An asymptotic expansion of the q-gamma function $\Gamma_q(x)$

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Received February 20, 2006; Accepted June 5, 2006

Abstract

In this paper, we get an asymptotic expansion of the q-gamma function $\Gamma_q(x)$. Also, we deduced q-analogues of Gauss’ multiplication formula and Legendre’s relation which give the known results when $q$ tends to 1.

1 Introduction

Analogous to Gauss’ infinite product representation for the gamma function [6]

$$\Gamma(x) = x^{-1} \prod_{n=1}^{\infty} [(1 + 1/n)^x(1 + x/n)^{-1}]$$  \hspace{1cm} (1.1)

the $q$–gamma function $\Gamma_q(x)$ is defined by [4]

$$\Gamma_q(x) = \frac{(q, q)_\infty}{(q^x, q^x)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1,$$  \hspace{1cm} (1.2)

where the $q$–shifted factorials are defined by [5]

$$(a, q)_0 = 1,$$

$$(a_1, \ldots, a_r; q)_k = \prod_{i=1}^{r} \prod_{j=0}^{k-1}(1 - a_i q^j), \quad k = 0, 1, 2, \ldots ,$$

$$(a; q)_\infty = \prod_{i=0}^{\infty}(1 - a q^i).$$

This function is a $q$–anologue of the gamma function since we have

$$\lim_{q \to 1} \Gamma_q(x) = \Gamma(x)$$  \hspace{1cm} (1.3)

The $q$–gamma function satisfies the functional equation

$$\Gamma_q(x + 1) = (1 - q^x)/(1 - q)\Gamma_q(x), \quad \Gamma_q(1) = 1,$$  \hspace{1cm} (1.4)

which is a $q$–extension of the well-known functional equation

$$\Gamma(x + 1) = x\Gamma(x), \quad \Gamma(1) = 1.$$  \hspace{1cm} (1.5)

Boher, H. and Mollerup , J. (1922) proved the following theorem for $\Gamma(x)$ function

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Theorem 1. If a function $f(x)$ satisfies the following three conditions, then it is identical in its domain of definition with the gamma function:

1. $f(x + 1) = xf(x)$,
2. $f(1) = 1$,
3. The domain of definition of $f(x)$ contains all $x > 0$, and is log convex for these $x$.

For the proof of this theorem see [2]. In 1978, R. Askey [3] gave the $q$-analogy of this theorem.

Theorem 2. If a function $f(x)$ satisfies the following three conditions

1. $f(x + 1) = [x]q f(x)$ for some $q$, $0 < q < 1$,
2. $f(1) = 1$,
3. $\log f(x)$ is convex for $x > 0$,

then $f(x) = \Gamma_q(x)$, where $[x]q = \frac{1-q^x}{1-q}$.

2 The behavior of the function $\Gamma_q(x)$ for large $x$

In order to study the behavior of the function $\Gamma_q(x)$ for large $x$, we consider a function of the form

$$f(x) = (1 - q)^{1/2 - x} e^{\mu(x)}.$$  \hspace{1cm} (2.1)$$

Our goal is to make $f(x)$ satisfy the basic conditions for the gamma function by choosing $\mu(x)$ in an appropriate way.

$$\frac{f(x + 1)}{f(x)} = \frac{e^{\mu(x+1) - \mu(x)}}{1 - q}$$  \hspace{1cm} (2.2)$$

Then $f(x)$ satisfy condition (2) in Theorem (2) iff

$$\mu(x) - \mu(x + 1) = -\log(1 - q^x),$$  \hspace{1cm} (2.3)$$

holds for $\mu(x)$.

Let $g(x)$ is the write side of the equation (2.3). If we set

$$\mu(x) = \sum_{n=0}^{\infty} g(x + n)$$  \hspace{1cm} (2.4)$$

then equation (2.3) holds, provided that the series in equation (2.4) converges. In order to study the convergence, we will combine this with an approximation of the function $\mu(x)$. Let us begin by considering the expansion

$$-\log(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1$$  \hspace{1cm} (2.5)$$

If we put $z = q^x$ the expansion is valid whenever $x > 0$ and $0 < q < 1$.

$$g(x) = -\log(1 - q^x) = \sum_{n=1}^{\infty} \frac{q^{nx}}{n}.$$
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Now we can approximate $g(x)$. If the integers 1, 2, 3, ... are all replaced by 1, then the result is an infinite geometric series having $\frac{q^x}{1 - q^x}$. But $g(x)$ is positive, hence

$$0 < g(x) < \frac{q^x}{1 - q^x} \quad (2.6)$$

Since every term of the series in equation (2.4) is positive, it suffices to show the convergence of

$$\sum_{n=0}^{\infty} \frac{q^{x+n}}{1 - q^{x+n}} \quad (2.7)$$

which converges from the ratio test. This gives the approximation

$$0 < \mu(x) < \frac{q^x}{(1 - q) - q^x} \quad (2.8)$$

i.e.

$$\mu(x) = \frac{\theta q^x}{(1 - q) - q^x}, \quad (2.9)$$

where $\theta$ is a number independent of $x$ between 0 and 1.

