GRAPH COLORINGS APPLIED IN SCALE-FREE NETWORKS

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Abstract—**Building up graph models to simulate scale-free networks is an important method since graphs have been used in researching scale-free networks and communication networks, such as graph colorings can be used for distinguishing objects of communication and information networks. In this paper we determine the avdtc chromatic numbers of some models related with researching networks.**

Keywords—scale-free networks; nodes; links; models; vertex distinguishing total colorings

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I. INTRODUCTION

In the communication network, to prevent the network between different site signal frequency of resonance, it must ensure that between different sites have different emission frequency. Frequency assignment problem produced in the rapid development of mobile communication. Because of the customer have increased dramatically, leading to a confliction between the increasing customer and the limited expansion of communication network resources outstanding. To solve the frequency assignment problem, the domestic scholars put forward the concept of adjacent vertex distinguishing edge coloring for the first time [6]. In 1993, Burris introduced the notation of vertex distinguishing edge coloring in his Ph. D. Dissertation.

Graph coloring theory is one of the most actively branch in graph theory. It involves in many fields, such as such as physics, chemistry, computer science, network theory, social science, etc in a wide range of applications. For example, time tabling and scheduling, frequency assignment, register allocation, labeling a point set, computer security, electronic banking, coding theory, communication network, and logistics and so on. These coloring is presented in the environment about how to solve the practical problems meet in computer science (such as the description of contacting between point and point in space database), up to now, it is concerned by more and more researchers. Vertex-distinguishing total coloring and adjacent vertex distinguishing total coloring are studied in [5]-[7].

Conjecture 1. [7] Every simple graph G on order $n\geq 2$ has its adjacent vertex distinguishing total chromatic number (avdtc chromatic number) $\chi''_{as}(G)\leq\Delta(G)+3$.

It seems quite difficult to settle down this conjecture, since it has been verified by several special classes of graphs up to now. Meanwhile, no counterexamples to the conjecture were found. For simple graphs having maximum degree three. Chen [2], Wang [3] and Hulgan [4], independently, confirm positively the conjecture by showing the avdtc chromatic numbers of simple graphs having maximum three is at most 6. However, Hulgan [4] pointed out: although complete graphs of odd order show the conjecture bound is sharp for even maximum degree, many maximum degree three graphs, including the K_4,$K_{3,3}$ and Petersen graphs, have avdtcs with only 5 colors.

He proposed a problem: For a graph G with $\Delta(G)=3$, is the bound $\chi''_{as}(G)\leq\Delta(G)+3$ sharp? In our knowledge, no graphs having maximum degree three and avdtc chromatic number six were reported in current literature.

We use standard terminology and notation of graph theory. Graphs mentioned are finite, undirected, and have no multiple edges and loopless, we call them simple graphs herein. By $[m, n]$ we denote a set of consecutive integers $\{m, m+1, \ldots, n\}$ with $n>m\geq 0$; and $N(u)$ indicates the set of vertices adjacent to a vertex u, $N_e(u)$ is the set of edges incident to the vertex u in a graph. A Δ-vertex of a graph is a vertex of maximum degree. For a simple graph G we call a mapping $f: V(G)\cup E(G)\rightarrow\{1, 2, \ldots, k\}$ a proper total k-coloring of G if $f(u)\neq f(v)$ for adjacent or incident $u, v\in V(G)\cup E(G)$. Let $C(f, u)=\{f(e): e\in N_e(u)\}$, $C\langle f, u\rangle=\{f(x): x\in N(u)\}\cup\{f(u)\}$, $C[f, u]=C(f, u)\cup\{f(u)\}$, and $C_2[f, u]=C(f, u)\cup C\langle f, u\rangle$. Furthermore, let $C\{f, u\}=\{C(f, u), C\langle f, u\rangle, C[f, u], C_2[f, u]\}$ for every $u\in V(G)$. We define the distinguishing constraints as:

(1) Typical local-constraint: for an edge $uv\in E(G)$ the notation $C\{f;u\}\neq C\{f;v\}$ stands for the four proper distinguishing constraints $C(f, u)\neq C(f, v)$, $C\langle f, u\rangle\neq C\langle f, v\rangle$, $C[f, u]\neq C[f, v]$ and $C_2[f, u]\neq C_2[f, v]$ holding at same time.

