a}_j'(x) + 4\tilde{a}_{j+1}'(x) - 2u'(x)\tilde{a}_j(x) = 0$$

for some constants c_j $(j = 0, \dots, g)$, and we obtain the proposition. ∎

Set

$$\tilde{\Xi}(x, E) = \sum_{i=0}^{g} \tilde{a}_{g-i}(x)E^i. \tag{6.9}$$

It follows from Eq.(6.8) that

$$\left(\frac{d^3}{dx^3} - 4\left(u(x) - E\right)\frac{d}{dx} - 2u'(x)\right)\tilde{\Xi}(x, E) = 0. \tag{6.10}$$

If we also set

$$\tilde{Q}(E) = \tilde{\Xi}(x, E)^2\left(E - u(x)\right) + \frac{1}{2}\tilde{\Xi}(x, E)\frac{d^2\tilde{\Xi}(x, E)}{dx^2} - \frac{1}{4}\left(\frac{d\tilde{\Xi}(x, E)}{dx}\right)^2, \tag{6.11}$$

then it is shown by differentiating the right-hand side of Eq.(6.11) and applying Eq.(6.10) that $\tilde{Q}(E)$ is independent of x. With reference to the expression for $\tilde{\Xi}(x, E)$ given by Eq.(6.9), $\tilde{Q}(E)$ is a monic polynomial in E of degree $2g + 1$. In a similar way to that of Proposition 3.2 in [9], we can show

$$\left((-1)^g\tilde{A} - \sum_{j=0}^{g} c_j H^{g-j}\right)^2 = \left(\sum_{j=0}^{g}\left\{\tilde{a}_j(x)\frac{d}{dx} - \frac{1}{2}\left(\frac{d}{dx}\tilde{a}_j(x)\right)\right\}H^{g-j}\right)^2 = \tilde{Q}(H)^2.$$

Let us recall the function $\Xi(x, E)$ defined in [7]. It satisfies Eq.(6.10) and has an expression

$$\Xi(x, E) = c_0(E) + \sum_{i=0}^{3}\sum_{j=0}^{l_i-1} b_j^{(i)}(E)\wp(x + \omega_i)^{l_i-j}, \tag{6.12}$$

where the coefficients $c_0(E)$ and $b_j^{(i)}(E)$ are polynomials in E, they do not have common divisors and the polynomial $c_0(E)$ is monic. It is shown that the dimension of the functions which are doubly-periodic and satisfy Eq.(6.10) is one (see [11, Proposition 3.9]). Since $\tilde{\Xi}(x, E)$ is a polynomial with respect to the variable E and coefficients of $\Xi(x, E)$ are coprime, we have

$$\tilde{\Xi}(x, E) = \Xi(x, E)\left(\sum_{i=0}^{k} d_i E^{k-i}\right) \tag{6.13}$$

for a non-negative integer k and constants d_i $(i = 0, \dots, k)$ such that $d_0 = 1$. Upon writing

$$\Xi(x, E) = \sum_{i=0}^{g-k} a_{g-k-i}(x) E^i, \tag{6.14}$$

we have $a_0(x) = 1$ and

$$\tilde{a}_{g-i}(x) = \sum_{j=0}^{k} a_{g-i-k+j}(x) d_{k-j}.$$

The functions $a_i(x)$ $(i = 0, \dots, g-k)$ also satisfy Eq.(6.8), because $\Xi(x, E)$ satisfies Eq.(6.10). Set

$$A = \sum_{j=0}^{g-k} \left\{ a_j(x) \frac{d}{dx} - \frac{1}{2} \left(\frac{d}{dx} a_j(x) \right) \right\} H^{g-k-j}, \tag{6.15}$$

It then follows from Eq.(6.8) for $a_i(x)$ that $[A, H] = 0$.

Proposition 6.4.

$$(-1)^g \tilde{A} - \sum_{l=0}^{g} c_{g-l} H^l = A \left(\sum_{j=0}^{k} d_{k-j} H^j \right). \tag{6.16}$$

Proof.

$$\begin{aligned}
(-1)^g \tilde{A} - \sum_{l=0}^{g} c_{g-l} H^l &= \sum_{i=0}^{g} \left\{ \tilde{a}_i(x) \frac{d}{dx} - \frac{1}{2} \left(\frac{d}{dx} \tilde{a}_i(x) \right) \right\} H^{g-i} \\
&= \sum_{i=0}^{g} \left\{ \left(\sum_{j=0}^{k} a_{g-i-k+j}(x) d_{k-j} \right) \frac{d}{dx} - \frac{1}{2} \left(\frac{d}{dx} \sum_{j=0}^{k} a_{g-i-k+j}(x) d_{k-j} \right) \right\} H^{g-i} \\
&= \sum_{i=0}^{g} \sum_{j=0}^{k} \left\{ a_{g-i-k+j}(x) \frac{d}{dx} - \frac{1}{2} \left(\frac{d}{dx} a_{g-i-k+j}(x) \right) \right\} H^{g-i-j} d_{k-j} H^j \\
&= A \left(\sum_{j=0}^{k} d_{k-j} H^j \right).
\end{aligned}$$

■

Set

$$Q(E) = \Xi(x, E)^2 (E - u(x)) + \frac{1}{2} \Xi(x, E) \frac{d^2 \Xi(x, E)}{dx^2} - \frac{1}{4} \left(\frac{d\Xi(x, E)}{dx} \right)^2. \tag{6.17}$$

Then the right hand side of Eq.(6.17) is independent of x, and $Q(E)$ is a monic polynomial in E of degree $2(g-k)+1$. It is shown in [9, Proposition 3.2] that $A^2 = Q(H)$. By Eq.(6.13) and the definitions of $Q(E)$ and $\tilde{Q}(E)$, we have

$$\tilde{Q}(E) = Q(E) \left(\sum_{i=0}^{k} d_i E^{k-i} \right)^2, \quad \deg Q(E) \le \deg \tilde{Q}(E) = 2g + 1. \tag{6.18}$$

Regarding the zeros of $Q(E)$ and $P(E)$, the following proposition is proven in [7, Theorem 3.8] (see also [9, Proposition 2.4]):

Proposition 6.5. (c.f. [7, Theorem 3.8]) The set of zeros of $Q(E)$ coincides with the set of zeros of $P(E)$.

Proposition 6.6. (c.f. [12]) Roots of the equation $P(E) = 0$ are distinct for generic periods $(2\omega_1, 2\omega_3)$.

A proof of this proposition is given in Appendix B.

Proposition 6.7. For the periods $(2\omega_1, 2\omega_3)$ for which the roots of the equation $P(E) = 0$ are distinct, we have $P(E) = Q(E) = \tilde{Q}(E)$ and $\Xi(x, E) = \tilde{\Xi}(x, E)$.

