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Distribution of positive type in Quantum Calculus

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Abstract

In this paper, we study some remarkable spaces of $S'_{q,*}(\mathbb{R}_{q,+})$ space of the q-tempered distribution introduced by M.A. Olshanetsky and V.B.K. Rogov [14], namely the q-analogue of the pseudo-measure $\mathcal{F}_q L^\infty(\mathbb{R}_{q,+})$, the q-function of the positive type $\mathcal{F}_q\mathcal{M}'$, and we give a q-version of the Bochner-Shwartz theorem related to q-cosine Fourier transform.

1 Preliminaries

To make this paper self containing we begin by recalling some notions used in Quantum Calculus. For deep study the reader is invited to consult the Gasper-Rahman book [6] and the references joint with this work. We will assume $0 < q < 1$ and we will use the same notation in [12].
A q-shifted factorial is defined by

$$(a;q)_0 = 1 \qquad ,(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad ;n = 1,2,\cdots,\infty. \tag{1.1}$$

And more generally:

$$(a_1,\cdots,a_r;q)_n = \prod_{k=1}^{r}(a_k;q)_n. \tag{1.2}$$

The basic hypergeometric series or q-hypergeometric series is given for r , s integers by

$$_r\varphi_s(a_1,\cdots,a_r;b_1,\cdots,b_s;q,x) = \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(b_1,\cdots,b_s;q)_n(q,q)_n}[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r}x^n$$

The q-derivative $D_{q,x}f$ of a function f on an open interval is given by :

$$D_{q,x}f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0 \tag{1.3}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exist. The q-shift operators are

$$(\Lambda_{q,x}f)(x) = f(qx) \tag{1.4}$$
$$(\Lambda_{q,x}^{-1}f)(x) = f(q^{-1}x). \tag{1.5}$$

We consider the q-operator

$$\Delta_{q,x} = \Lambda_{q,x}^{-1} D_{q,x}^2. \tag{1.6}$$

The q-Jackson integral from 0 to a and to ∞ are respectively defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{1.7}$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{-\infty}^{+\infty} f(q^n)q^n. \tag{1.8}$$

The q-analogue of the elementary exponential functions are crucial, they are defined by :

$$E(x;q) = (-(1-q)x;q)_\infty = \sum_0^\infty q^{\frac{n(n-1)}{2}} \frac{(1-q)^n}{(q;q)_n} x^n \quad , x \in \mathbb{R}, \tag{1.9}$$

and

$$e(x;q) = \frac{1}{((1-q)x;q^2)_\infty} = \sum_0^\infty \frac{(1-q)^n}{(q;q)_n} x^n \quad , \mid x \mid < \frac{1}{1-q}. \tag{1.10}$$

Because of its product representation, $e(x;q^2)$ has an analytic continuation to $\mathbb{R}\backslash\{\frac{1}{1-q^2}q^{-k}, k \in \mathbb{N}\}$. Further these functions satisfy the identity :

$$e(x;q)E(-x;q) = 1 \quad . \tag{1.11}$$

Some q-functional spaces will be used in the remainder. We begin by putting

$$\mathbb{R}_{q,+} = \{+q^k, k \in \mathbb{Z}\}. \tag{1.12}$$

$$\widehat{\mathbb{R}}_{q,+} = \{+q^k, k \in \mathbb{Z}\} \cup \{0\}. \tag{1.13}$$

and we denote by

• $\mathcal{S}_{q,*}(\mathbb{R}_{q,+})$ the q-analogue of Schwartz space of even functions defined on $\mathbb{R}_{q,+}$ such that $D_{q,x}^k f(x)$ is continuous in 0 for all $k \in \mathbb{N}$ and

$$N_{q,n,k}(f) = \sup_{x \in \mathbb{R}_{q,+}} \mid (1+x^2)^n D_{q,x}^k f(x) \mid < +\infty \tag{1.14}$$

• $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ the space of even functions infinitely q-differentiable on $\mathbb{R}_{q,+}$ with compact support in $\mathbb{R}_{q,+}$. We equip this space with the topology of the uniform convergence of the functions and their q-derivatives.

• $\mathcal{C}_{q,*,0}(\mathbb{R}_{q,+})$ the space of even functions f defined on $\mathbb{R}_{q,+}$ continuous on 0, infinitely q-differentiable and

$$\lim_{x\to\infty} f(x) = 0 \quad , \parallel f \parallel_{\mathcal{C}_{q,*,0}} = \sup_{x \in \mathbb{R}_{q,+}} \mid f(x) \mid < +\infty. \tag{1.15}$$

• $\mathcal{H}_{q,*}(\mathbb{R}_{q,+})$ the space of even functions f defined on $\mathbb{R}_{q,+}$ continuous on 0 with compact support such that

$$\parallel f \parallel_{\mathcal{H}_{q,*}} = \sup_{x \in \mathbb{R}_{q,+}} \mid f(x) \mid \quad < +\infty. \tag{1.16}$$

- $L_q^p(\mathbb{R}_{q,+})$, $p \in [1, +\infty[$, (resp $L_q^\infty(\mathbb{R}_{q,+})$)be the space of functions f such that,

$$\| f \|_{q,p} = \left(\int_0^\infty | f(x) |^p \, d_q x\right)^{\frac{1}{p}} < +\infty. \tag{1.17}$$

(resp

$$\| f \|_{\infty,q} = ess \sup_{x\in\mathbb{R}_{q,+}} | f(x) | < +\infty \quad .) \tag{1.18}$$

Jackson in [10] defined the q-analogue of the Gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad , 0 < q < 1; x \neq 0, -1, -2, ... \tag{1.19}$$

moreover the q-duplication formula holds

$$\Gamma_q(2x)\Gamma_{q^2}(\frac{1}{2}) = (1+q)^{2x-1}\Gamma_q^2(x)\Gamma_{q^2}(x+\frac{1}{2}). \tag{1.20}$$

We take the definition of q-trigonometric given by T.H.Koornwinder and R.F.Swarttouw (see [12]) with simple changes and we write q-cosine and q-sinus as a series of functions

$$\cos(x;q^2) = {}_1\varphi_1(0, q, q^2; (1-q)^2x^2) = \sum_{n=0}^\infty (-1)^n b_n(x;q^2) \tag{1.21}$$

$$\sin(x;q^2) = (1-q)x {}_1\varphi_1(0, q^3, q^2; (1-q)^2x^2) = \sum_{n=0}^\infty (-1)^n c_n(x;q^2) \tag{1.22}$$

where we have put

$$b_n(x;q^2) = b_n(1;q^2)x^{2n} = q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}x^{2n} \tag{1.23}$$

$$c_n(x;q^2) = c_n(1;q^2)x^{2n+1} = q^{n(n-1)}\frac{(1-q)^{2n+1}}{(q;q)_{2n+1}}x^{2n+1}. \tag{1.24}$$

