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Provenance of Type II hidden symmetries from nonlinear partial differential equations

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Abstract

The provenance of Type II hidden point symmetries of differential equations reduced from nonlinear partial differential equations is analyzed. The hidden symmetries are extra symmetries in addition to the inherited symmetries of the differential equations when the number of independent and dependent variables is reduced by a Lie point symmetry. These Type II hidden symmetries do not arise from contact symmetries or nonlocal symmetries as in the case of ordinary differential equations. The Lie point symmetries of a model equation and the two-dimensional Burgers' equation and their descendants are used to identify the hidden symmetries. The significant new result is the provenance of the Type II Lie point hidden symmetries found for differential equations reduced from partial differential equations. Two methods for determining the source of the hidden symmetries are developed.

1 Introduction

Partial differential equations (PDEs) describe many scientific and engineering problems. Nonlinear partial differential equations are difficult to solve analytically as initial-value problems except for high-symmetry solitons by inverse scattering and two-dimensional, nonlinear PDEs by a method of characteristics. However, symmetry methods are very useful in solving PDEs of intermediate symmetry for which special solutions can be determined (See [12].).

The classic symmetry method for differential equations is based on Lie group symmetries [14, 29, 28, 10, 30, 22, 23]. The number of independent and dependent variables of a partial differential equation can be reduced by one if the PDE is invariant under a Lie group. Further reductions by Lie group symmetries depend on the structure of the associated Lie algebra (See [17] for some initial developments in this regard.). With enough reductions of the number of variables the PDE is reduced to an ordinary differential equation (ODE). The order of this ODE can be reduced until quadratures is reached if there is a solvable Lie group of sufficient dimension.

It is very desirable to predict how many reductions in the number of variables (order in the case of an ODE) can be effected from the Lie symmetries of a PDE (ODE). This number may be more than initially thought due to the presence of hidden symmetries which were initially identified by Olver [28]. The hidden symmetries of ODEs have been extensively studied over the past decade [5, 6, 7, 1, 2, 3, 8, 18, 19, 16, 9, 26, 27]. If the ODE loses (gains) a symmetry in addition to the one used to reduce the order of the ODE, the ODE possesses a Type I (Type II) hidden symmetry. The Type I hidden symmetry of ODEs can be predicted from the structure of the Lie algebra at each stage of the order reduction. These hidden symmetries are nonlocal symmetries. The origin of the Type II hidden symmetries is less obvious. Some Type II hidden symmetries of ODEs have been shown to arise from contact symmetries when a derivative transforms to be part or all of a variable [9] while others arise from nonlocal symmetries [6]. The Type II hidden symmetries may also arise from particular three-dimensional subalgebras [3]. A Lie group point symmetry of the three-dimensional subalgebra 'disappears' in the first–order reduction to become a nonlocal symmetry. In the appropriate second–order reduction the nonlocal symmetry transforms to a Lie point symmetry.

Unlike the case of ODEs hidden symmetries of PDEs have not been studied extensively. There are some interesting treatments of hidden symmetries of PDEs from nonlocal transformations [25] and from conditional symmetries [31]. We confine our investigations here to hidden symmetries of PDEs for which the number of independent and dependent variables is reduced by Lie point symmetries. If a PDE loses (gains) a Lie point symmetry in addition to the Lie point symmetry used to reduce the numbers of independent and dependent variables, the PDE possesses a Type I (Type II) Lie point hidden symmetry. In this paper we assume that all hidden symmetries are Lie point hidden symmetries.

One of the earliest cases of Type I hidden symmetries was obtained by an analysis of a two-dimensional linear heat conduction equation [13]. Several examples of Type II hidden symmetries of PDEs have been reported although none was designated a 'hidden' symmetry [30, 15, 11]. An extra (not inherited) symmetry was reported in the ODE in the final step of the reduction path of the three-dimensional linear wave equation [30]. An extra (not inherited) symmetry was found in a reduction of the two-dimensional Burgers' equation [15]. In a recent conference presentation [4] Type II hidden symmetries were identified in the reduction path of the two-dimensional, linear wave equation as well as more Type II hidden symmetries of the three-dimensional linear wave equation. None of these reports suggested an origin for the Type II hidden symmetries found by reducing the number of variables of PDEs; the origin has been a mystery. It has been noted [4] that these Type II hidden symmetries do not arise from contact symmetries or nonlocal symmetries since the transformations to reduce the number of variables involve only variables. Thus the origin of these hidden symmetries *must* be in point symmetries.