Proof. By assumption, the equation $P(E) = 0$ has $2g + 1$ roots which are distinct. It follows from Proposition 6.5 that the number of roots of the equation $Q(E) = 0$ is greater than or equal to $2g+1$. Then we have $\deg Q(E) \geq 2g+1$. Combining with Eq.(6.18) and $\deg P(E) = 2g + 1$, we have $\tilde{Q}(E) = Q(E) = P(E)$, and the value k in Eq.(6.13) is zero. Hence $\Xi(x, E) = \tilde{\Xi}(x, E)$. ■

Proposition 6.8. All the constants c_j $(j = 0, \dots, g)$ are zero. That is, we have

$$\tilde{A} = (-1)^g \sum_{j=0}^{g} \left\{ \tilde{a}_j(x) \frac{d}{dx} - \frac{1}{2} \left(\frac{d}{dx} \tilde{a}_j(x) \right) \right\} H^{g-j}. \tag{6.19}$$

Proof. First, we assume that roots of the equation $P(E) = 0$ are distinct. Let $\{E_i\}_{i=1,\dots,2g+1}$ be the roots of the equation $P(E) = 0$. Then E_i $(i = 1, \dots, 2g + 1)$ are eigenvalues of the operator H on the space V. Let $\phi_i(x) \in V$ be the eigenfunction of the operator H corresponding to the eigenvalue E_i. It follows from Proposition 6.4 that $((-1)^g \tilde{A} - \sum_{l=0}^{g} c_{g-l} H^l)^2 = A^2 = Q(H) = P(H)$. Hence $(\tilde{A} - 2(-1)^g \sum_{l=0}^{g} c_{g-l} H^l)\tilde{A} + (\sum_{l=0}^{g} c_{g-l} H^l)^2 = P(H)$. We apply this operator to the function $\phi_i(x)$. Since the operator $\tilde{A}$ annihilates the space V, we have $(\sum_{l=0}^{g} c_{g-l}(E_i)^l)^2 \phi_i(x) = P(E_i)\phi_i(x) = 0$. Hence $\sum_{l=0}^{g} c_{g-l}(E_i)^l = 0$ for $i = 1, \dots, 2g + 1$. Since the polynomial of degree less that $g + 1$ cannot have $2g + 1$ zeros, we have $c_l = 0$ $(l = 0, \dots, g)$ for the case that roots of the equation $P(E) = 0$ are distinct.

Now consider the case that roots of the equation $P(E) = 0$ are not distinct. Since the operator $\tilde{A}$ is defined by Eq.(6.1) or Eq.(6.2), the coefficients c_i $(i = 0, \dots, g)$ are continuous as a function of periods $(2\omega_1, 2\omega_3)$. Since $c_i = 0$ $(i = 0, \dots, g)$ for a dense set of periods, we have $c_i = 0$ $(i = 0, \dots, g)$ for all periods. ■

For the case that roots of the equation $P(E) = 0$ are distinct, it has been shown that $P(E) = Q(E)$, the value k in Eq.(6.13) is zero, $\Xi(x, E) = \tilde{\Xi}(x, E)$ and $\tilde{A} = (-1)^g A$.

We now consider the case that roots of the equation $P(E) = 0$ are not distinct. It has already been shown that $P(E) = \tilde{Q}(E)$ for a dense set of periods $(2\omega_1, 2\omega_3)$, and $\deg P(E) = \deg \tilde{Q}(E)$ for all periods. Coefficients of $P(E)$ and $Q(E)$ are continuous with respect to periods. Hence we have $P(E) = \tilde{Q}(E)$ for all periods. Assume that the value k in Eq.(6.13) is positive. By applying Eq.(6.18) and Proposition 6.5, all zeros of $(\sum_{i=0}^{k} d_i E^i)^2$ are zeros of $Q(E)$. Let E_0 be a zero of $\sum_{i=0}^{k} d_i E^i$, i.e. $\sum_{i=0}^{k} d_i (E_0)^i = 0$.

By Eq.(6.16) and Proposition 6.8, we have that if $f(x)$ satisfies $Hf(x) = E_0 f(x)$, then $\tilde{A}f(x) = 0$. Hence all solutions to $(H - E_0)f(x) = 0$ are contained in the space V. However, this is a contradiction since, as was essentially shown in [7, Theorem 3.8] (see also the proof of [11, Proposition 3.9]), all solutions to $(H-E_0)f(x) = 0$ cannot be doubly-periodic up to signs. Hence we have $k = 0$. Therefore $\tilde{A} = (-1)^g A$, $\deg P(E) = \deg Q(E)$ and $P(E) = Q(E)$ for all periods. It follows from Proposition 6.2 that the operator $(-1)^g A (= \tilde{A})$ is characterized by the monic operator of order $2g + 1$ which annihilates all elements in the space V. Summarizing, we obtain the following theorem:

Theorem 6.9. Let $\tilde{A}$ be the operator defined by composing the generalized Darboux transformation (see Eq.(6.1) or Eq.(6.2)) and A be the operator defined from the even doubly-periodic function $\Xi(x, E)$ (see Eqs.(6.14, 6.15)). Let $P(E)$ be the characteristic polynomial of the operator H on the space V, and $Q(E)$ be the polynomial defined from $\Xi(x, E)$ (see Eq.(6.17)).
(i) We have $\tilde{A} = (-1)^g A$ and $P(E) = Q(E)$.
(ii) The operator $(-1)^g A$ is also characterized by the monic operator of order $2g+1$ which annihilates all elements in the space V.

Note that we have proved Conjecture 1 in [9] by (i) and Conjecture 2 in [9] by (ii). Since $P(E) = Q(E)$, the genus of the spectral curve $\nu^2 = -Q(E)$ is g, where g is defined in Eq.(4.5) ($l_0 + l_1 + l_2 + l_3$: even) or Eq.(4.10) ($l_0 + l_1 + l_2 + l_3$: odd), in agreement with the results in [3, 13].

7 Functions related to monodromy

We now reconsider the isomonodromic property for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$. Let l_i^e, l_i^o $(i = 0, 1, 2, 3)$ be the numbers defined in Eqs.(4.6, 4.11). We have shown in section 4 that, if $l_0 + l_1 + l_2 + l_3$ is even (resp. odd), then the operator $H^{(l_0,l_1,l_2,l_3)}$ is linked to $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$ by the generalized Darboux transformation, where $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$ is any operator listed in Eq.(4.7) (resp. Eq.(4.12)). We now show that functions related to monodromy for the operator $H^{(l_0,l_1,l_2,l_3)}$ coincide with the corresponding functions for the operator $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$. For this purpose, we recall functions defined in [7, 9, 10].