The reader will notice that the previous definition (1.21) derived from those given in [12] with minor change, and we have

$$\lim_{x\longrightarrow+\infty} \left\{ \begin{array}{l} \cos(x;q^2) \\ \\ \sin(x;q^2) \end{array} \right. = 0 \quad . \tag{1.25}$$

These functions are bounded and for every $x \in \mathbb{R}_q$ we have

$$|\cos(x;q^2)| \leq \frac{1}{(q;q^2)_\infty^2}, \tag{1.26}$$

$$|\sin(x;q^2) \leq \frac{1}{(q;q^2)_\infty^2}. \tag{1.27}$$

More generally in [5], the q-Bessel function is written as

$$j_\alpha(x;q^2) = \sum_{n=0}^\infty (-1)^n b_{n,\alpha}(x,q^2) \tag{1.28}$$

with

$$b_{n,\alpha}(x,q^2) = b_{n,\alpha}(1,q^2)x^{2n} = \frac{\Gamma_{q^2}(\alpha+1)q^{n(n-1)}}{(1+q)^{2n}\Gamma_{q^2}(n+1)\Gamma_{q^2}(\alpha+n+1)}x^{2n} \quad , \tag{1.29}$$

$$j_\alpha(x;q^2) = \Gamma_{q^2}(\alpha+1)\frac{q^\alpha(1+q)^\alpha}{x^\alpha}J_\alpha((1-q)x;q^2) \tag{1.30}$$

where $J_\alpha(x;q^2)$ is the q-Bessel Han Exton [16], defined by

$$J_\alpha(x;q) = \left(\frac{x}{1-q}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} q^k}{\Gamma_q(k+1)\Gamma_q(\alpha+k+1)} \left(\frac{x}{1-q}\right)^{2k}. \tag{1.31}$$

and

$$b_{n,-\frac{1}{2}}(x;q^2) = b_n(x;q^2). \tag{1.32}$$

The q-j_α Bessel function $j_\alpha(x;q^2)$ is defined on $\mathbb{R}$ and tends to the j_α Bessel function as $q \longrightarrow 1^-$.
By simple computation using (1.19) and (1.20) we obtain

$$j_{-\frac{1}{2}}(x;q^2) = \cos(x;q^2), \tag{1.33}$$

$$j_{\frac{1}{2}}(x;q^2) = \frac{\sin(x;q^2)}{x}. \tag{1.34}$$

Finally, let f be a function in $L_q^1(\mathbb{R}_{q,+})$, the q-even translation operator $T_{q,x}$ is defined (see [4]) by

$$T_{q,x}f(y) = \int_0^\infty f(t)d_q\mu_{x,y}(t) \quad , \tag{1.35}$$

where $d_q\mu(t)$ is the measure defined for x and y in $\mathbb{R}_{q,+}$ by

$$d_q\mu_{x,y}(t) = \sum_{-\infty}^{+\infty}(\frac{x}{y})^{2s}\frac{(q(\frac{x}{y})^2;q)_\infty}{(q;q)_\infty}{}_1\phi_1(0,q(\frac{x}{y})^2,q;q^{1+2s})q^s\delta_{yq^s}(t) \tag{1.36}$$

and δ_u is the mass unit supported at u.

Note that in [4], for f in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ the authors proved that the q-even translation $T_{q,x}$ can be written in the following form

$$T_{q,x}f(y) = \sum_{k=0}^{\infty} q^k(\frac{x}{y})^{2k}\sum_{s=-k}^{s=k}\frac{(-1)^{k-s}q^{(k-s)(k-s-1)/2}}{(q;q)_{k+s}(q;q)_{k-s}}f(q^s y), \quad y \neq 0 \tag{1.37}$$

and also written as the form

$$T_{q,x}f(x) = \sum_{n=0}^{\infty} b_n(x;q^2)\Delta_{q,x}^n f(x), \tag{1.38}$$

where $\Delta_{q,x}$ given by (1.6).

Furthermore for f and g be two functions in $L^1_q(\mathbb{R}_{q,+})$, we have

$$\int_0^\infty T_{q,x}f(y)d_qy = \int_0^\infty f(y)d_qy, \tag{1.39}$$

$$\int_0^\infty T_{q,x}f(y)g(y)d_qy = \int_0^\infty f(y)T_{q,x}g(y)d_qy, \tag{1.40}$$

in particular the following product formula holies

$$T_{q,y}\cos(tx;q^2) = \cos(tx;q^2)\cos(ty;q^2). \tag{1.41}$$

The q-convolution and the q-cosine Fourier transform studied and given in [4], for $f, g \in L^1_q(\mathbb{R}_{q,+})$ by:

$$f *_q g(x) = \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}\int_0^\infty T_{q,x}f(y)g(y)d_qy, \tag{1.42}$$

$$\mathcal{F}_q(f)(\lambda) = \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}\int_0^\infty f(t)\cos(\lambda t;q^2)d_qt. \tag{1.43}$$

Note that from ([4],[5],[13],...) the q-translation operators and the q-cosine Fourier transform satisfies the following properties

i. $T_{q,x}f(y) = T_{q,y}f(x)$.

ii. $\Delta_{q,x}T_{q,x}f(y) = \Delta_{q,y}T_{q,y}f(x)$.

iii. $T_{q,x}$ tends to σ_x whenever q tends to 1^-,where

$$\sigma_x(f)(y) = \frac{1}{2}[f(x+y) + f(x-y)], \quad y \in [0, +\infty[.$$

iv. $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_{q,*}(\mathbb{R}_{q,+})$ onto itself and $\mathcal{F}_q^2 = Id$.

v. $\mathcal{F}_q$ can be extended to a one to one map from $L^1(\mathbb{R}_{q,+})$ into $\mathcal{C}_{q,*,0}(\mathbb{R}_{q,+})$ and we have $\| \mathcal{F}_q(f) \|_{\mathcal{C}_{q,*,0}} \leq \frac{1}{(q(1-q))^{\frac{1}{2}}}||f||_{q,1}$.

vi. <u>Inversion formula</u>
For $f \in L^1(\mathbb{R}_{q,+})$ such that $\mathcal{F}_q(f) \in L^1(\mathbb{R}_{q,+})$, we have $f = \mathcal{F}_q(\mathcal{F}_q(f))$.