Now let us consider the condition (3) of theorem (2). The factor $(1 - q)^{1/2 - x}$ in equation (2.1) is log convex because the second derivative of its logarithm equal to zero for all $x$. If the factor $e^{\mu(x)}$ is log convex, in other words $\mu(x)$ is convex, then $f(x)$ also satisfies condition (2.4). The function $\mu(x)$ is convex if the general term of the series $g(x + n)$ is convex. To show this, it suffices to prove the convexity of $g(x)$ itself. But we have

$$g''(x) = \frac{q^x \log^2(q)}{(1 - q)^2} > 0 \quad (2.10)$$

By a suitable choice of the constant $a$, we get

$$\Gamma_q(x) = a(1 - q)^{1/2 - x} e^{\frac{\theta q^x}{(1 - q) - q^x}} \quad (2.11)$$

If we let $x$ be an integer $n$, we get the approximation

$$[n]_q! = \Gamma_q(n + 1) = a(1 - q)^{-1/2 - n} e^{\frac{\theta q^{n+1}}{(1 - q) - q^{n+1}}} \quad (2.12)$$

Now we will determine the exact value of the constant $a$. Let $p$ be a positive integer. We consider the function

$$f(x) = [p]_q^p \Gamma_q^p((x/p)/p) \Gamma_q^p((x + 1)/p) \ldots \Gamma_q^p((x + p - 1)/p), \quad x > 0 \quad (2.13)$$

The second derivative of $\log[p]_q^x$ is zero, and $\Gamma_q^p((x + k)/p)$ is log convex $\forall k = 1, 2, 3, ...$ then $f(x)$ is log convex. Also,

$$f(x + 1) = [p]_q^p \Gamma_q^p((x + p)/p) \Gamma_q^p((x + p)/p) f(x)$$

Then the function $f(x)$ satisfies the conditions (1) and (3) in theorem (2). Then

$$[p]_q^p \Gamma_q^p((x + 1)/p) \Gamma_q^p((x + 1)/p) \ldots \Gamma_q^p((x + p - 1)/p) = a_p \Gamma_q(x), \quad (2.14)$$
where \( a_p \) is a constant depending on \( p \) and by putting \( x = 1 \), we get

\[
a_p = [p]_q \Gamma_q(1/p) \Gamma_q(2/p) \cdots \Gamma_q(p/p).
\] (2.15)

But the \( q \)-gamma function \( \Gamma_q(x) \) function is defined by

\[
\Gamma_q(x) = \lim_{n \to \infty} \frac{(q, q)_n}{(q^x, q)_n} (1 - q)^{1-x},
\] (2.16)

then

\[
\Gamma_q(k/p) = \lim_{n \to \infty} \frac{(q^k, q^p)_n}{(q, q^p)_n} (1 - q^p)^{1-k/p}
\] (2.17)

and

\[
a_p = [p]_q (1 - q^p)^{(p-1)/p} \lim_{n \to \infty} \frac{((q^p, q^p)_n)^p}{\prod_{k=1}^p (q^k, q^p)_n}
\] (2.18)

By using equation (2.12) and the relation

\[
[q]^! = \frac{(q, q)_n}{(1 - q)^n},
\] (2.19)

then

\[
((q^p, q^p)_n)^p = a^p (1 - q^p)^{-p/2} e^{\frac{\theta q^p(n+1)}{(1-q) - q^p(n+1)}}.
\] (2.20)

Also,

\[
\prod_{k=1}^p (q^k, q^p)_n = (q, q)_n p
\]

\[= a(1 - q)^{-1/2} e^{\frac{\theta q^{p+1}}{(1-q) - q^{p+1}}}.
\]

Then

\[
a_p = [p]^{1/2}_q a^{p-1}
\] (2.21)

From the equations (2.15) and (2.21), we have

\[
a_2 = [2]^{1/2}_q a, \quad a_2 = [2]_q \Gamma_q(1/2),
\] (2.22)

then

\[
a = [2]^{1/2}_q \Gamma_q(1/2)
\] (2.23)

and

\[
a_p = [p]^{1/2}_q ([2]_q \Gamma_q^2(1/2))^{(p-1)/2}
\] (2.24)

Now in this paper we get the following expressions

\[
\Gamma_q(x) = [2]^{1/2}_q \Gamma_q(1/2)(1 - q)^{1/2-x} e^{\frac{\theta q^x}{(1-q) - q^x}}, \quad 0 < \theta < 1,
\] (2.25)

\[
[q]^! = [2]^{1/2}_q \Gamma_q(1/2)(1 - q)^{-1/2-n} e^{\frac{\theta q^{n+1}}{(1-q) - q^{n+1}}}.
\] (2.26)
An asymptotic expansion of the q-gamma function $\Gamma_q(x)$.

\[ \Gamma_q(x/p)\Gamma_q((x+1)/p)\ldots\Gamma_q((x+p-1)/p) = [p]_q^{1/2-x}(2q\Gamma^2_q(1/2))^{(p-1)/2}\Gamma_q(x). \] (2.27)

In particular, for $p = 2$

\[ \Gamma_q(x/2)\Gamma_q((x+1)/2) = [2]_q^{1-x}\Gamma_q(1/2)\Gamma_q(x). \] (2.28)

The formulas in equations (2.25) and (2.26) are similar to Stirling’s formulas in the usual case. The functional equation (2.27) is called $q$–Gauss’ multiplication formula. Also, equation (2.28) is called $q$–Legendre’s relation.

If we take the limit as $q \rightarrow 1$, then we get

\[ \Gamma((x+p-1)/p) = p^{1/2-x}(2\Gamma^2(1/2))^{(p-1)/2}\Gamma(x), \] (2.29)

and

\[ \Gamma(x+1/2) = 2^{1-x}\Gamma(1/2)\Gamma(x). \] (2.30)

These relations are called Gauss’ multiplication formula and Legendre’s relation (resp.) [1].

References