Let $\Xi^{(l_0,l_1,l_2,l_3)}(x, E)$ be the function defined in Eq.(6.12), $Q^{(l_0,l_1,l_2,l_3)}(E)$ be the polynomial defined in Eq.(6.17) and $P^{(l_0,l_1,l_2,l_3)}(E)$ be the polynomial $P(E)$ defined in section 6. Upon setting

$$\Lambda^{(l_0,l_1,l_2,l_3)}(x, E) = \sqrt{\Xi^{(l_0,l_1,l_2,l_3)}(x, E)} \exp \int \frac{\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(E)}dx}{\Xi^{(l_0,l_1,l_2,l_3)}(x, E)}, \qquad (7.1)$$

the function $\Lambda^{(l_0,l_1,l_2,l_3)}(x, E)$ is a solution to the differential equation

$$(H^{(l_0,l_1,l_2,l_3)} - E)f(x) = 0.$$

Upon further setting

$$\Phi_i(x, \alpha) = \frac{\sigma(x + \omega_i - \alpha)}{\sigma(x + \omega_i)} \exp(\zeta(\alpha)x), \qquad (i = 0, 1, 2, 3),$$

where $\sigma(x)$ is the Weierstrass sigma function, the function $\Lambda^{(l_0,l_1,l_2,l_3)}(x, E)$ admits an expression in terms of Hermite-Krichever Ansatz. That is, we have the following proposition:

Proposition 7.1. (c.f. [10, Theorem 2.3]) The function $\Lambda^{(l_0,l_1,l_2,l_3)}(x,E)$ can be expressed in the form

$$\Lambda^{(l_0,l_1,l_2,l_3)}(x,E)=\exp(\kappa x)\left(\sum_{i=0}^{3}\sum_{j=0}^{l_i-1}\tilde{b}_j^{(i)}\left(\frac{d}{dx}\right)^j\Phi_i(x,\alpha)\right) \tag{7.2}$$

for some values $\tilde{b}_j^{(i)}$ $(i=0,\dots,3,\ j=0,\dots,l_i-1)$ except for finitely-many eigenvalues E. The values α and κ are expressed in the form

$$\wp(\alpha)=R_1^{(l_0,l_1,l_2,l_3)}(E),\quad \wp'(\alpha)=R_2^{(l_0,l_1,l_2,l_3)}(E)\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(E)},$$
$$\kappa=R_3^{(l_0,l_1,l_2,l_3)}(E)\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(E)},$$

where $R_1^{(l_0,l_1,l_2,l_3)}(E), R_2^{(l_0,l_1,l_2,l_3)}(E), R_3^{(l_0,l_1,l_2,l_3)}(E)$ are rational functions in E.

It follows from Eq.(7.2) that

$$\Lambda^{(l_0,l_1,l_2,l_3)}(x+2\omega_k,E)=\exp(-2\eta_k\alpha+2\omega_k\zeta(\alpha)+2\kappa\omega_k)\Lambda^{(l_0,l_1,l_2,l_3)}(x,E) \tag{7.3}$$

for $k=1,2,3$. On the other hand, a monodromy formula in terms of a hyperelliptic integral was obtained in [9].

Proposition 7.2. (c.f. [9, Theorem 3.7]) Let $k\in\{1,2,3\}$, $q_k\in\{0,1\}$ and E_0 be the eigenvalue such that $\Lambda^{(l_0,l_1,l_2,l_3)}(x+2\omega_k,E_0)=(-1)^{q_k}\Lambda^{(l_0,l_1,l_2,l_3)}(x,E_0)$. Then

$$\Lambda^{(l_0,l_1,l_2,l_3)}(x+2\omega_k,E)=(-1)^{q_k}\Lambda^{(l_0,l_1,l_2,l_3)}(x,E)\cdot \tag{7.4}$$
$$\exp\left(-\frac{1}{2}\int_{E_0}^{E}\frac{-2\eta_k a^{(l_0,l_1,l_2,l_3)}(\tilde{E})+2\omega_k c^{(l_0,l_1,l_2,l_3)}(\tilde{E})}{\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(\tilde{E})}}d\tilde{E}\right),$$

where $a^{(l_0,l_1,l_2,l_3)}(E)$ and $c^{(l_0,l_1,l_2,l_3)}(E)$ are polynomials defined in [10].

The following proposition states the coincidence of functions:

Proposition 7.3. Let $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$ be any operator listed in Eq.(4.7) ($l_0+l_1+l_2+l_3$: even) or Eq.(4.12) ($l_0+l_1+l_2+l_3$: odd). Then we have

$$P^{(l_0,l_1,l_2,l_3)}(E)=P^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}(E),\quad Q^{(l_0,l_1,l_2,l_3)}(E)=Q^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}(E),$$
$$R_i^{(l_0,l_1,l_2,l_3)}(E)=R_i^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}(E),\qquad (i=1,2,3),$$
$$a^{(l_0,l_1,l_2,l_3)}(E)=a^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}(E),\quad c^{(l_0,l_1,l_2,l_3)}(E)=c^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}(E).$$

Proof. We prove the proposition for the case when $l_0+l_1+l_2+l_3$ is even and $(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)=(l_3^e,l_2^e,l_1^e,l_0^e)$. The proofs are similar for the other cases.

The space V for the operator $H^{(l_0,l_1,l_2,l_3)}$ is given by

$$U_{-l_0,-l_1,-l_2,-l_3}\oplus U_{-l_0,-l_1,l_2+1,l_3+1}\oplus U_{-l_0,l_1+1,-l_2,l_3+1}\oplus U_{-l_0,l_1+1,l_2+1,-l_3},$$

while the corresponding space for the operator $H^{(l_3^e,l_2^e,l_1^e,l_0^e)}$ is given by

$$U_{-l_3^e,-l_2^e,-l_1^e,-l_0^e} \oplus U_{-l_3^e,-l_2^e,l_1^e+1,l_0^e+1} \oplus U_{-l_3^e,l_2^e+1,-l_1^e,l_0^e+1} \oplus U_{l_3^e+1,-l_2^e,-l_1^e,l_0^e+1}.$$