(2) Complete local-constraint $C^2\{f; uv\}=\{A\neq B: A\in C\{f;u\}, B\in C\{f;v\}\}$ for every edge $uv\in E(G)$.

Definition 1. Let $f: V(G)\cup E(G)\rightarrow[1, k]$ be a proper total coloring of a simple graph G. If $C[f, u]\neq C[f, v]$ for every edge $uv\in E(G)$, then we called f an adjacent vertex distinguishing total k-coloring (k-*avdtc*) of G. The minimum number of k colors required for which G admits a k-avdtc is denoted as $\chi''_{as}(G)$.

Definition 2. A bicyclic graph $G=G[C_m, C_n]$ is a graph which it satisfies:

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(a) G is planar and contains a cycle $C_m= x_1x_2\ldots x_mx_1$ and a cycle $C_n=y_1y_2\ldots y_my_1$, $|V(C_m)|=m$, $|V(C_n)|=n$;

(b) $E(G)= E(C_m)\cup E(C_n)$ and $|V(G)|\leq m+n-2$;

(c) If $u\in V(C_m)\cap V(C_n)$), then $d_G(u)=4$, otherwise, $d_G(u)=2$.

Obviously, a bicyclic graph G has $\Delta(G)=4$, $\delta(G)=2$ and no vertex of degree 1 and vertex of degree 3.

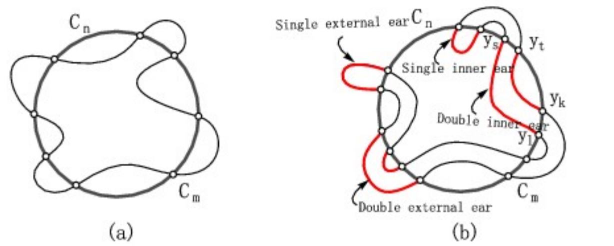


Figure 1. (a) A bicyclic graph has no double ear; (b) A intersecting cycle $G=G[C_m,C_n]$.

Definition 3. A bicyclic graph $G=G[C_m,C_n]$ is an intersecting cycle if it holds:

(a) If $|V(G)|=m+n-2$, then G contains a single inner ear and a single external ear;

(b) If $|V(G)|<m+n-2$, G contains two single ears, others are all double ears.

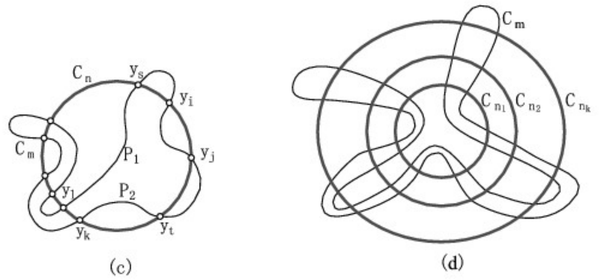


Figure 2. (c) A bicyclic graph has double ear and at least 3 single ears; (d) A intersecting k-cycle.

Definition 4. An intersecting k-cycle $H_k=H_k[C_m,Cn_i]$ ($i\in[1, k]$, $k\geq 2$) holds:

(a) H_k is a connected planar graph and contains cycles Cn_1, Cn_2 , … , Cn_k and C_m; $\Delta(H_k)=4$;

(b) $Cn_i\subseteq Cn_{i+1}$, $i\in[1,k-1]$, $V(Cn_i)\cap V(Cn_j)=\emptyset$, $i\neq j$.

(c) $E(H_k)=(\cup E(Cn_i))\cup E(C_m)$, $|V(H_k)|\leq m-2k+\sum n_i$; if $u\in V(C_m)\cap V(Cn_i)$, then $d_G(u)=4$, otherwise $d_G(u)=2$;

(d) $G=G[C_m,Cn_j]=H_k-\cup_{i\neq j}(E(Cn_i)\cup V(Cn_i))$, for each cycle Cn_j .

