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$Vect(S^1)$ Action on Pseudodifferential Symbols on S^1 and (Noncommutative) Hydrodynamic Type Systems

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Abstract

The standard embedding of the Lie algebra $Vect(S^1)$ of smooth vector fields on the circle $Vect(S^1)$ into the Lie algebra $\Psi D(S^1)$ of pseudodifferential symbols on S^1 identifies vector field $f(x)\frac{\partial}{\partial x} \in Vect(S^1)$ and its dual as $\pi(f(x)\frac{\partial}{\partial x}) = f(x)\xi$ $\pi(u(x)dx^2) = u(x)\xi^{-2}$. The space of symbols can be viewed as the space of functions on T^*S^1. The natural lift of the action of $Diff(S^1)$ yields $Diff(S^1)$-module. In this paper we demonstate this construction to yield several examples of dispersionless integrable systems. Using Ovsienko and Roger method for nontrivial deformation of the standard embedding of $Vect(S^1)$ into $\Psi D(S^1)$ we obtain the celebrated Hunter-Saxton equation. Finally, we study the Moyal quantization of all such systems to construct noncommutative systems.

Dedicated to Professor Dieter Mayer on his 60th birthday

1 Introduction

It is known that to every pseudodifferential operator F we associate its symbol – a formal Laurent series [5, 13, 14, 22]. Let $\Psi D(S^1)$ be the Lie algebra of pseudodifferential symbols on S^1 [18, 19]. It is known that the order of $\Psi D(S^1)$ is defined by ord $(F) = \{$ sup $k \in \mathbb{Z} | f_k(x) \neq 0\}$ for an element

$$F(x, \xi, \xi^{-1}) = \sum_{k\in\mathbf{Z}} \xi^k f_k(x) \in \Psi D(S^1),$$

where $f_k(x) = f_k(x + 2\pi)$ with $f_n = 0$ for sufficiently large n.

There are two natural derivations we can define on $\Psi D(S^1)$:

$$\partial_\xi : \sum_k f_k\xi^k \longmapsto \sum_k k f_k \xi^{k-1} \qquad \partial : \sum_k f_k\xi^k \longmapsto \sum_k f'_k\xi^k. \tag{1.1}$$

These allow us to define a natural Poisson bracket for $F, G \in \Psi D(S^1)$

$$\{F, G\} = \frac{\partial F}{\partial \xi}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial \xi} \tag{1.2}$$

Therefore, we define a Hamiltonian vector field associated to Poisson bracket as

$$\mathcal{X} = \{X_F \mid X_F = \frac{\partial F}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial}{\partial \xi}\}, \tag{1.3}$$

and these vector fields satisfy

$$[X_F, X_G] = X_{\{F,G\}}.$$

The Lie algebra operation on functions is just Poisson bracket.

The associative algebra of pseudodifferential symbols has a different multiplication rule defined by Adler [1]

$$(F \circ G) = \sum_{k\geq 0} \frac{1}{k!} : \frac{\partial^k F}{\partial \xi^k}\frac{\partial^k G}{\partial x^k} :$$

where : : stands for "Normal Ordering" defined as

$$: f(x)\xi^k g(x)\xi^l := f(x)g(x)\xi^{k+l}.$$

This is a natural generalization of the Wick product.

One can also define a Lie bracket with respect to $\circ$–product

$$\{F, G\} = F \circ G - G \circ F.$$

The composition law $\circ$ recovers the basic multiplication law of the pseudodifferential operators. For example, $\xi \circ u = u\xi + u'$ which recovers the basic Leibniz rule: $\partial u = u\partial + u'$.

Let us define the residue of a pseudodifferential symbol $F(x,\xi) = \sum_{j\leq N} f_j(x)\xi^j$ by res $F(x,\xi) = f_{-1}(x)$. Then one defines the Adler trace as

$$Tr(F) = \int_{S^1} Res\ F dx = \int_{S^1} f_{-1} dx, \tag{1.4}$$

where $\int$ is any linear map which annihilates derivatives. It is easy to see that $Tr\ [F, G] = 0$, since the residue of a commutator is a total derivative. The Adler trace can be used to define a symmetric bilinear form on pseudodifferential symbols

$$(F, G) \equiv \text{ Tr } F \circ G,$$

one can check that (F, G) defines an invariant bilinear nondegenerate form. Therefore, the Lie algebra $\Psi D(S^1)$ admits an analogue of the Killing form with respect to the Adler trace formula.

Our main aim is to study the action of $Vect(S^1)$ on the space of $\Psi DO(S^1)$. It is readily observable that the Lie algebra $Vect(S^1)$ is a subalgebra of $\Psi DO(S^1)$. Therefore, $\Psi DO(S^1)$ is a $Vect(S^1)$-module.

1.1 Noncommutative hydrodynamical systems

Using three entities, namely, Wigner function, Weyl transformation and Moyal $\star$ product, we can construct the Wigner-Moyal-Weyl formalism. It is a backbone of semi-classical physics.