It has been seen that the space $U_{-l_0,-l_1,-l_2,-l_3}$ (resp. $U_{-l_0,-l_1,l_2+1,l_3+1}$, $U_{-l_0,l_1+1,-l_2,l_3+1}$, $U_{-l_0,l_1+1,l_2+1,-l_3}$) is linked to the space $U_{-l_3^e,-l_2^e,-l_1^e,-l_0^e}$ (resp. $U_{-l_3^e,-l_2^e,l_1^e+1,l_0^e+1}$, $U_{-l_3^e,l_2^e+1,-l_1^e,l_0^e+1}$, $U_{l_3^e+1,-l_2^e,-l_1^e,l_0^e+1}$) by the generalized Darboux transformation $\tilde{L}_{-l_0,-l_1,-l_2,-l_3}$ (resp. $\tilde{L}_{-l_0,-l_1,l_2+1,l_3+1}$, $\tilde{L}_{-l_0,l_1+1,-l_2,l_3+1}$, $\tilde{L}_{-l_0,l_1+1,l_2+1,-l_3}$) and the shift $x \to x+\omega_3$ (resp. $x \to x+\omega_1$, $x \to x+\omega_2$, $x \to x$). It follows from Proposition 3.4 that the characteristic polynomial of $H^{(l_0,l_1,l_2,l_3)}$ on the space $U_{-l_0,-l_1,-l_2,-l_3}$ (resp. $U_{-l_0,-l_1,l_2+1,l_3+1}$, $U_{-l_0,l_1+1,-l_2,l_3+1}$, $U_{-l_0,l_1+1,l_2+1,-l_3}$) is equal to that of $H^{(l_3^e,l_2^e,l_1^e,l_0^e)}$ on the space $U_{-l_3^e,-l_2^e,-l_1^e,-l_0^e}$ (resp. $U_{-l_3^e,-l_2^e,l_1^e+1,l_0^e+1}$, $U_{-l_3^e,l_2^e+1,-l_1^e,l_0^e+1}$, $U_{l_3^e+1,-l_2^e,-l_1^e,l_0^e+1}$). Since the polynomial $P(E)$ is given by the product of characteristic polynomials of invariant subspaces, we have $P^{(l_0,l_1,l_2,l_3)}(E) = P^{(l_3^e,l_2^e,l_1^e,l_0^e)}(E)$. It further follows from Theorem 6.9 that $Q^{(l_0,l_1,l_2,l_3)}(E) = Q^{(l_3^e,l_2^e,l_1^e,l_0^e)}(E)$.

Let $\tilde{\alpha}$ and $\tilde{\kappa}$ be the corresponding values in Eq.(7.3) for the parameters $(l_3^e, l_2^e, l_1^e, l_0^e)$, that is, these values satisfy

$$\Lambda^{(l_3^e,l_2^e,l_1^e,l_0^e)}(x+2\omega_k, E) = \exp(-2\eta_k\tilde{\alpha} + 2\omega_k\zeta(\tilde{\alpha}) + 2\tilde{\kappa}\omega_k)\Lambda^{(l_3^e,l_2^e,l_1^e,l_0^e)}(x, E)$$

for $k = 1, 2, 3$. The monodromy matrix is preserved by the generalized Darboux transformation for almost all eigenvalues E. Since the values $\exp(\pm(-2\eta_k\alpha + 2\omega_k\zeta(\alpha) + 2\kappa\omega_k))$ (resp. $\exp(\pm(-2\eta_k\tilde{\alpha} + 2\omega_k\zeta(\tilde{\alpha}) + 2\tilde{\kappa}\omega_k)))$ are eigenvalues of the monodromy matrix, we have

$$\begin{aligned}
&-2\eta_1\alpha + 2\omega_1(\zeta(\alpha)+\kappa) = \pm(-2\eta_1\tilde{\alpha} + 2\omega_1(\zeta(\tilde{\alpha})+\tilde{\kappa})) + 2n_1\pi\sqrt{-1}, \\
&-2\eta_3\alpha + 2\omega_3(\zeta(\alpha)+\kappa) = \pm(-2\eta_3\tilde{\alpha} + 2\omega_3(\zeta(\tilde{\alpha})+\tilde{\kappa})) + 2n_3\pi\sqrt{-1},
\end{aligned}$$

for some integers n_1 and n_3. By the Legendre's relation $\eta_1\omega_3 - \eta_3\omega_1 = \pi\sqrt{-1}/2$ and the relation $\zeta(x+2\omega_k) = \zeta(x) + 2\eta_k$ $(k = 1, 3)$, it follows that

$$\alpha = \pm\tilde{\alpha} - (2n_1\omega_3 - 2n_3\omega_1), \quad \kappa = \pm\tilde{\kappa}.$$

It follows from the asymptotic behavior of κ (see [10, Proposition 3.2]) that the sign $\pm$ is plus. Hence $\wp(\alpha) = \wp(\tilde{\alpha})$, $\wp'(\alpha) = \wp'(\tilde{\alpha})$ and $\kappa = \tilde{\kappa}$ for almost E. Since $R_i^{(l_0,l_1,l_2,l_3)}(E)$ and $R_i^{(l_3^e,l_2^e,l_1^e,l_0^e)}(E)$ $(i = 1, 2, 3)$ are rational functions, we have $R_i^{(l_0,l_1,l_2,l_3)}(E) = R_i^{(l_3^e,l_2^e,l_1^e,l_0^e)}(E)$ for $i = 1, 2, 3$.

Let $\tilde{E}_0$ be the corresponding values in Eq.(7.4) for the parameters $(l_3^e, l_2^e, l_1^e, l_0^e)$. By applying a similar discussion for Eq.(7.4), we obtain that the integrals

$$\begin{aligned}
&\int_{E_0}^{E} \frac{a^{(l_0,l_1,l_2,l_3)}(\tilde{E})}{\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(\tilde{E})}} d\tilde{E} - \int_{\tilde{E}_0}^{E} \frac{a^{(l_3^e,l_2^e,l_1^e,l_0^e)}(\tilde{E})}{\sqrt{-Q^{(l_3^e,l_2^e,l_1^e,l_0^e)}(\tilde{E})}} d\tilde{E}, \\
&\int_{E_0}^{E} \frac{c^{(l_0,l_1,l_2,l_3)}(\tilde{E})}{\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(\tilde{E})}} d\tilde{E} - \int_{\tilde{E}_0}^{E} \frac{c^{(l_3^e,l_2^e,l_1^e,l_0^e)}(\tilde{E})}{\sqrt{-Q^{(l_3^e,l_2^e,l_1^e,l_0^e)}(\tilde{E})}} d\tilde{E},
\end{aligned}$$

are constants. By differentiating these integrals in the variable E and using the relation $Q^{(l_0,l_1,l_2,l_3)}(E) = Q^{(l^e_3,l^e_2,l^e_1,l^e_0)}(E)$, we have $a^{(l_0,l_1,l_2,l_3)}(E) = a^{(l^e_3,l^e_2,l^e_1,l^e_0)}(E)$ and $c^{(l_0,l_1,l_2,l_3)}(E) = c^{(l^e_3,l^e_2,l^e_1,l^e_0)}(E)$. ■

It follows from Proposition 7.3 that the hyperelliptic-to-elliptic integral formulae obtained in [10] for the parameters (l_0,l_1,l_2,l_3) coincide with those for the parameters $(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)$.