vii. <u>q-Plancherel theorem type</u>
The q-cosine Fourier transform $\mathcal{F}_q$ is an isometric isomorphism of $L^2(\mathbb{R}_{q,+})$ onto itself. The inverse $\mathcal{F}_q^{-1}$ coincides with $\mathcal{F}_q$.

viii. For $f, g \in L^1(\mathbb{R}_{q,+}), \mathcal{F}_q(f *_q g) = \mathcal{F}_q(f)\mathcal{F}_q(g)$.

ix. $\mathcal{F}_q : S'_{*,q}(\mathbb{R}_{q,+}) \longrightarrow S'_{*,q}(\mathbb{R}_{q,+})$ is an isomorphism satisfying $\mathcal{F}_q = \mathcal{F}_q^{-1}$; and we have $\langle \mathcal{F}_q(T), \varphi\rangle = \langle T, \mathcal{F}_q(\varphi)\rangle; \quad T \in S'_{*,q}(\mathbb{R}_{q,+}),\ \varphi \in \mathcal{S}_{q,*}(\mathbb{R}_{q,+})$.

x. $\displaystyle\int_0^\infty \mathcal{F}_q(f)(\xi)g(\xi)d_q\xi = \int_0^\infty f(\xi)\mathcal{F}_q(g)(\xi)d_q\xi; \quad f, g \in L^1(\mathbb{R}_{q,+})$.

xi. $\mathcal{F}_q(T_{q,x}f)(\xi) = cos(x; q^2)\mathcal{F}_q(f)(\xi); \quad f \in L^1(\mathbb{R}_{q,+})$.

In the remainder of this work we choose q such that $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$ and we put

$$c_q = \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}. \tag{1.44}$$

2 The q-pseudo-measure $\mathcal{F}_q L^\infty$ space

In this section, we introduce the notion of the q-pseudo-measure, taking in the account of the fact that $L^\infty(\mathbb{R}_{q,+}) \subset S'_{q,*}(\mathbb{R}_{q,+})$ and via the inversion theorem we have $\mathcal{F}_q L^\infty(\mathbb{R}_{q,+}) \subset \mathcal{F}_q S'_{q,*}(\mathbb{R}_{q,+}) \subset S'_{q,*}(\mathbb{R}_{q,+})$, we obtain the following definition

Definition 1. Let T in $S'_{q,*}(\mathbb{R}_{q,+})$ a q-tempered distribution. If T is in $\mathcal{F}_q L^\infty$ then it's called a q-pseudo-measure.

Definition 2. Let T be a q-distribution in $\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$ and let f in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, we define the q-convolution product $T *_q f$ for all φ in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ by

$$< T *_q f, \varphi > = < T, f *_q \varphi > . \tag{2.1}$$

Proposition 1. *Let T be a q-Tempered distribution in $\mathcal{F}_q L^\infty(\mathbb{R}_{q,+})$ then for all f in $L^2(\mathbb{R}_{q,+})$, we have*

1. *The operator* $\mathrm{L}(f)$ *defined by*

$$\mathrm{L}(f) = T *_q f = \mathcal{F}_q[(\mathcal{F}_q T)(\mathcal{F}_q f)] \tag{2.2}$$

is continued in $L^2(\mathbb{R}_{q,+})$ and we have for all x in $\mathbb{R}_{q,+}$

$$T *_q (T_{q,x} f) = T_{q,x}(T *_q f). \tag{2.3}$$

2. *for φ in $L^\infty(\mathbb{R}_{q,+})$, let the operator $\mathrm{L}_\varphi : f \longmapsto (\mathcal{F}_q\varphi) *_q f$ defined in $L^2(\mathbb{R}_{q,+})$ then we have*

$$||| \mathrm{L}_\varphi |||_q = \| \varphi \|_{\infty,q} \tag{2.4}$$

where

$$||| \mathrm{L}_\varphi |||_q = \sup_{\substack{f \in L^2(\mathbb{R}_{q,+}) \\ f \neq 0}} \frac{\| \mathrm{L}_\varphi(f) \|_{2,q}}{\| f \|_{2,q}} \tag{2.5}$$

Proof. Let f in $L^2(\mathbb{R}_{q,+})$ so $\mathcal{F}_q f \in L^2(\mathbb{R}_{q,+})$ and $\mathcal{F}_q T \in L^\infty(\mathbb{R}_{q,+})$ then we obtain $(\mathcal{F}_q f)(\mathcal{F}_q T)$ in $L^2(\mathbb{R}_{q,+})$ further the q-Plancherel theorem give

$$\begin{aligned} \| T *_q f \|_{2,q} &= \| \mathcal{F}_q(T *_q f) \|_{2,q} = \| (\mathcal{F}_q T)(\mathcal{F}_q f) \|_{2,q} \\ &\leq \| \mathcal{F}_q T \|_{\infty,q} \| \mathcal{F}_q f \|_{2,q} \\ &\leq Cst \| f \|_{2,q} \quad , \quad Cst = \| \mathcal{F}_q T \|_{\infty,q} \end{aligned}$$

Now we prove the second propriety,

$$\begin{aligned} \| \mathrm{L}_\varphi(f) \|_{2,q} &= \| \mathcal{F}_q((\mathcal{F}_q \varphi) *_q f) \|_{2,q} \\ &\leq \| \mathcal{F}_q f \|_{2,q} \| \varphi \|_{\infty,q} \\ &\leq \| f \|_{2,q} \| \varphi \|_{\infty,q} \end{aligned}$$

■

2.1 The q-Function of positive type, q-Bochner theorem

In this subsection, we characterize the q-cosine Fourier Transform of a positive bounded measure $\mathcal{F}_q \mathcal{M}'_+(\mathbb{R}_{q,+})$.

Definition 3. A measure μ is called bounded if for all f in $\mathcal{H}_{q,*}(\mathbb{R}_{q,+})$, we have

$$\mu(f) \leq C_q \| f \|_{\mathcal{H}_{q,*}} \tag{2.6}$$

where $C_q > 0$ is a positive constant.
We note by $\mathcal{M}'(\mathbb{R}_{q,+})$ the set of bounded measure on $\mathbb{R}_{q,+}$.