The state of a quantum system can be represented by a real valued function of the canonical cooordinates, the Wigner function W. It is known that the $\star$ squared of a Wigner function W is proportional to itself, i.e.,

$$W \star W = \frac{1}{\hbar} W.$$

The time evolution of the system state is governed by the Wigner-Moyal equation. For a Hamiltonian of the form $H(q,p) = \frac{p^2}{2m} + V(q)$ the Wigner-Moyal equation is defined as

$$W_t = \frac{2}{\hbar}(H \star W - W \star H).$$

The equation of motion of Wigner function W is defined as

$$\partial_t W(p,q,t) = \{H, W\}_\star(p,q,t), \tag{1.5}$$

where $H = \frac{1}{2m}p^2 + V(q)$ and

$$\{H, W\}_\star = \frac{1}{\hbar}(\frac{\partial H}{\partial q} \star \frac{\partial W}{\partial p} - \frac{\partial W}{\partial q} \star \frac{\partial H}{\partial p}). \tag{1.6}$$

Equation (1.5) is also known as the symbol map of the von Neumann's equation. This can be viewed as a Moyal deformation of the classical equation. In fact, in the quantum world, the time evolution of the system's state is governed by the Wigner-Moyal equation, closely related to Eqn. (1.5).

In the classical limit, the Moyal bracket reduces to Poisson bracket. Thus, the following equation

$$\partial_t W(p,q,t) = \{H, W\}_{PB}(p,q,t), \tag{1.7}$$

captures the Liouville equation of clasical mechanics. Physically it encodes the motion of classical Fermi fluid.

Let us recall again the associative algebra of pseudodifferential symbols once again. Let us denote by $\Psi D_\hbar(S^1)$ the algebra of pseudodifferential symbols equipped with the multiplication $\star_\hbar$. The associative algebras $\Psi D_\hbar(S^1)$ are isomorphic to each other and the commutator is given by

$$\{F, G\}_\hbar := F \star_\hbar G - G \star_\hbar F, \tag{1.8}$$

where

$$F \star_\hbar G = e^{[i\hbar(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})]} F(x_1, y_1) G(x_2, y_2)|_{(x,y)},$$

and this becomes Poisson bracket for

$$lim_{\hbar \to 0}\{F, G\}_\hbar = \{F, G\}.$$

Here the Lie algebra $\Psi D(S^1)$ contracts to the Poisson algebra of Laurent series on S^1, $C^\infty(S^1) \otimes \mathbb{C}[[\xi, \xi^{-1}]$. This notion of contraction was introduced by Wigner and Inönu.

1.2 Motivation of our paper

We will take a different route to study hydrodynamical type of integrable systems [3, 13, 14, 26]. We embed smooth vector fields on the circle $Vect(S^1)$ and its dual into the Lie algebra of pseudodifferential symbols $\Psi D(S^1)$. It is known that the $\Psi D(S^1)$ is a $Vect(S^1)$ module, and the standard coadjoint action is replaced by the Poisson action of $Vect(S^1)$ on $\Psi D(S^1)$. This allows us to construct several dispersionless systems [3,23]. One of main motivation of this article is to study dispersionless type systems through Virasoro action.

Ovsienko and Roger [18, 19] formulated the theory of deformations of the standard embedding of the Lie algebra of smooth vector fields on the circle $Vect(S^1)$ into the Lie algebra of pseudodifferential symbols $\Psi D(S^1)$. This again allows us to study different Hamiltonian flows under action of this deformed embedding on its dual, in particular, we obtain the Hunter-Saxton equation [7, 8].

One of the main thrust of this paper is to study the noncommutative framework [4, 6] of all these hydrodynamical systems. It is difficult to construct noncommutative systems from the orbit theory method. Since deformation quantization [16, 20] does not incorporate dynamics, it is a homotopy theory. In this paper we bypass this problem in following way: We lift the action of $Vect(S^1)$ on functions of T^*S^1. It is defined by Poisson actions. Finally we quantize these Poisson brackets via deformation of quantization.

1.3 Organization

This paper is organised as follows: In Section 2 we summarize the main ideas of $Vect(S^1)$ action on the space tensor densities and its action on T^*S^1. In Section 3 we describe the construction of dispersionless KdV equation. We study the Lax pair of dKdV equation. We extend this method to coupled dispersionless systems in Section 4. In Section 5 we study the deformation of standard embedding la Ovsienko and Roger, and this leads to the construction of the Hunter-Saxton equation. In Section 6 we study the noncommutative analogue of our construction, and this yields various noncommutative integrable systems.

2 Preliminaries: $Vect(S^1)$ action on tensor densities

Let us denote $\mathcal{F}_\mu(S^1)$ be the space of μ-densities on S^1, given as

$$\varphi = \phi(x)(dx)^\mu,$$

where μ denotes weight and $\phi(x) \in C^\infty(S^1)$. Thus, as a vector space $\mathcal{F}_\mu(S^1)$ is isomorphic to $C^\infty(S^1)$.