Let us consider the case $l_0 = 2$, $l_1 = l_2 = l_3 = 0$. For this case the operator $H^{(2,0,0,0)}$ is simply the operator H_1 in Eq.(1.1). We have $l^e_3 = l^e_2 = l^e_1 = 1$ and $l^e_0 = -1$ (see Eq.(4.6)), and the operator $H^{(1,1,1,0)}(= H^{(1,1,1,-1)})$ coincides with the operator H_2 in Eq.(1.2). It follows from Proposition 3.6 that

$$H^{(1,1,1,0)}L_{-2,1,1,0} = L_{-2,1,1,0}H^{(2,0,0,0)},$$

and $L_{-2,1,1,0}$ is given by

$$L_{-2,1,1,0} = \frac{d}{dx} - \frac{\wp'(x)}{2(\wp(x)-e_1)} - \frac{\wp'(x)}{2(\wp(x)-e_2)}.$$

Hence the operator $H_1(= H^{(2,0,0,0)})$ is connected to the operator $H_2(= H^{(1,1,1,0)})$ by a Darboux transformation, and we recover the isomonodromic property of these operators.

The functions $\Xi^{(2,0,0,0)}(x,E)$ and $\Xi^{(1,1,1,0)}(x,E)$ defined in Eq.(6.12) are given by

$$\begin{aligned}\Xi^{(2,0,0,0)}(x,E) =& 9\wp(x)^2 + 3E\wp(x) + E^2 - 9g_2/4,\\ \Xi^{(1,1,1,0)}(x,E) =& (E-3e_3)\wp(x) + (E-3e_2)\wp(x+\omega_1)\\ &+ (E-3e_1)\wp(x+\omega_2) + E^2 - 3g_2/2,\end{aligned}$$

while the functions $Q^{(2,0,0,0)}(x,E)$ and $Q^{(1,1,1,0)}(x,E)$ are given by

$$Q^{(2,0,0,0)}(E) = Q^{(1,1,1,0)}(E) = (E^2-3g_2)(E^3 - 9g_2E/4 - 27g_3/4).$$

The function $\Lambda^{(2,0,0,0)}(x,E)$ (resp. $\Lambda^{(1,1,1,0)}(x,E)$) defined by Eq.(7.1) is an eigenfunction of $H^{(2,0,0,0)}$ (resp. $H^{(1,1,1,0)}$) corresponding to the eigenvalue E. With reference to their expressions however, $\Lambda^{(2,0,0,0)}(x,E)$ and $\Lambda^{(1,1,1,0)}(x,E)$ are different functions. Note that the function $\Lambda^{(2,0,0,0)}(x,E)$ (resp. $\Lambda^{(1,1,1,0)}(x,E)$) corresponds to the function $\Lambda_1(x,E)$ (resp. $\Lambda_2(x,E)$) in the introduction. Although the functions $\Lambda^{(2,0,0,0)}(x,E)$ and $\Lambda^{(1,1,1,0)}(x,E)$ differ, we have from Proposition 7.3 that several other functions do coincide. In particular we have

$$\begin{aligned}&R_1^{(2,0,0,0)}(E) = R_1^{(1,1,1,0)}(E) = \frac{-(E^3-27g_3)}{9(E^2-3g_2)},\\ &R_2^{(2,0,0,0)}(E) = R_2^{(1,1,1,0)}(E) = \frac{2(E^3-9g_2E+54g_3)}{27(E^2-3g_2)^2},\\ &R_3^{(2,0,0,0)}(E) = R_3^{(1,1,1,0)}(E) = \frac{2}{3(E^2-3g_2)},\\ &a^{(2,0,0,0)}(E) = a^{(1,1,1,0)}(E) = 3E,\\ &c^{(2,0,0,0)}(E) = c^{(1,1,1,0)}(E) = E^2 - \frac{3}{2}g_2.\end{aligned}$$

Note that these functions are related with monodromy by Eqs.(7.3, 7.4). Upon applying Proposition 7.2, we recover Eqs.(1.3, 1.4).

8 Commuting operator for Lamé operator

We obtain two expressions of the commuting operator A for the Lamé operator $H^{(g,0,0,0)}$ by using the generalized Darboux transformation.

It follows from Proposition 6.1 and Theorem 6.9 (i) that the commuting operator A for the Lamé operator $H^{(g,0,0,0)}$ can be expressed in the form

$$A=\begin{cases} L_{1+\frac{g}{2},-\frac{g}{2},-\frac{g}{2},1-\frac{g}{2}}L_{1,-g,1,0}L_{1-\frac{g}{2},1+\frac{g}{2},-\frac{g}{2},-\frac{g}{2}}L_{-g,0,0,0} & (g:\ \text{even}); \\ -L_{\frac{1+g}{2},\frac{1-g}{2},-\frac{1+g}{2},\frac{1-g}{2}}L_{0,-g,1,0}L_{-\frac{1+g}{2},\frac{1+g}{2},\frac{1-g}{2},\frac{1-g}{2}}L_{-g,1,1,1} & (g:\ \text{odd}), \end{cases} \tag{8.1}$$

(see Eq.(3.2)), and the genus of the spectral curve for the Lamé operator $H^{(g,0,0,0)}$ is g.

To obtain another expression of A for $H^{(g,0,0,0)}$, we calculate the commuting operator A explicitly for the case $l_0=l_1=l_2=l_3=g\in\mathbb{Z}_{\geq 1}$. With this approach, the commuting operator A of $H^{(g,g,g,g)}$ can be expressed in the form

$$A=(-1)^g L_{-g,-g,-g,-g}=(-1)^g\wp'(x)^{2g+1}\widehat{\Phi}(\wp(x))\circ\left(\frac{1}{\wp'(x)}\frac{d}{dx}\right)^{2g+1}\circ\widehat{\Phi}(\wp(x))^{-1},$$

in which $\widehat{\Phi}(\wp(x))=((\wp(x)-e_1)^{-g/2}(\wp(x)-e_2)^{-g/2}(\wp(x)-e_3))^{-g/2}=(\wp'(x)/2)^{-g}$. Hence we have

$$A=(-1)^g\wp'(x)^{g+1}\circ\left(\frac{1}{\wp'(x)}\frac{d}{dx}\right)^{2g+1}\circ\wp'(x)^g. \tag{8.2}$$

Thus, applying the transformation $2x\to x$ and the relation $\sum_{i=0}^{3}\wp(x+\omega_i)=4\wp(2x)$, we recover the Lamé operator $4H^{(g,0,0,0)}(=4(-d^2/dx^2+g(g+1)\wp(x)))$ together with its commuting operator. The commuting operator A of $H^{(g,0,0,0)}$ is given by

$$A=\frac{1}{2}\left(\frac{-1}{4}\right)^g\wp'\left(\frac{x}{2}\right)^{g+1}\circ\left(\frac{2}{\wp'\left(\frac{x}{2}\right)}\frac{d}{dx}\right)^{2g+1}\circ\wp'\left(\frac{x}{2}\right)^g. \tag{8.3}$$

We therefore obtain the following proposition:

Proposition 8.1. The commuting operator A for the Lamé operator $H^{(g,0,0,0)}$ $(g\in\mathbb{Z}_{\geq 1})$ has two expressions given by Eq.(8.1) and Eq.(8.3).