Definition 4. The q-cosine Fourier transform of measure μ in $\mathcal{M}'(\mathbb{R}_{q,+})$, is defined : for all $\varphi \in S_q(\mathbb{R}_{q,+})$ by

$$< \mathcal{F}_q \mu, \varphi > = < \mu, \mathcal{F}_q \varphi > = \int_0^{+\infty} \mathcal{F}_q \varphi(\lambda) d_q \mu(\lambda). \tag{2.7}$$

Remark 1. In theory of measure, for μ in $\mathcal{M}'(\mathbb{R}_{q,+})$ the q-Jackson integral $< \mu, \varphi > = \int_0^{+\infty} \varphi(x) d_q \mu(x)$ have a sense if φ is a continuous and bounded function on $\mathbb{R}_{q,+}$(for example $\varphi = \cos(\lambda.; q^2)$, λ in $\mathbb{R}_{q,+}$ and relation (1.26)). More else, taking in the account of the fact that $L^1(\mathbb{R}_{q,+}) \subset \mathcal{M}'(\mathbb{R}_{q,+}) \subset S'_{q,+}(\mathbb{R}_{q,+})$, the q-cosine Fourier transform $\mathcal{F}_q$ in $L^1(\mathbb{R}_{q,+})$ given by (1.43) can be generalized to $\mathcal{M}'(\mathbb{R}_{q,+})$. We obtain the following proposition:

Proposition 2. *1. The q-cosine Fourier transform of a measure μ in $\mathcal{M}'(\mathbb{R}_{q,+})$ is the q-tempered distribution $\mathcal{F}_q \mu$ given by :*

$$\mathcal{F}_q \mu(\lambda) = c_q \int_0^{+\infty} \cos(\lambda x; q^2) d_q \mu(x). \tag{2.8}$$

2. for all $x, \lambda \in \mathbb{R}_{q,+}$ we have

$$T_{q,x} \mathcal{F}_q \mu(\lambda) = c_q \int_0^{+\infty} \cos(xt; q^2) \cos(\lambda t; q^2) d_q \mu(t). \tag{2.9}$$

Proof. for all φ in $S_{q,*}(\mathbb{R}_{q,+})$,

$$\begin{aligned} <\mu, \mathcal{F}_q\varphi> &= c_q \int_0^{+\infty}\int_0^{+\infty} \varphi(\lambda)\cos(\lambda t; q^2) d_q\lambda d_q\mu(t) \\ &= \int_0^{+\infty} \varphi(\lambda)(c_q \int_0^{+\infty} \cos(\lambda t; q^2) d_q\mu(t)) d_q\lambda \\ &= \int_0^{+\infty} \varphi(\lambda)\mathcal{F}_q\mu(\lambda) d_q\lambda \\ &= <\mathcal{F}_q\mu, \varphi> \end{aligned}$$

the result follows immediately. We prove (2) in the same way as (1). ■

Definition 5. A measure μ is called positive if for all f in $\mathcal{H}_{q,*}(\mathbb{R}_{q,+})$, $f \geq 0$ we have $\mu(f) \geq 0$.

Definition 6. Let f in $L^\infty(\mathbb{R}_{q,+})$, f is called a q-function of positive type if for all φ in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, we have

$$\int_0^{+\infty} \varphi *_q \varphi(x) f(x) d_qx \geq 0 \quad . \tag{2.10}$$

Proposition 3. *Let $f \in L^\infty(\mathbb{R}_{q,+}) \cap L^1(\mathbb{R}_{q,+})$.*
f is a q-function of positive type if and only if there exist $c_i, c_j \geq 0$ such that

$$(1-q)^2 \sum_{i,j=0}^{+\infty} c_i c_j T_{q,x_i} f(x_j) \geq 0. \tag{2.11}$$

Proof. Let φ_λ a q-approximation of unity we can show $\psi_\lambda = \varphi_\lambda *_q \varphi_\lambda$ is a q-approximation of unity . Consider $\theta_\lambda = \sum_{i=0}^{+\infty} c_i T_{q,x_i}\varphi(x)$, we have for $f \in L^\infty(\mathbb{R}_{q,+}) \cap L^1(\mathbb{R}_{q,+})$,

$$\begin{aligned} \sum_{i,j=0}^{+\infty} c_i c_j T_{q,x_i} f(x_j) &= \lim_{\lambda \longrightarrow 0} \sum_{i,j=0}^{+\infty} c_i c_j T_{q,x_i} f *_q \psi_\lambda(x_j) = \lim_{\lambda \longrightarrow 0} \sum_{i,j=0}^{+\infty} c_i c_j f *_q T_{q,x_i}\psi_\lambda(x_j) \\ &= \lim_{\lambda \longrightarrow 0} \sum_{i,j=0}^{+\infty} c_i c_j f *_q T_{q,x_i}(\varphi_\lambda *_q \varphi_\lambda)(x_j) \\ &= \lim_{\lambda \longrightarrow 0} \sum_{i,j=0}^{+\infty} c_i c_j \int_0^{+\infty} f(y) T_{q,y}((T_{q,x_i}\varphi_\lambda) *_q \varphi_\lambda(x_j)) d_qy \\ &= \lim_{\lambda \longrightarrow 0} \sum_{i,j=0}^{+\infty} c_i c_j \int_0^{+\infty} f(y)(T_{q,x_i}\varphi_\lambda *_q T_{q,x_j}\varphi_\lambda)(y) d_qy \\ &= \lim_{\lambda \longrightarrow 0} \int_0^{+\infty} f(y)\theta_\lambda *_q \theta_\lambda(y) d_qy \geq 0. \end{aligned}$$

Conversely, for all φ in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ with *supp* $\varphi = [0,h]$, $h > 0$, we have

$$\begin{aligned}
\int_0^{+\infty} \varphi *_q \varphi(x) f(x) d_q x &= \int_0^{+\infty}\int_0^{+\infty} T_{q,y}\varphi(x)\varphi(y) f(x) d_q x d_q y \\
&= \int_0^{+\infty}\int_0^{+\infty} \varphi(x)\varphi(y) T_{q,y} f(x) d_q x d_q y \\
&= \int_0^{h}\int_0^{h} \varphi(x)\varphi(y) T_{q,y} f(x) d_q x d_q y \\
&= (1-q)^2 \sum_{i,j=0}^{+\infty} h^2 q^{i+j} T_{q,q^j} f(q^i h)\varphi(q^i h)\varphi(q^j h) \\
&= (1-q)^2 \sum_{i,j=0}^{+\infty} c_i c_j T_{q,x_i} f(x_j) \quad \geq 0,
\end{aligned}$$

where $c_k = x_k\varphi(x_k), x_k = q^k h \quad ; k = i,j$. ■

Proposition 4. *Let μ a positive measure in $\mathcal{F}_q L^\infty(\mathbb{R}_{q,+})$ then μ is in $\mathcal{M}(\mathbb{R}_{q,+})$.*