There exists a pairing between $\mathcal{F}_\mu$ and $\mathcal{F}_{1-\mu}$, defined as

$$< f(x)(dx)^\mu, g(x)(dx)^{1-\mu} >:= \int_{S^1} f(x)g(x)\, dx. \tag{2.1}$$

This inner product is $Diff(S^1)$ invariant. This exactly coincides with Virasoro pairing when $\mu = 2$, where we have identified $(dx)^{-1} = \frac{d}{dx}$. It must be noticed that the space $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_{-1}$ are isomorphic to $C^\infty(S^1)$, the space of 1-form and the space of vector fields on S^1 respectively.

Let us consider the space of linear differential operators

$$L : \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\mu$$

with arbitrary weights λ and μ. It is customary to denote this space by $\mathcal{D}_{\lambda,\mu}(S^1)$. The subspace of differential operators of order $\leq n$ is denoted by $\mathcal{D}^k_{\lambda,\mu}(S^1)$.

Let us consider the natural $Vect(S^1)$ action on $\mathcal{D}^k_{\lambda,\mu}(S^1)$. The module $\mathcal{D}^2_{-\frac{1}{2},\frac{3}{2}}(S^1)$ is a well known module of second order differential operators, the Sturm-Liouville operators belong this class.

Definition 2.1. *The action of $Vect(S^1)$ action on the space of nth-order differential operators is given by the commutator of $A \in \mathcal{D}^k_{\lambda,\mu}(S^1)$ and the Lie derivative*

$$\mathcal{L}^{\lambda,\mu}_X(A) = L^\mu_X \circ A - A \circ L^\lambda_X, \tag{2.2}$$

where $X = f(x)\frac{d}{dx} \in Vect(S^1)$ and the Lie derivative is defined as

$$L^\mu_X = f(x)\frac{d}{dx} + \mu f'(x).$$

The classification of invariant bilinear operators on tensor fields is due to P. Grozman. Suppose we concentrate on first order operators. Then, for every first order operator on one dimensional manifold M

$$\{.,.\} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\lambda+\mu+1}$$

we obtain

$$\{f(x)(dx)^\lambda, g(x)(dx)^\mu\} = (\lambda f(x)g'(x) - \mu f'(x)g(x))(dx)^{\lambda+\mu+1}, \tag{2.3}$$

where x is a local coordinate on M. This operator is given by the Poisson bracket on T^*S^1.

2.1 Lifting of $Vect(S^1)$ action on T^*S^1

The space of symbols can be viewed as the space of functions on T^*S^1/S^1. The natural lifting of $Diff(S^1)$ action on T^*S^1 equips the space of functions $P = \sum_{p=1}^k a_{k-p}\xi^{k-p}$ with a structure of $Diff(S^1)$-module; this action coincides with the $Diff(S^1)$-action on the space of symbols.

The space $\mathcal{F}_\lambda$ can also be viewed as the space of functions on the cotangent bundle T^*S^1/S^1 homogeneous of degree $-\lambda$. This can be expressed in Darboux coordinates (x, ξ) as follows:

$$f(x)dx^\lambda \longleftrightarrow f(x)\xi^{-\lambda}$$

Therefore, Equation (2.3) can be computed via (1.2), which yields

$$\{f(x)\xi^{-\lambda}, g(x)\xi^{-\mu}\} = (\lambda f(x)g'(x) - \mu f'(x)g(x))\xi^{-(\lambda+\mu+1)}.$$

It is clear now that the action $Vect(S^1)$ on the space of symbol is expressed in terms of Poisson action.

3 Toy Model: Construction of dispersionless KdV equation

Let us study few examples of this construction. A 1-parameter family of $Vect(S^1)$ actions on $C^\infty(S^1)$ is given by

$$L^{(\lambda)}_{f(x)\frac{d}{dx}} a(x) = f(x)a'(x) + \lambda f'(x)a(x), \tag{3.1}$$

where $\lambda \in \mathbf{R}$. $L^{(\lambda)}_{f(x)\frac{d}{dx}}$ is considered to be the action of Lie derivative on $\mathcal{F}_\lambda$, tensor densities of degree λ, i.e. $a = a(x)dx^\lambda \in \mathcal{F}_\lambda$. This action can be realized also from the Poisson action.

Let us consider the standard embedding $\pi(X) = X\xi$ of $Vect(S^1)$ into the associative algebra of pseudodifferential symbols $\Psi D(S^1)$. The dual space of $Vect(S^1)$ can be identified to the set

$$\{udx^2)|\ u \text{ is a quadratic differential }\}.$$

Then a pairing between a point $f\frac{d}{dx} \in Vect(S^1)$ and a point $udx^{\otimes 2} \in (Vect(S^1))^*$ is given by

$$< f\frac{d}{dx}, udx^{\otimes 2} >_{L^2} = \int_{S^1} f(x)u(x)\ dx. \tag{3.2}$$

The standard embedding of the dual of $Vect(S^1)$ becomes

$$u(x)dx^2 \longleftrightarrow u(x)\xi^{-2} \in \mathcal{F}_2$$

and the coadjoint action of $Vect(S^1)$ on its dual $u(x)dx^2$ is replaced by the Poisson action of $f(x)\xi$ on $u(x)\xi^{-2}$.