Definition 5. Let $f: V(G)\cup E(G)\rightarrow[1, k]$ be a proper total coloring of a simple graph G. We call f to be a [4]-*adjacent vertex distinguishing total k-coloring* (k-[4]-avdtc for short) of G if $C\{f, x\}\neq C\{f, y\}$ for every edge $xy\in E(G)$. The minimum number of k colors required for which G admits a k-[4]-avdtc is denoted by $\chi''_{[4]as}(G)$, and called the [4]-*avdtc chromatic number* of G.

Definition 6. Let f: $V(G)\cup E(G)\rightarrow[1, k]$ be a proper total coloring of a simple graph G. We call f to be a *complete adjacent vertex distinguishing total* k-*coloring* (complete k-avdtc for short) of G if it has $C^2\{f; xy\}$ for Every edge $xy\in E(G)$ The minimum number of k colors required for which G admits a complete k-avdtc is denoted by $\chi''_{ca}(G)$, and called the *complete avdtc chromatic number* of G. Clearly, for a simple graph G, we have

$$\chi''_{as}(G)\leq\chi''_{[4]as}(G),\ \chi''_{cas}(G)\leq\chi''_{cas}(G)$$

by the above definitions.

Definition 7. A *generalized Petersen graph* $P(n, k)$ for $n\geq3$ and $1\leq k<n/2$ is a graph with vertex set $V=\{u_0, u_1, \ldots ,u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$, and edge set $E=\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \mid i\in[0, n-1]$, where subscripts are taken modulo n. Observe that the generalized Petersen graph induced over the set $\{v_1, \ldots, v_{n-1}\}$ is the union of g disjoint cycles of length p, where $g=\gcd(n, k)$ and $p=n/g$.

II. Main Results

Lemma 1. [5] Let G be a simple graph, if G contains adjacent Δ-vertices, then $\chi''_{as}(G)\geqslant\Delta(G)+2$.

Lemma 2. Let uv be an edge of an intersecting cycle $G=G[C_m,C_n]$. Replacing uv by a path uwv, w is not in $V(G)$, results a new graph, denoted as H. If G has no adjacent Δ-vertices, then $\chi''_{as}(H)\leq\chi''_{as}(G)$.

Lemma 3. Let uv be an edge of an intersecting k-cycle H_k. Replacing edge uv by a path uwv, w is not in $V(H_k)$, results a new graph, denoted as H'. If H_k has no adjacent Δ-vertices, then $\chi''_{as}(H)\leq\chi''_{as}(H_k)$.

Lemma 4. If an intersecting cycle $G=G[C_m,C_n]$ has no adjacent Δ-vertices and each vertex of degree 2 is adjacent to two vertices of degree Δ, then $\chi''_{as}(G)=5$.

Lemma 5. If an intersecting k-cycle H_k ($k\geq1$) has no adjacent Δ-vertices and each vertex of degree 2 is adjacent to two vertices of degree Δ, then $\chi''_{as}(H_k)=5$.

Lemma 6. [8] $\chi''_{as}(P(n,k))=5$.

By the definition of a generalized Petersen graph, we can obtain Lemmas 7, 8 and 9.

Lemma 7. For a generalized Petersen graph $P(n,k)$, $n\geq3$ and $1\leq k<n/2$, if n is odd, then p and g are odd.

Lemma 8. For a generalized Petersen graph $P(n,k)$, $n\geq3$ and $1\leq k<n/2$, if n is even, p is even, then g is either even or odd.

Lemma 9. For a generalized Petersen graph $P(n,k)$, $n\geq3$ and $1\leq k<n/2$, if n is even, p is odd, then g is even.

Lemma 10. [9] Let G be a simple graph with $n\geq3$ vertices and no isolated edges as well as at most one isolated vertex. Then G admits a total coloring f with $C\langle f, u\rangle=C\langle f, v\rangle$ for distinct vertices $u,v\in V(G)$ (resp. for every edge $uv\in E(G)$) if and only if $N(u)\cup\{u\}\neq N(v)\cup\{v\}$ for distinct $u,v\in V(G)$ (resp. for every edge $uv\in E(G)$).