Proof. Let $\mathbf{L}_\mu : f \longmapsto \mu *_q f$ for $L^2(\mathbb{R}_{q,+})$ in $L^2(\mathbb{R}_{q,+})$ and let f be the indicator function of the set $[0,r] \quad ; r \in \mathbb{R}_{q,+}$ defined by

$$f(x) = 1_{[0,r]}(x) = \begin{cases} 1 \quad , \quad x \in [0,r] \\ 0 \quad , \quad otherwise \end{cases} . \tag{2.12}$$

for all $y \in [0,r]$, we have

$$\begin{aligned}
f *_q f(y) &= c_q \int_0^r T_{q,x} 1_{[0,r]}(y) d_q y \\
&= c_q T_{q,x}\left(\int_0^r 1_{[0,r]}(y) d_q y\right) \quad \geq c_q \frac{r}{2}
\end{aligned}$$

to prove the proposition, it is suffices to notice that for all h in $\mathcal{H}_{q,*}(\mathbb{R}_{q,+})$

$$\sup_{\|h\|_{\infty,q}\leq 1} \mid \mu(h) \mid < +\infty, \tag{2.13}$$

but, when *supp* $h \subset [0,r]$, we obtain

$$\begin{aligned}
\mu(f *_q f) &= c_q \int_0^{+\infty} f *_q f(y) d_q\mu(y) = c_q^2 \int_0^{+\infty}\int_0^{+\infty} f(x) T_{q,y} f(x) d_q\mu(y) d_q x \\
&= c_q \int_0^{+\infty} f(x) \mu *_q f(x) d_q x \\
&\leq c_q \parallel \mu *_q f \parallel_{2,q} \parallel f \parallel_{2,q} \\
&\leq c_q \mid\mid\mid \mathbf{L}_\mu \mid\mid\mid_q \parallel f \parallel_{2,q} \parallel f \parallel_{2,q} = c_q r \mid\mid\mid \mathbf{L}_\mu \mid\mid\mid_q .
\end{aligned}$$

On the other hand

$$\mu((f *_q f) \mid h \mid) = c_q \int_0^{+\infty} f *_q f(y) \mid h(y) \mid d_q\mu(y) \geq c_q \frac{r}{2} \mid \mu(h) \mid$$

then

$$\frac{r}{2} \mid \mu(h) \mid \leq \mu((f *_q f) \mid h \mid) \leq \parallel h \parallel_{\infty,q} \mu(f *_q f) \leq r \mid\mid\mid \mathbf{L}_\mu \mid\mid\mid_q$$

i.e

$$\mid \mu(h) \mid \leq 2 \mid\mid\mid \mathbf{L}_\mu \mid\mid\mid_q \quad < +\infty \quad . \tag{2.14}$$

Hence the result follows. ■

Lemma 1. *For x_i, x_j in $\mathbb{R}_{q,+}$ such that $x_i \neq x_j$, we have :*

$$\int_0^{+\infty} \cos(\lambda x_i; q^2) \cos(\lambda x_j; q^2) d_q\lambda = 0 \qquad , \lambda \in \mathbb{R}_{q,+}. \tag{2.15}$$

Indeed, using (1.41) and (1.39), we deduce that

$$\begin{aligned} \int_0^{+\infty} \cos(\lambda x_i; q^2) \cos(\lambda x_j; q^2) d_q\lambda &= \int_0^{+\infty} T_{q,x_i} \cos(\lambda x_j; q^2) d_q\lambda \\ &= \int_0^{+\infty} \cos(\lambda x_j; q^2) d_q\lambda \\ &= \Big[\frac{\sin(\lambda x_j; q^2)}{x_j}\Big]_0^{+\infty} \\ &= 0 \end{aligned}$$

the result follows by (1.25).

Proposition 5. *If $\mu \in \mathcal{M}'_+(\mathbb{R}_{q,+})$, his q-cosine Fourier transform $\mathcal{F}_q\mu = f$ is a q-function of positive type.*

Indeed,

$$\begin{aligned} (1-q)^2 \sum_{i,j=0}^{+\infty} c_i c_j T_{q,x_i} f(x_j) &= (1-q)^2 \sum_{i,j=0}^{+\infty} c_i c_j T_{q,x_i} \mathcal{F}_q\mu(x_j) \\ &= (1-q)^2 c_q \sum_{i,j=0}^{+\infty} c_i c_j \int_0^{+\infty} \cos(\lambda x_i; q^2) \cos(\lambda x_j; q^2) d_q\mu(\lambda) \\ &= (1-q)^2 c_q \sum_{i=0}^{+\infty} c_i^2 \int_0^{+\infty} \cos^2(\lambda x_i; q^2) d_q\mu(\lambda) \quad \geq 0 \end{aligned}$$

3 Examples

In this section we give some basic functions where are q-function of positive type :

Example 1. *The function $x \longmapsto e(-tx^2; q^2)$ (see [4]) is a q-function of positive type since :*

$$\mathcal{F}_q(G(., t; q^2))(\lambda) = e(-t\lambda^2; q^2) \tag{3.1}$$

where

$$G(x,t;q^2) = A^{-1}(t,q)e(-\frac{x^2}{qt(1+q)^2};q^2) \quad (3.2)$$

$$A(t,q) = q^{-\frac{1}{2}}(1-q)^{\frac{1}{2}}\frac{(-\frac{1-q}{1+q}\frac{1}{t},-\frac{1+q}{1-q}q^2t;q^2)_\infty}{(-\frac{1-q}{1+q}\frac{1}{qt},-\frac{1+q}{1-q}q^3t;q^2)_\infty} \quad (3.3)$$

which is a positive function in $L^1(\mathbb{R}_{q,+})$.

Example 2. *The function* $x \longmapsto j_\alpha(x;q^2)$ *(see [5] , [2]) is a q-function of positive type, indeed it's the q-cosine Fourier transform of :*

$$\mathcal{F}_q(\frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\alpha+\frac{1}{2})}W_\alpha(.;q^2)1_{[0,1]}(.))(\lambda) = j_\alpha(\lambda;q^2) \quad (3.4)$$

where $W_\alpha(x;q^2)$ *defined in [5] by :*

$$W_\alpha(x;q^2) = \frac{(x^2q^2;q^2)_\infty}{(x^2q^{2\alpha+1};q^2)_\infty} \quad (3.5)$$

which is a positive function in $L^1(\mathbb{R}_{q,+})$.