Let us consider the subspace of $\Psi D(S^1)$ consisting of functions $\{a(x)\xi^\lambda\}$ of degree λ in ξ. This subspace is $Vect(S^1)$ module and isomorphic to $\mathcal{F}_{-\lambda}$. Hence, one can rewrite Equation (3.1) as

$$\{f(x)\xi, a(x)\xi^\lambda\} = (fa' - \lambda f'a)\xi^\lambda = L^{(-\lambda)}_{f(x)\frac{d}{dx}}(a)\xi^\lambda, \tag{3.3}$$

where $\pi(f(x)\partial) = f(x)\xi$. This embedding attributes a passage from Virasoro algebra to the algebra of groups of area preserving diffeomorphisms.

Definition 3.1. *An evolution equation is called Hamiltonian if it can be written in the form*

$$u_t = \mathcal{O}\frac{\delta H}{\delta u}[u],$$

where $\mathcal{O}$ is a Hamiltonian operator. The two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are said to form a Hamiltonian pair if every linear combination $a\mathcal{O}_1 + b\mathcal{O}_2$ for a, b constants is a Hamiltonian operator.

Given a pair of Hamiltonian operators $\mathcal{O}_1$, $\mathcal{O}_2$ it can be shown that the recursion operator

$$\mathcal{R} = \mathcal{O}_2 \mathcal{O}_1^{-1}$$

generates hierarchies of local symmetries for the integrable system of PDEs. In fact, this recursion operator maps symmetries to symmetries, offers a natural way to construct the whole infinite hierarchy fron a single seed system.

Lemma 3.1. *The Hamiltonian operator $\mathcal{O}$ corresponding to the action of $Vect(S^1)$ on $u(x)\xi^{-2} \in \Psi D(S^1)$ yields*

$$\mathcal{O} = -(\partial u + u\partial). \tag{3.4}$$

Proof. It is known that the space $\Psi D(S^1)$ is a $Vect(S^1)$ module. Hence, it acts on $\Psi D(S^1)$ by

$$ad^*_{f(x)\xi} u(x)\xi^{-2} = -\{f(x)\xi, u(x)\xi^{-2}\}$$

$$= -(\frac{\partial}{\partial \xi}(f(x)\xi)\frac{\partial}{\partial x}(u\xi^{-2} - \frac{\partial}{\partial x}(f(x)\xi)\frac{\partial}{\partial \xi}(u\xi^{-2})$$

$$= -(fu' + 2f'u)\xi^{-2}.$$

Thus we obtain the Hamiltonian operator from this. ■

3.1 Euler-Poincaré flow

Let us give a rapid introduction to Euler-Poincaré framework [2, 15]. Let I be an inertia operator, defined as

$$I : \mathfrak{g} \longrightarrow \mathfrak{g}^*.$$

Therefore, a point $\mu \in \mathfrak{g}^*$ evolves by

$$\frac{d\mu}{dt} = (I^{-1}\mu).\mu, \tag{3.5}$$

where the right hand side denotes the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This equation is called the Euler-Poincaré equation.

Definition 3.2. *The Euler-Poincaré equation on $\mathfrak{g}^*$ corresponding to the Hamiltonian $H(\mu) = \frac{1}{2} < I^{-1}\mu, \mu >$ is given by*

$$\frac{d\mu}{dt} = -ad^*_{I^{-1}\mu}\mu.$$

It characterizes an evolution of a point $\mu \in \mathfrak{g}^*$.

Thus the inertia operator maps

$$I \ : \ u(x)\xi \longmapsto u(x)\xi^{-2},$$

and the corresponding Hamiltonian is given by

$$H = \frac{1}{2} Res < u(x)\xi, u(x)\xi^{-2} > .$$

We have to rewrite the definition of Euler-Poincaré flow for our case.

Definition 3.3. *The Euler-Poincaré equation induced by the action of $Vect(S^1)$ on its dual is given by*

$$u_t = -ad^*_{(u(x,t)\xi)} \frac{\delta H}{\delta u} \equiv \{u(x,t)\xi, \frac{\delta H}{\delta u}\}. \tag{3.6}$$

We have already computed the Hamiltonian operator or coadjoint operator induced from the action of $f(x)\xi \in Vect(S^1)$ on its dual $u(x)\xi^{-2}$ in Lemma 1. Thus we obtain:

Proposition 3.1. *The Euler-Poincaré flow on $\mathcal{F}_2$, dual space of $Vect(S^1)$, yields the dispersionless KdV equation*

$$u_t + 3uu_x = 0, \tag{3.7}$$

for $H = \frac{1}{2}\int_{S^1} u^2 \, dx$.