Generalized Petersen graphs $P(n,k)$ have complete k-avdtcs by Lemma 10.

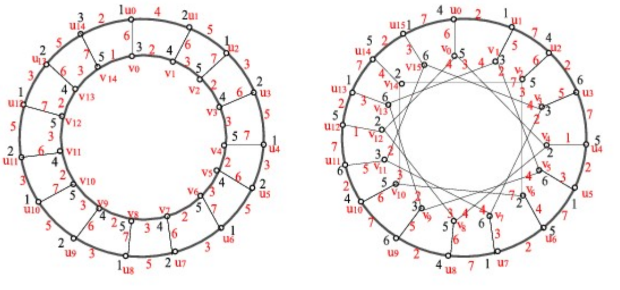
Lemma 11. $\chi''_{cas}(P(3,1))=6$.

Theorem 12. Let $G=G[C_m,C_n]$ be an intersecting cycle, if G has no adjacent Δ-vertices, then $\chi''_{as}(G)=5$.

Theorem 13. Let H_k ($k\geq2$) be an intersecting k-cycle, if H_k has no adjacent Δ-vertices, then $\chi''_{as}(H_k)=5$.

Theorem 14. If $n\geq3$, $1\leq k<n/2$, then $6\leq\chi''_{[4]as}(G)\leq7$.

Theorem 15. If $n \geq 3$, $1 \leq k < n/2$, then $6 \leq \chi''_{cas}(G) \leq 8$.



P(15, 1) P(16, 4)

Figure 3. An example of Theorem 15, $\chi''_{cas}(P(15,1))=7$ and $\chi''_{cas}(P(16,4))=7$.

III. Proofs

Proof of Lemma 2. Let f be an avdtc of an intersecting cycle $G=G[C_m,C_n]$, and $\min f(V(G) \cup E(G)) = \chi''_{as}(G)=|C|$. Set $f(uv)=\alpha$. Consider the following two cases.

Case 1. G contains a path xuv, $d_G(x)=d_G(v)=4$ and $d_G(u)=2$. Replacing edge uv by a path uwv, obtained a new graph H, w is not in $V(G)$. In H, we consider the coloring only in vertex w and its two incidence edges uw,wv, and define a proper total coloring g of H as: $g(uw)=a_1$, $a_1 \in \overline{C}(u,f)$; $g(wv)=\alpha$; $g(w) \in C \backslash \{\alpha, a_1, g(u), g(v)\}$; $g(t)=f(t)$, $t \in (S_1 \backslash \{uv\}) \subseteq (S_2 \backslash \{uw, w, wv\})$, where $S_1=V(G) \cup E(G)$, $S_2=V(H) \cup E(H)$. Clearly, $C(t_0,g)=C(t_0, f)$, $t_0 \in V(H) \backslash \{u,w\}$, $C(u, g)=\{a_1, g(u), g(ux)\}$, $C(w, g)=\{\alpha, a_1, g(w)\}$, since $\alpha \in C(w, g)$, but α is not in $C(u,g)$. Therefore, $C(u,g) \neq C(w,g)$. Since $d_H(x)=d_H(v)=4$, $d_H(u)=d_H(w)=2$, so $C(w,g) \neq C(v,g)$, $C(x,g) \neq C(u,g)$. Hence, g is an avdtc of H, and $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Case 2. G contains a path $xuvy$, $d_G(x)=d_G(y)=4$ and $d_G(u)=d_G(v)=2$. Replacing uv by a path uwv enables us to obtain a new graph H, w is not in $V(G)$.In H, we consider the coloring only in vertex w and its two incidence edges uw,wv, and then define a proper total coloring h of H as: $h(wv)=b_1$, $b_1 \in \overline{C}(v,f)$; $h(w)=\alpha$; $h(uw) \in C \backslash \{\alpha, b_1, h(u), h(ux)\}$; $h(t)=f(t)$, $t \in (S_1 \backslash \{uv\}) \subseteq (S_2 \backslash \{uw, w, wv\})$, where $S_1=V(G) \cup E(G)$, $S_2=V(H) \cup E(H)$. For such a structure h, $C(t_0,h)=C(t_0, f)$, $t_0 \in V(H) \backslash \{w,v\}$, $C(u,h)=\{h(ux), h(u), h(uw)\}$, $C(w,h)=\{\alpha, b_1, h(uw)\}$, $C(v,h)=\{b_1, h(v), h(vy)\}$, since $\alpha \in C(w,h)$, but α is not in $C(u,h)$, therefore, $C(u,h) \neq C(w,h)$. And $\alpha \in C(w,h)$, but α is not in $C(v,h)$, therefore, $C(w,h) \neq C(v,h)$, $C(x, h) \neq C(u,h)$ and $C(y,h) \neq C(v,h)$ for $d_H(x)=d_H(y)=4$ and $d_H(u)=d_H(v)=2$, Hence, h is an avdtc of H, and $\chi''_{as}(H) \leq \chi''_{as}(G)$. The Lemma is covered.