Indeed ,

$$\int_0^{+\infty}\frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\alpha+\frac{1}{2})}W_\alpha(x;q^2)1_{[0,1]}(x)d_qx = \frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\alpha+\frac{1}{2})}\int_0^1 W_\alpha(x;q^2)d_qx$$
$$= \frac{(1+q^{-1})}{\Gamma_{q^2}(\frac{1}{2})}j_\alpha(0;q^2) = \frac{(1+q^{-1})}{\Gamma_{q^2}(\frac{1}{2})}.$$

Proposition 6. *Let* T *in* $\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$*, these assertions are equivalents:*

1. *for all* $\varphi \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+})$*, we have* $< T, f^2 > \geq 0$.
2. *T is a positive q-distribution*
   (i.e for all $\varphi \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+})$; $\varphi \geq 0$ *implies that* $< T,\varphi > \geq 0$ *).*
3. *T is a positive measure.*

Indeed ,
(1)$\Longrightarrow$ (2), it is sufficient to say that for all $\varphi \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+}); \varphi \geq 0$, is a limit of functions f_k^2 where $f_k \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+})$. Let $f_k(x) = \chi_q(x)\sqrt{\varphi(x)+\frac{1}{k}}$, where χ_q in $\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$ positive equal to 1 in the support of φ then :

$$f_k^2(x) - \varphi(x) = \frac{\chi_q^2(x)}{k} \longrightarrow 0 \quad ,k \to \infty \quad in\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$$

and the result follows.
(3)$\Longrightarrow$ (1) evident.

(2)$\Longrightarrow$(3), it is sufficient to prove that $T \in \mathcal{H}'_{q,*}(\mathbb{R}_{q,+})$. Let K a compact of $\mathbb{R}_{q,+}$, consider $\psi_K \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ such that $\psi_K \geq 0$ and $\psi_K \equiv 1$ on K, then for all $\varphi \geq 0$, *supp* $\varphi \subset K$,

$$- \parallel \varphi \parallel_\infty \psi_K \leq \parallel \varphi \parallel_\infty \psi_K \tag{3.6}$$

then

$$|< T, \varphi >| \leq C_K \parallel \varphi \parallel_\infty \quad ; C_K =< T, \psi_K > \tag{3.7}$$

then $T \in \mathcal{H}'_{q,*}(\mathbb{R}_{q,+})$.

Theorem 1. *(of Bochner) Let $f \in L^\infty(\mathbb{R}_{q,+})$, if f is a q-function of positive type, there exist $\mu \in \mathcal{M}'_+(\mathbb{R}_{q,+})$ such that*

$$f = \mathcal{F}_q \mu. \tag{3.8}$$

Proof. Let $f \in L^\infty(\mathbb{R}_{q,+})$, of positive type and putting $T = \mathcal{F}_q f$.
for all $g \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, we have : $\mathcal{F}_q g$ in $S_{q,*}(\mathbb{R}_{q,+}) \subset L^1(\mathbb{R}_{q,+})$ then

$$\begin{aligned} < T, g^2 > &= < \mathcal{F}_q f, g^2 >=< f, \mathcal{F}_q(g^2) > \\ &= < f, \mathcal{F}_q g *_q \mathcal{F}_q g > \quad \geq 0 \end{aligned}$$

thus T is a positive q-distribution. Again, by using proposition 6 it's a measure of positive type. But since $T \in \mathcal{F}_q L^\infty(\mathbb{R}_{q,+})$, by proposition 4 this measure is bounded, the result follows after minor computation. ■

Remark 2. the following result leads that for all f in $L^\infty(\mathbb{R}_{q,+})$,
$\mathcal{F}_q \mathcal{H}'_{q,*}(\mathbb{R}_{q,+}) = \Big\{ \ q\text{-function of positive type} \ \Big\} = \mathcal{P}(\mathbb{R}_{q,+})$.

In the following, we shall give some properties

Proposition 7. *We have :*

1. *If $f_1, f_2, \cdots, f_k \in \mathcal{P}(\mathbb{R}_{q,+})$ then $f_1 + f_2 + \cdots + f_k \in \mathcal{P}(\mathbb{R}_{q,+})$.*
2. *If $f \in \mathcal{P}(\mathbb{R}_{q,+}), \lambda \in \mathbb{R}_{q,+}$ then $\lambda f \in \mathcal{P}(\mathbb{R}_{q,+})$.*
3. *If $f_1, f_2 \in \mathcal{P}(\mathbb{R}_{q,+})$ then $f = f_1 f_2 \in \mathcal{P}(\mathbb{R}_{q,+})$.*

Indeed ,
If μ_1, μ_2 are two bounded measures in $\mathbb{R}_{q,+}$, $\mu = \mu_1 *_q \mu_2$ defined by :
for all φ in $\mathcal{H}_{q,*}(\mathbb{R}_{q,+})$

$$< \mu, \varphi >=< \mu_1 *_q \mu_2, \varphi >= c_q \int_0^{+\infty} \int_0^{+\infty} T_{q,x} \varphi(y) d_q\mu_1(x) d_q\mu_2(y) \tag{3.9}$$

defined a bounded measure in $\mathbb{R}_{q,+}$. If we take $\varphi = c_q \cos(\lambda x; q^2)$, we obtain :

$$\begin{aligned} < \mu, \varphi > &= c_q^2 \int_0^{+\infty} \int_0^{+\infty} T_{q,x} \cos(\lambda y; q^2) d_q\mu_1(x) d_q\mu_2(y) \\ &= c_q^2 \int_0^{+\infty} \int_0^{+\infty} \cos(\lambda x; q^2) \cos(\lambda y; q^2) d_q\mu_1(x) d_q\mu_2(y) \\ &= \mathcal{F}_q(\mu_1)(\lambda) \mathcal{F}_q(\mu_2)(\lambda) \\ &= \mathcal{F}_q(\mu)(\lambda) \\ &= \mathcal{F}_q(\mu_1 *_q \mu_2)(\lambda). \end{aligned}$$

Moreover if μ_1, μ_2 are positive then $\mu_1 *_q \mu_2$ too, the q-Bochner theorem leads that the product of two functions of positive type is of positive type too.