We have identified a common provenance for the Type II hidden symmetries of differential equations reduced from PDEs that covers the PDEs studied. The crucial point is that the differential equation that is reduced from a PDE and possesses a Type II hidden symmetry is also a reduced differential equation from one or more other PDEs. The inherited symmetries from these other PDEs are a larger class of Lie point symmetries that includes the Type II hidden symmetries. The Type II hidden symmetries are actually inherited symmetries from one or more of the other PDEs. The other PDEs are assumed to have the same independent and dependent variables and to be reduced by the same symmetry as the original PDE. Of course for any given PDE the reduction in the number of variables may produce the maximum number of inherited symmetries possible such that all other reduced differential equations have the same or fewer inherited symmetries.

The crucial question is whether we can identify the PDEs from which the Type II hidden symmetries are inherited. In most cases there appear to be an indefinite number of these PDEs; we present some possibilities. Some PDEs may be identified by inspection; others may be constructed by calculating the invariants by reverse transformations. Both methods are developed in the next section.

2 Model equation

We begin by introducing a model equation to demonstrate the properties of Type II hidden symmetries. The model equation is

$$u_{xxx} + u\,(u_t + cu_x) = 0, \tag{2.1}$$

where c is a constant and the subscripts denote differentiation with respect to the variable indicated. This nonlinear PDE is similar to but not the same as the KdV equation [24]. The reduced differential equation from the KdV equation does not possess a Type II hidden symmetry for the traveling-wave transformation assumed below. The Lie point symmetries of (2.1) are represented by the Lie group generators

$$U_1 = \frac{\partial}{\partial t} \tag{2.2a}$$

$$U_2 = \frac{\partial}{\partial x} \tag{2.2b}$$

$$U_3 = 3t\frac{\partial}{\partial t} + (x + 2ct)\frac{\partial}{\partial x} \tag{2.2c}$$

$$U_4 = t\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{2.2d}$$

These symmetries are determined by the classical method or more easily by a computer program such as LIE [20, 21]. The symmetry $cU_2 + U_1$ is used to reduce the number of variables of (2.1) via $w = u, y = x - ct$ and $w(y)$ is the traveling-wave solution sought. The reduced differential equation is the ODE

$$w_{yyy} = 0 \tag{2.3}$$

which has seven Lie point symmetries. The Lie group generators that represent its seven-dimensional Lie algebra are

$$X_1 = \frac{\partial}{\partial y} \tag{2.4a}$$

$$X_2 = \frac{\partial}{\partial w} \tag{2.4b}$$

$$X_3 = y^2\frac{\partial}{\partial w} \tag{2.4c}$$

$$X_4 = y\frac{\partial}{\partial y} \tag{2.4d}$$

$$X_5 = y\frac{\partial}{\partial w} \tag{2.4e}$$

$$X_6 = w\frac{\partial}{\partial w} \tag{2.4f}$$

$$X_7 = \tfrac{1}{2}y^2\frac{\partial}{\partial y} + yw\frac{\partial}{\partial w}. \tag{2.4g}$$

The inherited symmetries are $U_1 \to X_1$, $U_3 \to X_4$, $U_4 \to X_6$, all of which can be inferred by looking at the Lie algebra of (2.2). The other symmetries are Type II hidden symmetries.

Two possible methods have been identified for finding possible PDEs the symmetries of which are inherited in the transformation $w = u, y = x - ct$ in (2.4). The first method is to guess possible PDEs, evaluate their Lie point symmetries by LIE and then check if the group generators reduce to (2.4). That is the easiest method if it works. The second method is to perform reverse transformations for each group generator in (2.4) and determine the PDE from common invariants for the group generators.