Proof of Lemma 4. Case 1. Two cycles C_m and C_n intersect only one time. To distinguish, C_m is represented by C_m^1, C_n is represented by C_n^1. Let $C_m^1=x_1x_2x_3x_4x_1$, $C_n^1=y_1y_2y_3y_4y_1$, $d_G(x_i)=d_G(y_i)=4$, odd $i \in [1,4]$; $d_G(x_i)=d_G(y_i)=2$, even $i \in [1,4]$. Since $\Delta(G)=4$ and G contains no adjacent Δ-vertices, we can see $\chi''_{as}(G) \geq \Delta(G)+1=5$. We show that $\chi''_{as}(G) \leq 5$ in the following. Define a proper total coloring h of the intersecting cycle G as: $h(x_1)=h(y_1)=1$, $h(x_3)=h(y_3)=2$, $h(y_2)=h(y_4)=4$, $h(x_2)=h(x_4)=3$, $h(y_1y_2)=h(y_3y_4)=3$, $h(y_2y_3)=1$, $h(y_1y_4)=2$, $h(x_1x_2)=h(x_3x_4)=4$, $h(x_2x_3)=h(x_1x_4)=5$.

We have $\overline{C}(x_1,h)=\overline{C}(x_3,h)=\overline{C}(y_1,h)=\overline{C}(y_3,h)=\emptyset$, $\overline{C}(x_2,h)=\overline{C}(x_4,h)=\{1,2\}$, $\overline{C}(y_2,h)=\{2, 5\}$, $\overline{C}(y_4,h)=\{1, 5\}$. Obviously, $C(y_i,h) \neq C(y_{i+1}, h)$, $i \in [1,3]$; $C(x_i,h) \neq C(x_{i+1}, h)$, $i \in [1, 3]$; $C(y_1, h) \neq C(y_4, h)$; $C(x_1, h) \neq C(x_4, h)$. It follows that h is an avdtc of G, therefore, $\chi''_{as}(G)=5$.

Case 2. Suppose two cycles C_m and C_n intersect β (β>1) times, and $\chi''_{as}(G)=5$. To distinguish, C_m is represented by C_m^2, C_n is represented by C_n^2. Let $C_m^2=x_1x_2 \ldots x_{4\beta}x_1$, $C_n^2=y_1y_2 \ldots y_{4\beta}y_1$.By the inductive hypothesis, G has an avdtc ϕ that $\min \phi(V(G) \cup E(G))= \chi''_{as}(G)=5$. Without loss of generality, we give a ϕ as follows: $\phi(x_i)=\phi(y_j)=1$, $i, j \in [1,4\beta]$ and $i, j \equiv 1 \pmod 4$; $\phi(x_i)=\phi(y_j)=2$, $i, j \in [1, 4\beta]$ and $i, j \equiv 3 \pmod 4$; $\phi(y_j)=4$, for even $j \in [1,4\beta]$; $\phi(x_i)=3$, for even $i \in [1,4\beta]$; $\phi(y_jy_{j+1})=3$, for odd $j \in [1,4\beta]$; $\phi(y_jy_{j+1})=1$, $j \in [1,4\beta]$ and $j \equiv 2 \pmod 4$; $\phi(y_1y_{4\beta})=\phi(y_jy_{j+1})=2$, $j \in [1,4\beta]$ and $j \equiv 0 \pmod 4$; $\phi(x_ix_{i+1})=4$, for odd $i \in [1,4\beta]$; $\phi(x_ix_{i+1})=\phi(x_1x_{4\beta})=5$, for even $i \in [1,4\beta]$. Now we consider two cycles C_m and C_n intersect $\beta+1$ (β>1) times. Denote new cycles as C_m^3, C_n^3, let