Proof. This is a straight forward calculation. ∎

Remark 3.1. *The dispersionless KdV equation is also known as Riemann equation, the prototype for the hyperbolic systems. This equation is a integrable Hamiltonian system. Dispersionless equations can be obtained by construction or as a quasi-classical limit of integrable ones. In the latter case we perform scaling $\frac{\partial}{\partial t} \to \epsilon\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x} \to \epsilon\frac{\partial}{\partial x}$ and take the limit $\epsilon \to 0$.*

3.2 Lax Pair

If we take a close look to equation (3.6) then it is clear that it is a special case of a Adler-Kostant-Symes flow [1, 21, 25] on the space of pseudo differential symbols

$$\frac{\partial \Omega}{\partial t} = \{\Psi, \Omega\}, \tag{3.8}$$

where $\Omega \in \Pi^-(\Psi D(S^1))$ and $\Psi \in \Pi^+(\Psi D(S^1))$.

The Lax pair of the equation (3.8) is given by

$$L\phi = \lambda\phi, \qquad \partial_t\phi + A\phi = 0, \tag{3.9}$$

where

$$L\phi = \{\Omega, \phi\}, \quad A\phi + \{\Psi, \phi\} = 0.$$

Compatibility of the above yields

Proposition 3.2.

$$\partial_t L + [A, L] = \partial_t \Omega - \{\Psi, \Omega\} = 0.$$

Proof. (Outline):

$$(\partial_t L)\phi = \{(\partial_t \Omega), \phi\},$$
$$[A, L]\phi = -\{\Psi, \{\Omega, \phi\}\} + \{\Omega, \{\Psi, \phi\}\}$$
$$= \text{ Jacobi identity } + \{\phi, \{\Psi, \Omega\}\}.$$

■

4 Virasoro action and coupled dispersionless systems

Let us consider the Virasoro action on the dual of a semi-direct product spaces $Vect(S^1) \ltimes C^\infty$,

Let us study first the action of $Vect(S^1)$ on $u(x)\xi^{-2} + v$.

Proposition 4.1. *The Euler-Poincaré flow induced by the action of $f(x)\xi \in Vect(S^1)$ on dual space of the semi-direct product space $Vect(S^1) \ltimes C^\infty(S^1)$ yields*

$$\begin{aligned} u_t + 3uu_x &= 0, \\ v_t + uv_x &= 0 \end{aligned} \tag{4.1}$$

where $u(x)\xi^{-2} + v \in (Vect(S^1) \ltimes C^\infty(S^1))^*$ and the Hamiltonian $H = \frac{1}{2}u^2$.

Proof. Let us first compute

$$\{f\xi, u\xi^{-2} + v\} =$$
$$\frac{\partial(f\xi)}{\partial \xi}\frac{\partial(u\xi^{-2} + v)}{\partial x} - \frac{\partial(f\xi)}{\partial x}\frac{\partial(u\xi^{-2} + v)}{\partial \xi}$$
$$= [(\partial u + u\partial)f]\xi^{-2} - [v_x f]$$

Thus,

$$\frac{\partial(u(x)\xi^{-2} + v)}{\partial t} = -ad^*_{u\xi}(u\xi^{-2} + v) \equiv -\{u(x)\xi, u(x)\xi^{-2} + v\}$$

yields the above equation. ■

Proposition 4.2. *The Euler-Poincaré flow induced by the action of $f(x)\xi \in Vect(S^1)$ on $u(x)\xi^{-2\alpha} + v(x)\xi^{-2\alpha-1}$ yields*

$$\begin{aligned} u_t + (2\alpha + 1)uu_x &= 0, \\ v_t + uv_x + (2\alpha + 1)vu_x &= 0 \end{aligned} \tag{4.2}$$

If we substitute

$$\frac{v}{u} = \lambda \tag{4.3}$$

in above equation, we obtain a two component dispersionless KdV equation

$$\begin{aligned} u_t + (2\alpha + 1)uu_x &= 0, \\ \lambda_t + (\lambda u)_x &= 0 \end{aligned} \tag{4.4}$$

Proof. By direct computation. ■

This system is a bihamiltonian system with operators

$$\mathcal{O}_1 = -\begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}$$

$$\mathcal{O}_2 = -\begin{pmatrix} \partial u + u\partial & 0 \\ 0 & \partial u \end{pmatrix}$$

In principle we can extend the Poisson action of $f(x)\xi$ to more general space, say, $u\xi^{-2} + v + w\xi$. This yields coupled equations of three variables. These are dispersionless analog of multicomponent KdV systems. In [24] Svinolupov studied N-component KdV-type equations of the form

$$u^i_t = u^i_{xxx} + a^i_{jk}u^j u^k, \qquad i, j = 1 \cdots N, \tag{4.5}$$

where the fields u^i depend on x and t alone and the a^i_{jk} are constants, symmetric in the lower indices. These systems are examples of equations of hydrodynamic type.

Remark 4.1. *It must be worth to say that all the two component hydrodynamic systems we have obtained here do not fall into the category of two component hyperbolic systems. The two component hyperbolic systems are given as [17]*

$$u_t = H_{uv}u_x + H_{vv}v_x$$

$$v_t = H_{uu}u_x + H_{uv}v_x,$$

where H is the Hamiltonian. The Born-Infeld equation and the polytropic gas dynamics equation are the most important equations of this class.

4.1 The dispersionless Ito equations

In this section we consider the action of $Vect(S^1) \ltimes C^\infty(S^1)$ on the space of pseudo differential symbols. We give a derivation of the dispersionless Ito equation starting from the Virasoro action on the space of differential operators.