Some other likely PDEs that reduce to (2.3) by using the variables y and w are

$$u_{xxx} = 0, \qquad u_{ttt} = 0, \qquad u_{xxt} = 0, \qquad u_{xtt} = 0, \tag{2.5}$$

where $u = u(x,t)$ still. The Lie group generators for the first equation in (2.5) are

$$U_1 = F_1(t)\frac{\partial}{\partial x} \tag{2.6a}$$

$$U_2 = F_2(t)\frac{\partial}{\partial u} \tag{2.6b}$$

$$U_3 = F_3(t)\frac{\partial}{\partial t} \tag{2.6c}$$

$$U_4 = F_4(t)x^2\frac{\partial}{\partial u} \tag{2.6d}$$

$$U_5 = F_5(t)x\frac{\partial}{\partial x} \tag{2.6e}$$

$$U_6 = F_6(t)x\frac{\partial}{\partial u} \tag{2.6f}$$

$$U_7 = F_7(t)u\frac{\partial}{\partial u} \tag{2.6g}$$

$$U_8 = F_8(t)\left(\tfrac{1}{2}x^2\frac{\partial}{\partial x} + ux\frac{\partial}{\partial u}\right). \tag{2.6h}$$

The functions $F_j(t), j = 1, \ldots, 8$, are arbitrary here. However, by appropriate choices of polynomials in t for $F_j(t)$ (and also taking combinations) the group generators become

$$V_1 = \frac{\partial}{\partial x} \tag{2.7a}$$

$$V_2 = \frac{\partial}{\partial u} \tag{2.7b}$$

$$V_3 = \frac{\partial}{\partial t} \tag{2.7c}$$

$$V_4 = (x - ct)^2 \frac{\partial}{\partial u} \tag{2.7d}$$

$$V_5 = (x - ct) \frac{\partial}{\partial x} \tag{2.7e}$$

$$V_6 = (x - ct) \frac{\partial}{\partial u} \tag{2.7f}$$

$$V_7 = u \frac{\partial}{\partial u} \tag{2.7g}$$

$$V_8 = \tfrac{1}{2}(x - ct)^2 \frac{\partial}{\partial x} + (x - ct)u \frac{\partial}{\partial u} \tag{2.7h}$$

and reduce to the seven group generators in (2.4). (The choice of polynomial is dictated purely by the need to recover (2.4).) The symmetry X_7 in (2.4) is not inherited for $u_{xxt} = 0$ in (2.5) although the other six symmetries are inherited. By interchanging x and t we see that the symmetries of $u_{ttt} = 0$ in (2.5) are inherited to give the symmetries in (2.4), but the symmetry of X_7 in (2.4) is not inherited for $u_{xtt} = 0$. It is clear then that the 'extra' symmetries obtained for (2.4) do not have any connection with the original PDE (2.1), but rather have come from the fact that (2.4) could be obtained from the reduction of another PDE. Thus the origin of the reduced ODE itself is not a simple matter.

Remark: In the above we have assumed that the PDEs (2.5) are all reduced by using the same variables as the original PDE (2.1). This does not have to be the case. For example one could envision a PDE of the form

$$w_{xxx} + w_{xx}\left(x w_{xx} + t w_{tx}\right) = 0 \tag{2.8}$$

or

$$w_{xxx} + w_{xx}\left(w_x + \frac{t w_t}{x}\right) = 0, \tag{2.9}$$

both of which reduce to (2.3) when we use the new independent variable $y = x/t$ with the dependent variable unchanged. Indeed this approach could yield fruitful results by relating new (or unknown) PDEs to known integrable PDEs via common reduced PDEs (or ODEs). We shall explore this elsewhere.

The second method to determine possible partial differential equations other than (2.1) involves reverse transformations. In this approach one starts with the Lie group generators in (2.4) and calculates the Lie generators in the original variables of (2.1) that reduce to the group generators in (2.4). Once the generators are computed, the common invariants are calculated. The other possible PDEs are combinations of these invariants. To illustrate the procedure we find the group generator that reduces to X_7 in (2.4). Let

$$U_a = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.10}$$

For U_a to reduce to X_7 we require that

$$U_a(u) = \eta = yw = (x-ct)u \tag{2.11a}$$
$$U_a(y) = \xi^x - c\xi^t = \tfrac{1}{2}y^2 = \tfrac{1}{2}(x-ct)^2. \tag{2.11b}$$