$$C_m^3=x_1x_2 \ldots x_{4\beta}x_{4\beta+1}x_{4\beta+2}x_{4\beta+3}x_{4\beta+4}x_1,$$
$$C_n^3=y_1y_2 \ldots y_{4\beta}y_{4\beta+1}y_{4\beta+2}y_{4\beta+3}y_{4\beta+4}y_1.$$

We define a proper total coloring φ of two cycles C_m^2 and C_n^2 that intersect $\beta+1$ (β>1) times as: $\varphi(x_i)=\varphi(y_j)=1$, $i, j \in [4\beta+1,4\beta+4]$ and $i, j \equiv 1 \pmod 4$; $\varphi(x_i))=\varphi(y_j)=2$, $i, j \in [4\beta+1,4\beta+4]$ and $i, j \equiv 3 \pmod 4$; $\varphi(y_j)=4$ for even $i \in [4\beta+1,4\beta+4]$; $\varphi(x_i)=3$, even $i \in [4\beta+1,4\beta+4]$, $\varphi(y_jy_{j+1})=3$ for odd $j \in [4\beta+1,4\beta+4]$; $\varphi(y_jy_{j+1})=1$, odd $j \in [4\beta+1,4\beta+4]$ and $i \equiv 2 \pmod 4$; $\varphi(y_1y_{4\beta+4})=\varphi(y_jy_{j+1})=2$, odd $j \in [4\beta+1, 4\beta+4]$ and $j \equiv 0 \pmod 4$; $\varphi(x_ix_{i+1})=4$, odd $i \in [4\beta+1,4\beta+4]$; $\varphi(x_ix_{i+1})=\varphi(x_1x_{4\beta+4})=5$, even $i \in [4\beta+1,4\beta+4]$. The colors of the rest vertices and edges keep no change in the coloring ϕ.

We observe that $\overline{C}(x_i, \varphi)=\overline{C}(y_j, \varphi)=\emptyset$, for odd $i, j \in [1, 4\beta+4]$; $\overline{C}(x_i, \varphi)=\{1,2\}$, for even $i \in [1,4\beta+4]$; $\overline{C}(y_j, \varphi)=\{2, 5\}$, $j \in [1,4\beta+4]$ and $j \equiv 2 \pmod 4$, $\overline{C}(y_j, \varphi)=\{1,5\}$, $j \in [1, 4\beta+4]$ and $j \equiv 0 \pmod 4$. Furthermore, $C(y_i, \varphi) \neq C(y_{i+1},\varphi)$, $i \in [1, 4\beta+3]$; $C(y_{4\beta+4}, \varphi) \neq C(y_1, \varphi)$; $C(x_i, \varphi) \neq C(x_{i+1}, \varphi)$, $i \in [1, 4\beta+3]$; $C(x_{4\beta+4}, \varphi) \neq C(x_1, \varphi)$. Therefore, φ is an avdtc of G with $\chi''_{as}(G)=5$.

Proof of Lemma 5. Case 1. Two cycles C_m and Cn_i ($i \in [1,k]$) intersect only one time. To distinguish, C_m is represented by C_m^4 and Cn_i is represented by Cn_i^4. Let $C_m^4=x_1x_2 \ldots x_{4k}x_1$, $Cn_i^4=y_{i,1}y_{i,2} \ldots y_{i,4}y_{i,1}$, $i \in [1,k]$. Since $\Delta(H_k)=4$, and H_k contains no adjacent Δ-vertices, thus, $\chi''_{as}(H_k) \geq \Delta(H_k)+1=5$.