We consider the case $g=1$. For this case, the operator defined by Eq.(8.1) is given by

$$\begin{aligned}&-L_{1,0,-1,0}L_{0,-1,1,0}L_{-1,1,0,0}=\\&-\left(\frac{d}{dx}+\frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_2}\right)\left(\frac{d}{dx}+\frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_1}-\frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_2}\right)\left(\frac{d}{dx}-\frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_1}\right),\end{aligned}$$

while the operator defined by Eq.(8.3) is written as

$$-\frac{1}{8}\wp'\left(\frac{x}{2}\right)^2\circ\left(\frac{2}{\wp'\left(\frac{x}{2}\right)}\frac{d}{dx}\right)^3\circ\wp'\left(\frac{x}{2}\right).$$

These operators can both be rewritten in the form

$$-\left\{ \left(\frac{d}{dx}\right)^3 - 3\wp(x)\frac{d}{dx} - \frac{3}{2}\wp'(x) \right\},$$

which coincides with the operator A defined by Eq.(6.15).

Next we consider the case $g = 2$. For this case, the operator defined by Eq.(8.1) is given by

$$\begin{aligned}
&L_{2,-1,-1,0}L_{1,-2,1,0}L_{0,2,-1,-1}L_{-2,0,0,0} = \\
&\left(\frac{d}{dx} + \frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_1} + \frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_2}\right)\left(\frac{d}{dx} + \frac{\wp'(x)}{\wp(x)-e_1} - \frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_2}\right) \\
&\left(\frac{d}{dx} - \frac{\wp'(x)}{\wp(x)-e_1} + \frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_2} + \frac{1}{2}\frac{\wp'(x)}{\wp(x)-e_3}\right) \\
&\left(\left(\frac{d}{dx}\right)^2 - \frac{1}{2}\left(\frac{\wp'(x)}{\wp(x)-e_1} + \frac{\wp'(x)}{\wp(x)-e_2} + \frac{\wp'(x)}{\wp(x)-e_3}\right)\frac{d}{dx}\right)
\end{aligned}$$

while the operator defined by Eq.(8.3) is given by

$$\frac{1}{32}\wp'\left(\frac{x}{2}\right)^3 \circ \left(\frac{2}{\wp'\left(\frac{x}{2}\right)}\frac{d}{dx}\right)^5 \circ \wp'\left(\frac{x}{2}\right)^2.$$

These operators can both be rewritten in the form

$$\left(\frac{d}{dx}\right)^5 - 15\wp(x)\left(\frac{d}{dx}\right)^3 - \frac{45}{2}\wp'(x)\left(\frac{d}{dx}\right)^2 - 9\left(5\wp(x)^2 - \frac{3}{4}g_2\right)\frac{d}{dx},$$

which coincides with the operator A defined by Eq.(6.15).

A Elliptic functions

This appendix presents the definitions of and the formulae for the elliptic functions. The Weierstrass $\wp$-function is defined by

$$\wp(x) = \frac{1}{x^2} + \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}}\left(\frac{1}{(x-2m\omega_1-2n\omega_3)^2} - \frac{1}{(2m\omega_1+2n\omega_3)^2}\right). \tag{A.1}$$

Setting $\omega_2 = -\omega_1 - \omega_3$ and $e_k = \wp(\omega_k)$ $(k = 1, 2, 3)$ yields the relations

$$\begin{aligned}
&e_1 + e_2 + e_3 = 0, \quad \wp'(x)^2 = 4(\wp(x)-e_1)(\wp(x)-e_2)(\wp(x)-e_3), \\
&\frac{\wp''(x)}{\wp'(x)^2} = \frac{1}{2}\left(\frac{1}{\wp(x)-e_1} + \frac{1}{\wp(x)-e_2} + \frac{1}{\wp(x)-e_3}\right), \\
&\wp(x+\omega_i) = e_i + \frac{(e_i-e_{i'})(e_i-e_{i''})}{\wp(x)-e_i} \quad (i = 1, 2, 3),
\end{aligned} \tag{A.2}$$

where $i', i'' \in \{1, 2, 3\}$ with $i' < i''$, $i \neq i'$ and $i \neq i''$.

B Proofs of Propositions 3.4 (ii), 3.5 and 6.6

Let α_i be a number such that $\alpha_i = -l_i$ or $\alpha_i = l_i + 1$ for all $i \in \{0, 1, 2, 3\}$. Set $\mathbf{a} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and

$$\begin{aligned}
v_r^{\mathbf{a}} &= (\wp(x) - e_1)^{\alpha_1/2}(\wp(x) - e_3)^{\alpha_3/2}(\wp(x) - e_2)^{\alpha_2/2+r},\\
a_{r+1,r}^{\mathbf{a}} &= -4(r + \gamma_1^{\mathbf{a}})(r + \gamma_2^{\mathbf{a}}),\\
a_{r-1,r}^{\mathbf{a}} &= -4r(r + \alpha_2 - 1/2)(e_2 - e_3)(e_2 - e_1),\\
a_{r,r}^{\mathbf{a}} &= -4r((e_2 - e_3)(r + \alpha_2 + \alpha_1) + (e_2 - e_1)(r + \alpha_2 + \alpha_3))\\
&\quad - 4e_2\gamma_1^{\mathbf{a}}\gamma_2^{\mathbf{a}} + e_1(\alpha_2 + \alpha_3)^2 + e_2(\alpha_1 + \alpha_3)^2 + e_3(\alpha_1 + \alpha_2)^2,\\
\gamma_1^{\mathbf{a}} &= (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)/2, \quad \gamma_2^{\mathbf{a}} = (-\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + 1)/2.
\end{aligned}$$

Then the action of the operator $H^{(l_0,l_1,l_2,l_3)}$ is given by

$$H^{(l_0,l_1,l_2,l_3)}v_r^{\mathbf{a}} = a_{r+1,r}^{\mathbf{a}}v_{r+1}^{\mathbf{a}} + a_{r,r}^{\mathbf{a}}v_r^{\mathbf{a}} + a_{r-1,r}^{\mathbf{a}}v_{r-1}^{\mathbf{a}}, \tag{B.1}$$

(see [8]). Set $d = -\sum_{i=0}^{3}\alpha_i/2$ and suppose that $d \in \mathbb{Z}_{\geq 0}$. Then the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is spanned by $v_0^{\mathbf{a}}, v_1^{\mathbf{a}}, \dots, v_d^{\mathbf{a}}$, and it follows from Eq.(B.1) that $H^{(l_0,l_1,l_2,l_3)}$ preserves the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$. The operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ also preserves the $d+1$-dimensional space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$.