4 The q-Distributions of positive Type : q-Bochner-Schwartz theorem

In this section, we summarize some of properties studied by A. Fitouhi, M. M. Hamza and F. Bouzeffour in [5]. The q-analogue of Kober-Erdely transform is given by :
For $\alpha \neq -\frac{1}{2}, -1, -\frac{3}{2}, ...$ and f in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$

$$\chi_{\alpha,q}(f)(x) = C(\alpha, q^2)\frac{1+q}{x}\int_0^x W_\alpha(\frac{t}{x}; q^2; q^2) f(xt) d_q t \quad , x \neq 0 \tag{4.1}$$

and

$$\chi_{\alpha,q}(f)(0) = f(0) \tag{4.2}$$

where

$$C(\alpha, q^2) = \frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \tag{4.3}$$

and

$$W_\alpha(x; q^2) = \frac{(x^2q^2; q^2)_\infty}{(x^2q^{2\alpha+1}; q^2)_\infty} = {}_1\phi_1(q^{1-2\alpha}, -, q^2, x^2q^{2\alpha+1}) \tag{4.4}$$

and the q-transposed operator ${}^t\chi_{\alpha,q}$ of $\chi_{\alpha,q}$ is given for f in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ and $\alpha \neq -\frac{1}{2}$, -1, $-\frac{3}{2}$,... by :

$$^t\chi_{\alpha,q}(f)(x) = \frac{q(1+q^{-1})^{-\alpha+\frac{1}{2}}\Gamma_{q^2}(\alpha+1)}{\Gamma^2_{q^2}(\alpha+\frac{1}{2})}\int_{qx}^{+\infty} W_\alpha(\frac{x}{t}; q^2) f(t) t^{2\alpha} d_q t. \tag{4.5}$$

The operators $\chi_{\alpha,q}$ and ${}^t\chi_{\alpha,q}$ define isomorphisms on $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ (see [5]).
The q-generalized Bessel translation can be defined via the q-transmutation operator by

$$T_x^\alpha f(y) = \chi_{\alpha,q,x}\chi_{\alpha,q,y}(T_{q,x}^{-\frac{1}{2}}\chi^{-1}_{\alpha,q,y}(f)(y)) \tag{4.6}$$

where $T_{q,x}^{-\frac{1}{2}}$ is the q-even translation defined by (1.35).

For f and g in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, the q-Bessel convolution and the Fourier transform are given by :

$$f *_\alpha g(x) = \frac{(1+q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}\int_0^{+\infty} T_x^\alpha f(y) g(y) y^{2\alpha+1} d_q y, \tag{4.7}$$

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = \frac{(1+q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}\int_0^{+\infty} f(x) j_\alpha(\lambda x; q^2) d_q x. \tag{4.8}$$

It satisfies

$$\begin{aligned}\chi_{\alpha,q}(f *_q g) &= \chi_{\alpha,q}(f) *_\alpha (g), &(4.9)\\ \mathcal{F}_{\alpha,q}(f *_\alpha g) &= \mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g), &(4.10)\\ \mathcal{F}_{\alpha,q} &= \mathcal{F}_q \circ {}^t\chi_{\alpha,q}. &(4.11)\end{aligned}$$

where $*_q$ design the q-even convolution given by (1.42).
If we proceed as in [5], we can show easily that

$$^t\chi_{\alpha,q}(f *_\alpha g) = {}^t\chi_{\alpha,q}(f) *_q {}^t\chi_{\alpha,q}(g). \tag{4.12}$$

Definition 7. Let T be in $\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$, T is called of positive type if for all φ in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, we have

$$< T, \varphi *_q \varphi > \quad \geq 0 \qquad . \tag{4.13}$$

Example 3. *The q-distribution T of $\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$ defined by :*

$$< T, f >= ({}^t\chi_{q,\alpha})^{-1}(f)(0) \qquad , f \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+}) \tag{4.14}$$

is a q-distribution of positive type
where $^t\chi_{q,\alpha}$ is given by (4.5).

Proof. Let f in $\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, using the relation (4.12), we obtain :

$$\begin{aligned} f *_\alpha f(0) &= < T, {}^t\chi_{q,\alpha}(f *_\alpha f) > \\ &= < T, {}^t\chi_{q,\alpha}(f) *_q {}^t\chi_{q,\alpha}(f) > \end{aligned}$$

on the other hand by (4.7)

$$f *_\alpha f(0) = c_q \int_0^{+\infty} f^2(y) x^{2\alpha+1} d_q y \quad \geq 0 \tag{4.15}$$

the result follows immediately. ■

Theorem 2. *(Bochner-Schwartz)*
Let T in $\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$, the following assertions are equivalent

1. *T is of positive type.*
2. *T is a q-tempered distribution, and it's the q-cosine Fourier transform of a q-tempered positive measure.*
3. *there exist a positive measure μ and integer $k \geq 0$ such that :*
   (a) $\int_0^{+\infty} (1+x^2)^{-k} d_q\mu(x) \quad < +\infty$

   (b) $T = \mathcal{F}_q\mu$.

Proof. (2)$\Longrightarrow$ (1) if $\mathcal{F}_qT=\mu\in\mathcal{H}'_{q,*}(\mathbb{R}_{q,+})\bigcap S'_{q,*}(\mathbb{R}_{q,+})$ we have, for all $\varphi\in\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$

$$\begin{aligned}<\mathcal{F}_q\mu,\varphi*_q\varphi> &= <\mu,\mathcal{F}_q(\varphi*_q\varphi)>\\ &= <\mu,(\mathcal{F}_q\varphi)^2> \quad \geq 0\end{aligned}$$

(3)$\Longrightarrow$ (2) evident.

(1)$\Longrightarrow$ (3) we remark that for all $\varphi\in\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, the function $\varphi\longmapsto T*_q\varphi*_q\varphi$ is of positive type, because for all $\psi\in\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$

$$\begin{aligned}<T*_q\varphi*_q\varphi,\psi*_q\psi> &= <T,\varphi*_q\varphi*_q\psi*_q\psi>\\ &= <T,(\varphi*_q\psi)*_q(\varphi*_q\psi)> \quad \geq 0;\end{aligned}$$

then by the theorem 1, there exist a measure $\mu_\varphi\in\mathcal{H}'_{q,*}(\mathbb{R}_{q,+})$ such that $\mu_\varphi=\mathcal{F}_q(T*_q\varphi*_q\varphi)$ we choose $\psi\in\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ such that $\mathcal{F}_q\psi(\lambda)\neq 0 \quad , \lambda\in\mathbb{R}_{q,+}$ and let $\mu=(\mathcal{F}_q\psi)^{-2}(\lambda)\mu_\psi$ then μ is a positive measure, we can write :

$$\mathcal{F}_q(T*_q\varphi*_q\varphi*_q\psi*_q\psi)=(\mathcal{F}_q\psi)^2\mu_\varphi=(\mathcal{F}_q\varphi)^2\mu_\psi \tag{4.16}$$