Then η is determined, but we need another condition to find ξ^t and ξ^x . The reduction of the PDE is assumed to be via the traveling-wave symmetry,

$$U_c = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \tag{2.12}$$

and the commutator

$$[U_c, U_a] = A_a U_c \tag{2.13}$$

holds where A_a is a constant that may be zero. This condition yields

$$\xi^t = A_a t + f^a(x-ct) \tag{2.14a}$$
$$\xi^x = \tfrac{1}{2}(x-ct)^2 + A_a ct + cf^a(x-ct) \tag{2.14b}$$

with f^a an arbitrary function of its argument. Depending upon the choices for A_a and f^a, we obtain different group generators. The simplest choice with A_a and f^a both zero works here so that $U_a = V_8$. The common invariants that reduce to $u_{xxx} = 0$ are found from (2.7). They are u_{xxx} for V_1, V_2, V_3, V_4, V_6 but $u_x^3 u_{xxx}$ for V_5, u_{xxx}/u for V_7 and $u^2 u_{xxx}$ for V_8. (We note that the characteristic equations may be difficult to integrate for some choices of group generators.) Since the invariants are set equal to zero, the PDE is $u_{xxx} = 0$.

3 Two-dimensional Burgers' equation

The original one-dimensional Burgers' equation in gas dynamics was simplified from the Navier-Stokes equations and was a model equation for shock waves. Some applications of the different Burgers' equations have been discussed previously [15]. The existence of an extra symmetry besides the inherited symmetries of the two-dimensional Burgers' equation under one particular symmetry reduction was noted. We discuss the origin of this Type II hidden symmetry.

The two-dimensional Burgers' equation is

$$u_t + uu_z = u_{xx} + u_{zz}, \qquad u = u(x,z,t). \tag{3.1}$$

The Lie group generators of (3.1) are

$$U_1 = \frac{\partial}{\partial x} \tag{3.2a}$$

$$U_2 = \frac{\partial}{\partial z} \tag{3.2b}$$

$$U_3 = \frac{\partial}{\partial t} \tag{3.2c}$$

$$U_4 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u} \tag{3.2d}$$

$$U_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}. \tag{3.2e}$$

We reduce the number of variables of the two-dimensional Burgers' equation by the transformation

$$u = w(t, \rho), \qquad \rho = z - \frac{x}{a} \tag{3.3}$$

found from the symmetry $U_a = aU_1 + U_2$. The reduced PDE is the one-dimensional Burgers' equation

$$w_t + ww_\rho = \frac{1 + a^2}{a^2} w_{\rho\rho}. \tag{3.4}$$

The symmetries of (3.4) are

$$X_1 = \frac{\partial}{\partial \rho} \tag{3.5a}$$

$$X_2 = \frac{\partial}{\partial t} \tag{3.5b}$$

$$X_3 = t\frac{\partial}{\partial \rho} + \frac{\partial}{\partial w} \tag{3.5c}$$

$$X_4 = 2t\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial \rho} - w\frac{\partial}{\partial w} \tag{3.5d}$$

$$X_5 = t^2\frac{\partial}{\partial t} + t\rho\frac{\partial}{\partial \rho} + (\rho - tw)\frac{\partial}{\partial w}. \tag{3.5e}$$

The symmetries $X_j, j = 1, \ldots, 4$, are inherited symmetries of the two-dimensional Burgers' equation but X_5 is a Type II hidden symmetry.

In order to determine the other possible PDEs the inherited symmetries of which include all the symmetries in (3.5) we need to make an educated guess. The results of §2 suggest that a good candidate is

$$u_t + uu_z = \frac{1 + a^2}{a^2} u_{zz}, \qquad u = u(x, z, t). \tag{3.6}$$

Using LIE we find the Lie group generators are

$$U_1 = F_1(x)\frac{\partial}{\partial x} \tag{3.7a}$$

$$U_2 = F_2(x)\frac{\partial}{\partial z} \tag{3.7b}$$

$$U_3 = F_3(x)\frac{\partial}{\partial t} \tag{3.7c}$$

$$U_4 = F_4(x)\left(t\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right) \tag{3.7d}$$

$$U_5 = F_5(x)\left(2t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}\right) \tag{3.7e}$$

$$U_6 = F_6(x)\left(t^2\frac{\partial}{\partial t} + zt\frac{\partial}{\partial z} + (z - tu)\frac{\partial}{\partial u}\right). \tag{3.7f}$$