Proposition B.1. (Proposition 3.4 (ii)) Set $d = -\sum_{i=0}^{3}\alpha_i/2$. If $d \in \mathbb{Z}_{\geq 0}$, then the characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ coincides with that of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$.

Proof. Set $-\mathbf{a} - \mathbf{d} = (-\alpha_0 - d, -\alpha_1 - d, -\alpha_2 - d, -\alpha_3 - d)$. The action of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$ is given by

$$H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}v_r^{-\mathbf{a}-\mathbf{d}} = a_{r+1,r}^{-\mathbf{a}-\mathbf{d}}v_{r+1}^{-\mathbf{a}-\mathbf{d}} + a_{r,r}^{-\mathbf{a}-\mathbf{d}}v_r^{-\mathbf{a}-\mathbf{d}} + a_{r-1,r}^{-\mathbf{a}-\mathbf{d}}v_{r-1}^{-\mathbf{a}-\mathbf{d}}.$$

Set $u_r^{\mathbf{a}} = ((-1)^r d!/(r!(d-r)!))v_{d-r}^{\mathbf{a}}$. By a direct calculation, Eq.(B.1) can be rewritten in the form

$$H^{(l_0,l_1,l_2,l_3)}u_r^{\mathbf{a}} = a_{r-1,r}^{-\mathbf{a}-\mathbf{d}}u_{r+1}^{\mathbf{a}} + a_{r,r}^{-\mathbf{a}-\mathbf{d}}u_r^{\mathbf{a}} + a_{r+1,r}^{-\mathbf{a}-\mathbf{d}}u_{r-1}^{\mathbf{a}}.$$

Hence the matrix representation of $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the basis $\langle v_0^{-\mathbf{a}-\mathbf{d}}, \dots, v_d^{-\mathbf{a}-\mathbf{d}}\rangle$ is the transposition of the matrix representing $H^{(l_0,l_1,l_2,l_3)}$ on the basis $\langle u_0^{\mathbf{a}}, \dots, u_d^{\mathbf{a}}\rangle$, and we obtain that the characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ coincides with that of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$. ■

Proposition B.2. (Proposition 3.5) Set $d = -\sum_{i=0}^{3}\alpha_i/2$. If $d \in \mathbb{Z}_{\geq 0}$ and $\alpha_i \neq \alpha_j$ for some $i, j \in \{0, 1, 2, 3\}$, then zeros of the characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ are distinct for generic periods $(2\omega_1, 2\omega_3)$.

Proof. The assumption $\alpha_i \neq \alpha_j$ for some $i, j \in \{0, 1, 2, 3\}$ implies that $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ do not satisfy $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$, and it is easy to show that there exist $i_0, i_1, i_2 \in \{0, 1, 2, 3\}$ such that $i_1 \neq i_2$, $\alpha_{i_0} \neq \alpha_{i_1}$ and $\alpha_{i_0} \neq \alpha_{i_2}$. By permutation of periods $\omega_1, \omega_2, \omega_3$ and shift transformations $x \to x + \omega_i$ $(i = 1, 2, 3)$, we can permute numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. Hence it is sufficient to prove the proposition under the assumption $\alpha_3 \neq \alpha_1$ and $\alpha_3 \neq \alpha_2$.

Set $\omega_1 = 1/2$, $\omega_3 = \tau/2$ and $p = \exp(\pi\sqrt{-1}\tau)$. Then e_1, e_2 and e_3 are expressed as power series in p and we have $e_1 = \pi^2(2/3 + O(p^2))$, $e_2 = \pi^2(-1/3 + 8p + O(p^2))$ and $e_3 = \pi^2(-1/3 - 8p + O(p^2))$. The matrix elements are expressed as

$$
\begin{aligned}
& a^{\mathbf{a}}_{r+1,r} = \tilde{a}^{(0)}_{r+1,r}, \quad a^{\mathbf{a}}_{r-1,r} = \tilde{a}^{(1)}_{r-1,r}(p + O(p^2)), \quad a^{\mathbf{a}}_{r,r} = \tilde{a}^{(0)}_{r,r} + \tilde{a}^{(1)}_{r,r}p + O(p^2), \\
& \tilde{a}^{(0)}_{r+1,r} = -4(r + \gamma^{\mathbf{a}}_1)(r + \gamma^{\mathbf{a}}_2), \quad \tilde{a}^{(1)}_{r-1,r} = 64r(r + \alpha_2 - 1/2), \\
& \tilde{a}^{(0)}_{r,r} = (2r + \alpha_2 + \alpha_3)^2 - \textstyle\sum_{i=0}^{3} l_i(l_i + 1)/3, \\
& \tilde{a}^{(1)}_{r,r} = -8\{r(12r + 8\alpha_1 + 12\alpha_2 + 4\alpha_3) + 4\gamma^{\mathbf{a}}_1\gamma^{\mathbf{a}}_2 - (\alpha_1 + \alpha_3)^2 + (\alpha_1 + \alpha_2)^2\}.
\end{aligned}
$$

If $p = 0$, then the operator $H^{(l_0,l_1,l_2,l_3)}$ acts triangularly and eigenvalues are given by $\tilde{a}^{(0)}_{r,r}$ $(r = 0, \dots, d)$. Since the eigenvalues are quadratic in r, the multiplicity of the eigenvalues is one or two. Hence it is sufficient to show that the eigenvalue with multiplicity two for the case $p = 0$ separates when p varies. Assume that $\tilde{a}^{(0)}_{r,r} (= \tilde{a}^{(0)}_{r',r'})$ is the eigenvalue with multiplicity two, $0 \leq r < r' \leq d$ and $r, r' \in \mathbb{Z}$. Then we have $r + r' = -(\alpha_2 + \alpha_3)$.

If $r + 1 < r'$, then the eigenvalue around $\tilde{a}^{(0)}_{r,r}$ is expanded as $E = \tilde{a}^{(0)}_{r,r} + c_1 p + \dots$, and c_1 satisfies

$$
\begin{aligned}
& \{(\tilde{a}^{(0)}_{r-1,r-1} - \tilde{a}^{(0)}_{r,r})(\tilde{a}^{(1)}_{r,r} - c_1)(\tilde{a}^{(0)}_{r+1,r+1} - \tilde{a}^{(0)}_{r,r}) - (\tilde{a}^{(0)}_{r-1,r-1} - \tilde{a}^{(0)}_{r,r})\tilde{a}^{(0)}_{r+1,r}\tilde{a}^{(1)}_{r,r+1} \\
& - \tilde{a}^{(0)}_{r,r-1}\tilde{a}^{(1)}_{r-1,r}(\tilde{a}^{(0)}_{r+1,r+1} - \tilde{a}^{(0)}_{r,r})\}\{(\tilde{a}^{(0)}_{r'-1,r'-1} - \tilde{a}^{(0)}_{r',r'})(\tilde{a}^{(1)}_{r',r'} - c_1)(\tilde{a}^{(0)}_{r'+1,r'+1} - \tilde{a}^{(0)}_{r',r'}) \\
& - (\tilde{a}^{(0)}_{r'-1,r'-1} - \tilde{a}^{(0)}_{r',r'})\tilde{a}^{(0)}_{r'+1,r'}\tilde{a}^{(1)}_{r',r'+1} - \tilde{a}^{(0)}_{r',r'-1}\tilde{a}^{(1)}_{r'-1,r'}(\tilde{a}^{(0)}_{r'+1,r'+1} - \tilde{a}^{(0)}_{r',r'})\} = 0,
\end{aligned}
\tag{B.2}
$$