then

$$\mu_\varphi=(\mathcal{F}_q\varphi)^2\mu \quad ,\varphi\in\mathcal{D}_{q,*}(\mathbb{R}_{q,+}). \tag{4.17}$$

we deduce that

$$\begin{aligned}<T,\varphi*_q\varphi> &= <T*_q\varphi,\varphi>=<T*_q\varphi,\varphi*_q\delta_q>=<T*_q\varphi*_q\varphi,\delta_q>\\ &= (\mathcal{F}_q\mu_\varphi)(0)\\ &= \int_0^{+\infty} d_q\mu_\varphi(t)\\ &= \int_0^{+\infty}(\mathcal{F}_q\varphi)^2(t)d_q\mu(t)\end{aligned}$$

i.e for all $\chi_q=\varphi*_q\varphi \quad ;\varphi\in\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ we have

$$<T,\chi_q>=\int_0^{+\infty}(\mathcal{F}_q\chi_q)(t)d_q\mu(t)=<\mathcal{F}_q\mu,\chi_q>. \tag{4.18}$$

so the result follows.
Now we prove (a), let $\chi\in\mathcal{D}_{q,*}(\mathbb{R}_{q,+})$ such that *supp* $\chi\subset[0,1]$ and $\mathcal{F}_q\chi(\lambda)$ be >0 in $\mathbb{R}_{q,+}$. Since for $0<\varepsilon\leq 1$, putting $\chi_\varepsilon(x)=\varepsilon^{-1}\chi(\varepsilon^{-1}x)$ and $m=\inf_{\lambda\leq 1}\mid\mathcal{F}_q\chi(\lambda)\mid$. Furthermore if

we use the theorem 3 in [13], there exist $k \geq 0$ and $C > 0$ such that

$$\begin{aligned}
\mu(0 \leq \lambda \leq \varepsilon^{-1}) \leq m^{-1} \int_0^{\varepsilon^{-1}} \mathcal{F}_q\chi(\varepsilon\lambda) d_q\mu(\lambda) &\leq m^{-1} \int_0^{+\infty} \mathcal{F}_q\chi(\varepsilon\lambda) d_q\mu(\lambda) \\
&= m^{-1} \left| < T, \chi_\varepsilon > \right| \\
&\leq C \sup_{\substack{p < k \\ x \in \mathbb{R}_{q,+}}} |\Delta_q^p \chi_\varepsilon(x)| \\
&\leq C_1 \varepsilon^{-1-2k} \sup_{\substack{p < k \\ x \in \mathbb{R}_{q,+}}} |\Delta_q^p \chi(x)| \\
&= C_2 \varepsilon^{-1-2k}.
\end{aligned}$$

This prove that for $R \to \infty$, the measure μ defined in $[0, R]$ is an $\Theta(R^{1+2k})$ this achieve the proof of (a). ■

Example 4. *The q-distribution* $x \longmapsto q^{\nu+\frac{1}{2}}(1+q)^{\nu+\frac{1}{2}} \dfrac{\Gamma_{q^2}(\frac{\nu+1}{2})}{\Gamma_{q^2}(-\frac{\nu}{2})} \mid x \mid^{-\nu-1}$, $Re\nu > -1$ *is a q-distribution of positive type.*

Indeed,
In [5] we have,

$$\mathcal{F}_q(\mid x \mid^\nu) = q^{\nu+\frac{1}{2}}(1+q)^{\nu+\frac{1}{2}} \frac{\Gamma_{q^2}(\frac{\nu+1}{2})}{\Gamma_{q^2}(-\frac{\nu}{2})} \mid x \mid^{-\nu-1} . \tag{4.19}$$

On the other hand : for all $\varphi \geq 0$

$$<\mid x \mid^\nu, \varphi >= \int_0^{+\infty} x^\nu \varphi(x) d_q x \quad \geq 0 \tag{4.20}$$

Theorem 3. *All q-distribution of positive type T defined in* $\mathcal{D}'_{q,*}(\mathbb{R}_{q,+})$, *can be written as:*

$$T = (1 - \Delta_{q,x})^k f(x) \quad , k \in \mathbb{N}$$

where f is a q-function of positive type.

Proof. We have for all $\varphi \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+})$, by theorem 2 there exist $k \in \mathbb{N}$ and μ a positive measure such that

$$< T, \varphi >=< \mathcal{F}_q T, \mathcal{F}_q\varphi >=< \mu, \mathcal{F}_q\varphi >= \int_0^{+\infty} \mathcal{F}_q\varphi(\lambda) d_q\mu(\lambda) \tag{4.21}$$

and

$$\int_0^{+\infty} \frac{1}{(1+\lambda^2)^k} d_q\mu(\lambda) \quad < +\infty$$

and putting $d_q\nu(\lambda) = (1+\lambda^2)^{-k} d_q\mu(\lambda)$, the measure ν is a positive measure, bounded. Then by proposition 5 we have $f_1(\lambda) = \mathcal{F}_q\nu(\lambda)$ is a q-function of positive type , furthermore for all $\varphi \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+})$,

$$\begin{aligned} < T, \varphi > &= \int_0^{+\infty} \mathcal{F}_q\varphi(\lambda) d_q\mu(\lambda) \\ &= c_q \int_0^{+\infty}\int_0^{+\infty} \cos(\lambda t; q^2)\varphi(t)(1+\lambda^2)^k d_q\nu(\lambda) d_qt \\ &= c_q \int_0^{+\infty}\int_0^{+\infty} (1-\Delta_{q,t})^k(\cos(\lambda t; q^2))\varphi(t) d_q\nu(\lambda) d_qt \\ &= c_q \int_0^{+\infty}\int_0^{+\infty} (1-\Delta_{q,t})^k(\varphi(t))\cos(\lambda t; q^2) d_q\nu(\lambda) d_qt \\ &= \int_0^{+\infty} (1-\Delta_{q,t})^k \varphi(t) f_1(t) d_qt. \end{aligned}$$

where

$$f_1(t) = c_q \int_0^{+\infty} \cos(\lambda t; q^2) d_q\nu(\lambda) = \mathcal{F}_q\nu(\lambda) \tag{4.22}$$

then

$$< T, \varphi > = < f_1, (1-\Delta_{q,t})^k\varphi > = < (1-\Delta_{q,t})^k f_1, \varphi > \quad ; \varphi \in \mathcal{D}_{q,*}(\mathbb{R}_{q,+}) \quad .$$

■

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