Definition-Proposition 1.

$$\begin{aligned} &ad^*_{(f(x)\xi, g(x))}(u(x)\xi^{-2} + v(x)\xi^{-1}) \\ &= \{f(x)\xi, u(x)\xi^{-2} + v(x)\xi^{-1}\} + \{g(x), v(x)\xi^{-1}\} \end{aligned} \tag{4.6}$$

Lemma 4.1.

$$\begin{aligned} &ad^*_{(f(x)\xi, g(x))}(u(x)\xi^{-2} + v(x)\xi^{-1}) \\ &= -[(\partial u + u\partial)f]\xi^{-2} - [(v\partial)g]\xi^{-2} - [(\partial v)f]\xi^{-1} \end{aligned} \tag{4.7}$$

Proof. By direct computation. ■

Lemma 4.2. *The Hamiltonian operator associated to (4.7) is given by*

$$\mathcal{O} = -\left(\begin{array}{c|c} \partial u + u\partial & v\partial \\ \hline \partial v & 0 \end{array}\right) \tag{4.8}$$

Theorem 4.1 (Folklore)**.** *The Euler-Poincaré flow on $\mathcal{F}_2 \oplus \mathcal{F}_1$ generated by action of $Vect(S^1)$ satisfies dispersionless Ito equation*

$$\begin{aligned} u_t + 3uu_x + vv_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned} \tag{4.9}$$

for the Hamiltonian $H = \frac{1}{2}\int(u^2 + v^2)\, dx$.

Proof. It follows directly from the following formula

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = -\begin{pmatrix} \partial u + u\partial & v\partial \\ \partial v & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

■

Remark 4.2. *The dispersionless Ito equation admits a bihamiltonian structure*

$$\mathcal{O}dH[u,v] = \mathcal{O}_1 dH_1[u,v],$$

where the first Hamiltonian operator is defined as

$$\mathcal{O}_1 = -\begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}$$

and $H_1 = \frac{1}{2}\int(u^3 + uv^2)\, dx$.

Theorem 4.2. *The Euler-Poincaré flow induced by the action of $Vect(S^1)$ on the dual space $\mathcal{F}_2 \oplus \mathcal{F}_3$ satisfies dispersionless coupled equation*

$$\begin{aligned} u_t + 3uv_x + u_x v &= 0 \\ v_t + 2uu_x + 4vv_x &= 0 \end{aligned} \tag{4.10}$$

where $u(x)\xi^{-2} + v(x)\xi^{-3} \in \mathcal{F}_2 \oplus \mathcal{F}_3$ and $H = \int uv\, dx$

Proof. Let us first compute the following action

$$\begin{aligned} &ad^*_{(f(x)\xi, g(x))}(u(x)\xi^{-2} + v(x)\xi^{-3}) \\ &= -\{f(x)\xi, (u(x)\xi^{-2} + v(x)\xi^{-3}\} - \{g(x), (u(x)\xi^{-2})\} \\ &= -[(\partial u + u\partial)f]\xi^{-2} - [(2v\partial + \partial v)f]\xi^{-3} - [(2u\partial)g]\xi^{-3}. \end{aligned}$$

Thus the Hamiltonian operator is

$$\mathcal{O} = -\left(\begin{array}{c|c} \partial u + u\partial & 0 \\ \hline 2v\partial + \partial v & 2u\partial \end{array}\right)$$

Using the theorem Euler-Poincaré we obtain our desired equation. ■

5 Deformation of standard Embedding of vector fields and integrable systems

In this Section we will study the Hunter-Saxton equation from flow associated to deformation of standard Embedding of vector fields. The Hunter-Saxton equation

$$u_{xt} = uu_{xx} + \kappa u_x^2 \tag{5.1}$$

has a number of applications in the nonlinear instability theory of a director field of a liquid crystal. It has also appeared in Einstein-Weyl geometry. Hunter and Zheng [8] have given a very thorough treatment of (5.1), including the introduction of a Hamiltonian structure and proof of complete integrability. Here we will rederive the Hunter-Saxton equation using another method.

Most recently, Ovsienko and Roger have studied linear maps

$$\pi^t : Vect(S^1) \longrightarrow \Psi D(S^1)[[t]]$$

to the Lie algebra of series of formal parameter t, where

$$\pi^t = \pi + t\pi_1 + t^2\pi_2 + \cdots ,$$

and $\pi_k : Vect(S^1) \longrightarrow \Psi D(S^1)$ are some linear maps. The condition of homomorphism reads

$$\pi^t([X,Y]) = [\pi^t(X), \pi^t(Y)].$$

A polynomial deformation of the following form

$$\pi(c) = \pi + \sum_{k\in\mathbb{Z}} \pi_k(c)\xi^k,$$

where $\pi_k(c) : Vect(S^1) \longrightarrow C^\infty(S^1)$ satisfy $\pi_k(0) = 0$ and $\pi_k \equiv 0$ for suffiently large k. It defines a Lie algebra homomorphism

$$\pi(c) : Vect(S^1) \longrightarrow \Psi D(S^1),$$

where $\Psi D(S^1)$ is the space of pseudo-differential operators on S^1.