We define a proper total coloring ψ of an intersecting cycle H_k as: $\psi(y_{i,1})=1$, $i \in [1,k]$; $\psi(y_{i,3})=2$, $i \in [1,k]$; $\psi(y_{i,2})=\psi(y_{i,4})=4$, $i \in [1,k]$; $\psi(y_{i,j}y_{i,j+1})=3$, $i \in [1,k]$, odd $j \in [1,4]$; $\psi(y_{i,1}y_{i,4})=2$, $i \in [1,k]$; $\psi(y_{i,2}y_{i,3})=1$, $i \in [1,k]$; $\psi(x_i)=1$, odd

$i\in[1,2k]$; $\psi(x_i)=2$, odd $i\in[2k, 4k]$; $\psi(x_i)=3$, even $i\in[1,4k]$; $\psi(x_ix_{i+1})=4$, odd $i\in[1,4k]$; $\psi(x_ix_{i+1})=\psi(x_1x_{4k})=5$, even $i\in[1, 4k]$. We have $\overline{C}(x_i, \psi)=\overline{C}(y_{i,j}, \psi)=\emptyset$, odd $i\in[1,4k]$, odd $j\in[1, 4k]$; $\overline{C}(x_i,\psi)=\{1,2\}$, even $i\in[1,4k]$; $\overline{C}(y_{i,2}, \psi)=\{2, 5\}$, $i\in[1, k]$, $j\equiv2$ (mod 4), $\overline{C}(y_{i,4}, \psi)=\{1,5\}$, $i\in[1,k]$, $j\equiv0$ (mod 4). Furthermore, $C(y_{i,j}, \psi)\neq C(y_{i,j+1}, \psi)$, $i\in[1,k]$, $j\in[1,3]$; $C(y_{i,4},\psi)\neq C(y_{i,1},\psi)$, $i\in[1,k]$; $C(x_h, \psi)\neq C(x_{h+1}, \psi)$, $h\in[1, 4k-1]$; $C(x_{4k}, \psi)\neq C(x_1, \psi)$. Thereby, ψ is an avdtc of H_k with $\chi''_{as}(H_k)=5$.

Case 2. Suppose two cycles C_m and Cn_i ($i\in[1,k]$) intersect β ($\beta>1$) times, and $\chi''_{as}(H_k)=5$. To distinguish, C_m is represented by C_m^5, and Cn_i is represented by Cn_i^5. Let

$$C_m^5=x_1x_2\ldots x_{4k\beta}x_1,\ Cn_i^5=y_{i,1}y_{i,2}\ldots y_{i,4\beta}y_{i,1},\ i\in[1,k].$$

By the inductive hypothesis, H_k has an avdtc ζ that min $\zeta(V(H_k)\cup E(H_k))=\chi''_{as}(H_k)=5$. Without loss of generality, we define a coloring ζ as: $\zeta(y_{i,j})=1$, $i\in[1,k]$, $j\in[1,4\beta]$ and $j\equiv1$(mod 4); $\zeta(y_i,3)=2$, $i\in[1,k]$, $j\in[1,4\beta]$ and $j\equiv3$(mod 4); $\zeta(y_{i,j})=4$, $i\in[1,k]$, even $j\in[1,4\beta]$; $\zeta(y_{i,j}y_{i,j+1})=3$, $i\in[1,k]$, odd $j\in[1,4\beta]$; $\zeta(y_{i,1}y_{i,4\beta})=\zeta(y_{i,j}y_{i,j+1})=2$, $i\in[1,k]$, $j\in[1,4\beta]$ and $j\equiv0$ (mod 4); $\zeta(y_{i,j}y_{i,j+1})=1$, $i\in[1,k]$, $j\in[1,4\beta]$ and $j\equiv2$ (mod 4); $\zeta(x_i)=1$, odd $i\in[1,2k\beta]$; $\zeta(x_i)=2$, odd $i\in[2k\beta, 4k\beta]$; $\zeta(x_i)=3$, even $i\in[1,4k\beta]$; $\zeta(x_ix_{i+1})=4$, odd $i\in[1,4k\beta]$; $\zeta(x_ix_{i+1})=\zeta(x_1x_{4k\beta})=5$, even $i\in[1,4k\beta]$.