which follows from expanding the characteristic polynomial of the matrix $\left(a^{\mathbf{a}}_{i,j}\right)_{i,j=0,\dots,d}$ in variable p and observing the coefficient of p^2. By a direct calculation, the condition that Eq.(B.2) for the variable c_1 has multiple roots is given by $(\alpha_3 - \alpha_1)(2r + \alpha_2 + \alpha_3) = 0$. If $2r + \alpha_2 + \alpha_3 = 0$, then $r = r'$ but this contradicts the relation $r < r'$. By the assumption $\alpha_3 \neq \alpha_1$, it follows that Eq.(B.2) for the variable c_1 does not have multiple roots, and the solution $E = \tilde{a}^{(0)}_{r,r} + c_1 p + \dots$ separates.

If $r + 1 = r'$, then $r = -(\alpha_2 + \alpha_3 + 1)/2$, the eigenvalue around $\tilde{a}^{(0)}_{r,r}$ is expanded as $E = \tilde{a}^{(0)}_{r,r} + c_{1/2}\sqrt{p} + \dots$, and $c_{1/2}$ is determined by

$$
c^2_{1/2} = \tilde{a}^{(0)}_{r+1,r}\tilde{a}^{(1)}_{r,r+1} = -16(\alpha_0 - \alpha_1)(\alpha_2 - \alpha_3)(\alpha_0 + \alpha_1 - 1)(\alpha_2 + \alpha_3 - 1). \tag{B.3}
$$

Since $0 \leq r + r' = -(\alpha_2 + \alpha_3) \leq 2d = -(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)$, we have $(1 - (\alpha_0 + \alpha_1))(1 - (\alpha_2 + \alpha_3)) > 0$. Combining with $\alpha_2 \neq \alpha_3$, it follows that, if $\alpha_0 \neq \alpha_1$, then Eq.(B.3) for the variable $c_{1/2}$ does not have multiple roots, and the solution $E = \tilde{a}^{(0)}_{r,r} + c_{1/2}\sqrt{p} + \dots$ separates. If $\alpha_0 = \alpha_1$, then we have $a^{\mathbf{a}}_{r+1,r} = \tilde{a}^{(0)}_{r+1,r} = 0$ for $r = -(\alpha_2 + \alpha_3 + 1)/2$. The eigenvalue around $\tilde{a}^{(0)}_{r,r}$ is expanded as $E = \tilde{a}^{(0)}_{r,r} + c_1 p + \dots$, and c_1 is determined by

$$
\begin{aligned}
& \{(\tilde{a}^{(0)}_{r-1,r-1} - \tilde{a}^{(0)}_{r,r})(\tilde{a}^{(1)}_{r,r} - c_1) - \tilde{a}^{(0)}_{r,r-1}\tilde{a}^{(1)}_{r-1,r}\} \cdot \\
& \quad \{(\tilde{a}^{(0)}_{r+2,r+2} - \tilde{a}^{(0)}_{r+1,r+1})(\tilde{a}^{(1)}_{r+1,r+1} - c_1) - \tilde{a}^{(0)}_{r+2,r+1}\tilde{a}^{(1)}_{r+1,r+2}\} = 0.
\end{aligned}
\tag{B.4}
$$

The condition that Eq.(B.4) for the variable c_1 has multiple roots is written as $(\alpha_2 - \alpha_3)(\alpha_2+\alpha_3-1)(2\alpha_0-1) = 0$. However, this is impossible since $(1-(\alpha_2+\alpha_3))(1-2\alpha_0) > 0$ and $\alpha_3 \neq \alpha_2$. Hence, if $\alpha_0 = \alpha_1$, then Eq.(B.4) for the variable c_1 does not have multiple roots, and the solution $E = \tilde{a}^{(0)}_{r,r} + c_1 p + \dots$ separates.

Thus we have found that the multiple root $E = \tilde{a}^{(0)}_{r,r}$ at $p = 0$ separates by expanding the eigenvalue as a series in p or $\sqrt{p}$.

The zeros of the characteristic polynomial equation are therefore distinct for generic periods $(2\omega_1, 2\omega_3)$. ■

Corollary B.3. (Proposition 6.6) Let l_0, l_1, l_2, l_3 be non-negative integers and V be the vector space expressed in the form of Eq.(4.4) ($l_0 + l_1 + l_2 + l_3$: even) or Eq.(4.9) ($l_0 + l_1 + l_2 + l_3$: odd). We denote the monic characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space V by $P(E)$. Then the roots of the equation $P(E) = 0$ are distinct for generic periods $(2\omega_1, 2\omega_3)$.

Proof. It is shown in the proof of [7, Theorem 3.2] (or [11, Proposition 3.9]) that any two characteristic polynomials of distinct subspaces in Eq.(4.3) or Eq.(4.8) do not have common roots. Hence it is sufficient to show that the characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on any space listed in Eq.(4.3) or Eq.(4.8) does not have multiple zeros for generic periods $(2\omega_1, 2\omega_3)$. From Proposition B.2 we have that if $\alpha_i \neq \alpha_j$ for some $i, j \in \{0, 1, 2, 3\}$, then the characteristic polynomial does not have multiple zeros for generic periods. In Eqs.(4.3, 4.8), the case $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$ for $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ appears only for the case $l_0 = l_1 = l_2 = l_3$ and $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = -l_0$. We set $g = l_0 (= l_1 = l_2 = l_3)$.

For the case $l_0 = l_1 = l_2 = l_3 = g$, the operator $H^{(g,g,g,g)}$ is expressed in the form

$$H^{(g,g,g,g)} = -\frac{d^2}{dx^2} + 4g(g+1)\wp(2x),$$

and the finite-dimensional space $V (= V_{-g,-g,-g,-g})$ for the case $l_0 = l_1 = l_2 = l_3 = g$ coincides with the space V for the case $l_0 = g$ and $l_1 = l_2 = l_3 = 0$ by replacing basic periods $(2\omega_1, 2\omega_3) \to (\omega_1, \omega_3)$. For the case $l_0 \neq 0$ and $l_1 = l_2 = l_3 = 0$, the corresponding proposition is proved in Proposition B.2 or [15, §23.4]. Thus Corollary B.3 is proved. ■

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