It was shown by Ovsienko and Roger [18, 19] that every infinitesimal deformation of the standard embedding of $Vect(S^1)$ into $\Psi D(S^1)$ is equivalent to

$$f(x)\partial \longmapsto f\xi + c_0 f' + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdots = \tilde{F},$$

where the standard canonical embedding is $\pi(f(x)\partial) = f(x)\xi$.

In this section we study the Euler-Poincaré flow induced by the action of deformation of vector fields for $c_0 = 0$.

Proposition 5.1. *The Euler-Poincaré flow induced by the action of the infinitesimal deformation of the embedding of* $Vect(S^1)$ $f(x)\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdot$ *on the dual of the* $Vect(S^1)$, $u(x)\xi^{-2}$ *yields a dispersionless KdV and a system of stationary partial differential equations with respect to Hamiltonian* $H = \frac{1}{2}\int_{S^1} u^2\, dx$.

Proof. Let us compute

$$[u(x)\xi^{-2}]_t = \{f(x)\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdots, u(x)\xi^{-2}\}$$
$$= \frac{\partial(f\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdot)}{\partial x}\frac{\partial(u\xi^{-2})}{\partial \xi} - \frac{\partial(f\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdot)}{\partial \xi}\frac{\partial(u\xi^{-2})}{\partial x}$$
$$= (-2uf' - fu')\xi^{-2} - c_1(2uf''' + f''u')\xi^{-4} + \cdots$$

Thus using the Euler-Poincaré theorem and $H = \frac{1}{2}\int_{S^1} u^2\, dx$ we obtain

$$u_t = 3uu_x \qquad \xi^{-2}$$
$$2uu_{xxx} + u_{xx}u_x = 0 \qquad \xi^{-4}$$

etc. ■

Corollary 1. *The second equation denotes the stationary Hunter-Saxton type equation* $u_{xxt} = 2uu_{xxx} + u_{xx}u_x$ *for* $\kappa = 1$.

5.1 Construction of the Hunter-Saxton equation

Our next aim is to compute the full Hunter-Saxton equation. In order derive this, we are compelled to change the action of the defomed embedding $f(x)\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdot$ on $u(x)\xi^{-2}$ to $u(x)\xi^{-2} + u''\xi^{-4} + \cdots$.

In fact,

$$u(x)dx^2 \longmapsto u(x)\xi^{-2} + u''\xi^{-4} + \cdots$$

can be viewed as an infinitesimal deformation of the dual of Virasoro algebra $u(x)dx^2 \in Vect(S^1)$.

Proposition 5.2. *The second member of the Euler-Poincaré flows induced by the action of the infinitesimal deformation of the embedding* $f(x)\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdot$ *on* $u(x)\xi^{-2} + u''\xi^{-4} + \cdots$ *yields the dispersionless KdV, the Hunter-Saxton type equation etc. with respect to Hamiltonian* $H = \frac{1}{2}\int_{S^1} u^2\, dx$.

Proof. It follows directly from

$$[u(x)\xi^{-2} + u''\xi^{-4} + \cdots]_t =$$
$$\{f(x)\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2} + \cdots, u(x)\xi^{-2} + u''\xi^{-4} + \cdots\}.$$

Again, using the Euler-Poincaré theorem and $H = \frac{1}{2}\int_{S^1} u^2\, dx$ we obtain

$$u_t = 3uu_x \qquad \xi^{-2}$$
$$u_{xxt} = 2uu_{xxx} + u_{xx}u_x = 0 \qquad \xi^{-4}$$

etc.

Therefore, the second equation (equating ξ^{-4}) yields the Hunter-Saxton equation. ■

Thus, we obtain a "new derivation" of the Hunter-Saxton equation. Hence we have given a nice application of the Ovsienko-Roger construction.

6 Construction of Noncommutative dispersionless KdV

In this Section we will show how to obtain noncommutative (dispersionless) integrable systems using our method. We will explain the construction through examples.

In the noncommutative regime we replace the ordinary multiplication by $\star$ product, and the Poisson bracket is replaced by Moyal bracket. Thus we obtain Moyal-Wigner type quantization of our systems.

The Moyal product is defined by

$$F \star G = \sum_{0}^{\infty} \frac{\kappa^m}{m!} \sum_{i=0}^{m} (-1)^i \frac{m!}{(m-i)!i!} \frac{\partial^m F}{\partial \xi^{m-i} \partial x^i} \frac{\partial^m G}{\partial x^{m-i} \partial \xi^i}$$

It is clear that all the associative algebras $\Psi D_M(S^1)$ are isomorphic to each other, but the commutator is given by Moyal bracket.