Now we consider the case of two cycles C_m^5 and Cn_i^5 ($i\in[1,k]$) intersect $\beta+1$ ($\beta>1$) times. Denote new cycles as C_m^6, Cn_i^6. For $i\in[1,k]$, let

$$C_m^6=x_1x_2\ldots x_{4k\beta}x_{4k\beta+1}x_{4k\beta+2}\ldots x_{4k\beta+4k}x_1,$$
$$Cn_i^6=y_{i,1}y_{i,2}\ldots y_{i,4\beta}y_{i,4\beta+1}\,y_{i,4\beta+2}\,y_{i,4\beta+3}\,y_{i,4\beta+4}y_{i,1}.$$

Let π be a proper total coloring of H_k. We define a coloring π as: $\pi(y_{i,j})=1$, $i\in[1,k]$, $j\in[4\beta+1,4\beta+4]$ and $j\equiv1$(mod 4); $\pi(y_{i,3})=2$, $i\in[1,k]$, $j\in[4\beta+1,4\beta+4]$ and $j\equiv3$ (mod 4); $\pi(y_{i,j})=4$, $i\in[1,k]$, even $j\in[4\beta+1, 4\beta+4]$; $\pi(y_{i,j}y_{i,j+1})=3$, $i\in[1,k]$, odd $j\in[4\beta+1,4\beta+4]$; $\pi(y_{i,1}y_{i,4\beta+4})=\pi(y_{i,j}y_{i,j+1})=2$, $i\in[1,k]$, $j\in[4\beta+1, 4\beta+4]$ and $j\equiv0$ (mod 4); $\pi(y_{i,j}y_{i,j+1})=1$, $i\in[1,k]$, $j\in[4\beta+1, 4\beta+4]$ and $j\equiv2$ (mod 4); $\pi(x_i)=1$, odd $i\in[4k\beta+1,4k\beta+2k]$; $\pi(x_i)=2$, odd $i\in[4k\beta+2k, 4k\beta+4k]$; $\pi(x_i)=3$, even $i\in[4k\beta+1, 4k\beta+4k]$; $\pi(x_ix_{i+1})=4$, odd $i\in[4k\beta+1,4k\beta+4k]$; $\pi(x_ix_{i+1})=\pi(x_1x_{4k\beta+4})=5$, even $i\in[1,4k\beta+4k]$, the colors of the rest vertices and edges keep no change in the coloring ζ. We observe that $\overline{C}(x_i, \pi)=\overline{C}(y_{i,j}, \pi)=\emptyset$, odd $i\in[1,4k]$, odd $j\in[1, 4\beta+4]$; $\overline{C}(x_i, \pi)=\{1,2\}$, even $i\in[1,4\beta+4]$; $\overline{C}(y_{i,j}, \pi)=\{2,5\}$, $i\in[1,k]$, $j\in[1,4\beta+4]$ and $j\equiv2$(mod 4); $\overline{C}(y_{i,4}, \pi)=\{1,5\}$, $i\in[1,k]$, $j\in[1,4\beta+4]$ and $j\equiv0$ (mod 4). Then we have $C(y_{i,j}, \pi)\neq C(y_{i,j+1},\pi)$, $i\in[1,k]$, $j\in[1,4\beta+3]$, $C(y_{i,4\beta+4}, \pi)\neq C(y_{i,1}, \pi)$, $i\in[1,k]$; $C(x_h, \pi)\neq C(x_{h+1}, \pi)$, $h\in[1, 4k\beta+4k-1]$; $C(x_{4k\beta+4k}, \pi)\neq C(x_1, \pi)$. Thereby, π is an avdtc of H_k, and $\chi''_{as}(H_k)=5$, which means that $\chi''_{as}(H_k)=5$.

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