The $F_j(x)$, $j = 1, \ldots, 6$, are arbitrary functions of x in general but here we find the symmetries that reduce to those in (3.5) via (3.3). In this case F_2 and F_4 both form part of two group generators. For F_2 one of the generators becomes V_2 and the other combines with U_5 to become V_5. Similarly for F_4 one of the generators becomes V_4 and the other combines with U_6 to form V_6. The group generators are then

$$V_1 = \frac{\partial}{\partial x} \tag{3.8a}$$

$$V_2 = \frac{\partial}{\partial z} \tag{3.8b}$$

$$V_3 = \frac{\partial}{\partial t} \tag{3.8c}$$

$$V_4 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u} \tag{3.8d}$$

$$V_5 = 2t\frac{\partial}{\partial t} + \left(z - \frac{x}{a}\right)\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} \tag{3.8e}$$

$$V_6 = t^2\frac{\partial}{\partial t} + \left(z - \frac{x}{a}\right)t\frac{\partial}{\partial z} + \left(z - \frac{x}{a} - tu\right)\frac{\partial}{\partial u}. \tag{3.8f}$$

The method of reverse transformations can also be applied to determine the group generators. Here

$$U_J = \xi_t^j(x,z,t,u)\frac{\partial}{\partial t} + \xi_x^j(x,z,t,u)\frac{\partial}{\partial x} + \xi_z^j(x,z,t,u)\frac{\partial}{\partial z} + \eta^j(x,z,t,u)\frac{\partial}{\partial u}, \quad j = 1, \ldots, 5, \tag{3.9}$$

and the sixth generator arises from $aU_1 + U_2$. We find in general, that

$$\xi_x^j = A_j x + f^j\left(z - \frac{x}{a}\right), \qquad \xi_z^j = U_j(\rho) + A_j\frac{x}{a} + \frac{f^j\left(z - \frac{x}{a}\right)}{a}. \tag{3.10}$$

Again we take $A_j = f^j = 0$. The resultant group generators are the same as in (3.8).

Invariants that are common to all six generators in (3.8) give the PDE by taking a linear combination:

$$u_t + uu_z = Cu_{zz} + D\left(u_{xz} + \frac{u_{zz}}{a}\right)\left(u_{xx} + \frac{2u_{xz}}{a} + \frac{u_{zz}}{a^2}\right), \tag{3.11}$$

where $C = \left(1+a^2\right)/a^2$ and D is a constant. There are factors of $t^{3/2}$ and t^3 multiplying each of the invariants of V_5 and V_6 respectively, but they cancel in the PDE. Now (3.11) reduces to (3.6) if D is zero. With D nonzero the Lie symmetries of (3.11), as determined by LIE, simplify to (3.8) without imposing conditions on the $F_j(x)$ in (3.7).

There is an alternate set of group generators with V_5 in (3.8) replaced by U_5, $F_5(x) = 1$ in (3.7). We require that the commutator for these generators

$$[U_5, V_6] = 2V_6 \qquad \text{or} \qquad [V_5, V_6] = 2V_6 \tag{3.12}$$

holds so that the relation reduces to

$$[X_4, X_5] = 2X_5 \tag{3.13}$$

for the group generators in (3.5). The resultant invariants for the alternate set of group generators excludes one invariant such that $D = 0$ in (3.11).

4 Conclusion

The provenance of Lie point Type II hidden symmetries has been illustrated for two nonlinear partial differential equations: a model equation and the two-dimensional Burgers' equation. These Type II hidden symmetries are extra symmetries that appear when the number of variables of a PDE is reduced by a variable transformation found from a Lie group symmetry of the PDE. The Type II hidden symmetries are inherited symmetries of other PDEs in the same independent and dependent variables. The reduction of the number of variables of these other PDEs is by the same Lie point symmetry used to reduce the original PDE. Two methods are presented for finding one or more other PDEs from which the Type II hidden symmetries are inherited.

The first method starts with a guess of the form of the other PDEs. For both examples the other PDEs are similar to the reduced PDE, but retain the original variables. The other PDEs are reduced by the same symmetry as the original PDE. Then the Lie group symmetries of these other PDEs are computed and their reduced form calculated to find the inherited symmetries.

The second method by reverse transformations starts from the symmetries of the reduced differential equation. Then the possible symmetries from which the reduced symmetries arose are computed. More symmetries may be found by this method than by the first method, but the second method is computationally more intensive. Also it may be difficult to compute the invariants for some choices of group generators. Finally we observe that there may be other possible PDEs for the two examples shown here.

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