The Moyal bracket is defined by

$$\{F, G\}_{Moyal} := \frac{F \star G - G \star F}{\kappa}. \tag{6.1}$$

In the noncommutative case $Vect(S^1)$ acts on the extended $\Psi D(S^1)$ by

$$\{f(x)\xi, u(x)\xi^{-2}\}_{Moyal} = -(f \star u' + 2f' \star u)\xi^{-2}.$$

Definition 6.1. *The Moyal deformed Euler-Poincaré flow on the space of quadratic differentials with respect to Hamiltonian*

$$H(u) = \frac{1}{2} Res < u(x)\xi, u(x)\xi^{-2} >$$

is defined as

$$(u(x)\xi^{-2})_t = \{u(x)\xi, u(x)\xi^{-2}\}_{Moyal}. \tag{6.2}$$

Therefore using Equation (6.2) we obtain the noncommutative dispersionless KdV

$$u_t + u \star u_x + 2u_x \star u = 0. \tag{6.3}$$

Hence, we have given a new construction of ncKdV equation.

Similarly we can compute the Moyal deformation of the other coupled dispersionless KdV systems. Let us give one example, Moyal deformation of the dispersionless Ito equation.

Let us consider the Moyal action of $Vect(S^1) \ltimes C^\infty(S^1)$ on its dual.

Definition 6.2. *the Moyal action of* $Vect(S^1) \ltimes C^\infty(S^1)$ *on* $\mathcal{F}_2 \oplus \mathcal{F}_1$ *is given by*

$$(f(x)\xi, g(x)) \star (u(x)\xi^{-2} + v(x)\xi^{-1})$$

$$= \{f(x)\xi, u(x)\xi^{-2} + v(x)\xi^{-1}\}_{Moyal} + \{g(x), v(x)\xi^{-1}\}_{Moyal}$$

From a simple calculation we obtain

$$\{f(x)\xi, u(x)\xi^{-2} + v(x)\xi^{-1}\}_{Moyal} + \{g(x), v(x)\xi^{-1}\}_{Moyal}$$

$$= -(2f_x \star u + f \star u_x + g_x \star v)\xi^{-2} - (f_x \star v + f \star v_x)$$

Proposition 6.1. *The Moyal deformed Euler-Poincaré flow on $\mathcal{F}_2 \oplus \mathcal{F}_1$ yields the Moyal defomed Ito system*

$$\begin{aligned} u_t + 2u_x \star u + u \star u_x + v_x \star v &= 0\,, \\ v_t + u_x \star v + u \star v_x &= 0 \end{aligned} \tag{6.4}$$

for $H = \frac{1}{2}(u^2 + v^2)\, dx$.

Proof. Our result follows from

$$(u\xi^{-2} + v(x)\xi^{-1})_t = \{u(x)\xi + v(x), u(x)\xi^{-2} + v(x)\xi^{-1}\}_{Moyal}.$$

Equating ξ^{-2} and ξ^{-1} terms we obtain our desired result. ■

6.1 Lax Pair of noncommutative dispersionless systems

Let us study the Lax pair of the following deformed equation

$$\frac{\partial \Omega}{\partial t} = \{\Psi, \Omega\}_{Moyal}, \tag{6.5}$$

for $\Psi = u(x)\xi$ and $\Omega = u(x)\xi^{-2}$ we recover the noncommutative analogue of dispersive KdV equation. Similarly, for noncommutative dispersionless Ito equation $\Psi = u(x)\xi + v(x)$ and $\Omega = u(x)\xi^{-2} + v(x)\xi^{-1}$.

In fact, Equation (6.5) is closely related to the Lax pair of the original equation.

$$L_\hbar \phi = \lambda \phi, \qquad \partial_t \phi + A_\hbar \phi = 0, \tag{6.6}$$

where

$$L_\hbar \phi = \{\Omega, \phi\}_{Moyal}, \quad A_\hbar \phi + \{\Psi, \phi\}_{Moyal} = 0.$$

Compatibility yields

Proposition 6.2.

$$\partial_t L_\hbar + [A, L]_\hbar = \partial_t \Omega - \{\Psi, \Omega\}_{Moyal} = 0.$$

Proof. It is clear that the Moyal deformation preserves the skew symmetric condition, then rest of the proof follows from Section (3.2). ■

7 Discussion and outlook

In this paper we have studied various hydrodynamic type systems (mostly integrable) from the action of $Vect(S^1)$ on the functions on T^*S^1. This is a Poisson action and gives rise to dispersionless KdV and coupled KdV equations.

This construction can be extended to study dispersionless superintegrable equations. For example, we know from the Kirillov's local Lie algebra theory [9, 10, 11] that an element of a vector field and its "square root" form a super algebra. Using this super algebra theory we are able to construct a dispersionless super-KdV systems [12].

We have examined the Ovsienko-Roger construction of nontrivial deformations of the standard embedding of $Vect(S^1)$ of smooth vector fields on a circle into the Lie algebra $\Psi D(S^1)$ of pseudodifferential symbols on S^1. This has enabled us to study Hunter-Saxton equation associated to such embeddings. In future, our aim will be to find the analogous embeddings of $Vect(S^{1|1})$ into the Lie algebra of (super) pseudodifferential symbols on supercircle $S^{1|1}$. We would also like to study the super analogue of the Ovsienko-Roger construction and its connection to super-hydrodynamic type systems.

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