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New Solvable Nonlinear Matrix Evolution Equations

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Abstract

We introduce an extension of the factorization-decomposition technique that allows us to manufacture new solvable nonlinear matrix evolution equations. Several examples of such equations are reported.

1 Introduction

Solvable and/or integrable nonlinear matrix equations are of course interesting "in se". However they are also important in the context of solvable and/or integrable nonlinear dynamical systems. Indeed recently some techniques were introduced to associate solvable (integrable) many body problems with solvable (integrable) matrix equations; namely one can obtain solvable dynamical equations for N particles on a line [1], or, via convenient parametrizations of matrices in terms of vectors (see [2],[3]), solvable (integrable) rotation-invariant Newtonian equations of motion for particles in an arbitrary n-dimensional space (see [2],[4],[5]).

In this paper we show how to construct new solvable nonlinear matrix evolution equations through a new extension of the *decomposition-factorization* techniques (see f.i. [6]). We illustrate this new technique only in the simplest case (LU decomposition-factorization and $2 \otimes 2$ block matrices). It is plain that this technique could be extended to different and more complex cases. A subsequent paper will be devoted to a deeper investigation (more equations, explicit solutions). In the following Section we set the notation and we give the explicit LU decomposition-factorization of $2 \otimes 2$ block matrices. In Section 3 we illustrate the technique to construct solvable nonlinear matrix equations. In Section 4 we give examples of such equations, namely systems of first order solvable nonlinear matrix equations and also second order solvable nonlinear matrix evolution equations (obtained through suitable reductions).

2 A parameterization of block matrices

Let us consider here and in the following $2 \otimes 2$ block matrices, namely:

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \ , \tag{1}$$

where all the entries M_k , $(k = 1, 2, 3, 4)$, are square matrices of arbitrary order.

Now consider the matrix-subspaces $\tilde{U}$ and $\tilde{L}$ of (respectively) *upper* (*lower*) type, say $U \in \tilde{U}$ if

$$U = \begin{pmatrix} U_1 & U_2 \\ 0 & U_4 \end{pmatrix} \ , \tag{2a}$$

$W \in \tilde{W}$ if

$$W = \begin{pmatrix} W_1 & 0 \\ W_3 & W_4 \end{pmatrix} \ . \tag{2b}$$

Let us assume that all the involved matrices depend on a parameter t (time). Moreover we assume that two of the six matrices U_k, W_k are *preassigned (constant known matrices or time dependent matrices whose evolution is known).* In the following we shall assume that W_1 and W_4 are *preassigned (* of course different choices could give different results).

Given an arbitrary $2 \otimes 2$ block matrix M, there is a *unique* way to decompose it as a *sum* of a pair of matrices (of *upper* and *lower* type), and as well a *unique* way to decompose it as a *product* (with a given order) of a pair of such matrices. Indeed

$$M = A + B \ , \tag{3a}$$

where $A \in \tilde{U}$:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \ , \tag{3b}$$

and $B \in \tilde{W}$:

$$B = \begin{pmatrix} B_1 & 0 \\ B_3 & B_4 \end{pmatrix} \ , \tag{3c}$$

with known (*preassigned)* B_1, B_4, clearly entails

$$M_1 = A_1 + B_1, \ M_2 = A_2, \ M_3 = B_3, \ M_4 = A_4 + B_4 \,, \tag{4a}$$

which are trivially inverted to read

$$A_1 = M_1 - B_1, A_2 = M_2, \ \ A_3 = M_3, \ A_4 = M_4 - B_4 \ . \tag{5a}$$

And likewise

$$M = YX \ , \tag{6a}$$

where $X \in \tilde{U}$:

$$X = \begin{pmatrix} X_1 & X_2 \\ 0 & X_4 \end{pmatrix} \ , \tag{6b}$$

and $Y \in \tilde{W}$:

$$Y = \begin{pmatrix} Y_1 & 0 \\ Y_3 & Y_4 \end{pmatrix} \ , \tag{6c}$$

with known (*preassigned*) Y_1, Y_4, clearly entails

$$M_1 = Y_1 X_1 \ , \tag{7a}$$

$$M_2 = Y_1 X_2 \ , \tag{7b}$$

$$M_3 = Y_3 X_1 \ , \tag{7c}$$

$$M_4 = Y_3 X_2 + Y_4 X_4 \ ; \tag{7d}$$

which can be easily inverted:

$$X_1 = Y_1^{-1} M_1 \ , \tag{8a}$$

$$X_2 = Y_1^{-1} M_2 \ , \tag{8b}$$

$$Y_3 = M_3 M_1^{-1} Y_1 \ , \tag{8c}$$

$$X_4 = Y_4^{-1} \left(M_4 - M_3 M_1^{-1} M_2 \right) \ . \tag{8d}$$

There are two obvious generalizations of the here introduced technique :

- one could consider block matrices of higher order,
- one could consider other factorization (f.i. QR instead of LU).

3 Derivation of solvable nonlinear matrix evolution equations

Let us consider a time dependent $2 \otimes 2$ block matrix $L(t)$

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \ . \tag{9}$$

Let us also consider $\tilde{L} = f(L)$, a *scalar*, but otherwise *arbitrary*, function of the matrix L. With no loss of generality we can assume

$$\tilde{L} = \sum_{n=-\infty}^{\infty} c_n L^n \ , \tag{10}$$

where the coefficients c_n are scalars, possibly known functions of time.

Now decompose $\tilde{L}$ as a *sum* of a pair of matrices (of *upper* and *lower* type):

$$\tilde{L} = \begin{pmatrix} \tilde{L}_1 & \tilde{L}_2 \\ \tilde{L}_3 & \tilde{L}_4 \end{pmatrix} = A + B = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ B_3 & B_4 \end{pmatrix} \ , \tag{11}$$

where the matrices B_1, B_4 are known (*preassigned, constant or possibly dependent on time).*

Let us also introduce the matrices $X(t), Y(t)$ *via* the evolution equations

$$\dot{X} = AX, \ \dot{Y} = YB \, , \tag{12a}$$

with the initial conditions

$$X(0) = I, \ Y(0) = I \ . \tag{12b}$$

Remark 1. *The above initial conditions are chosen just for sake of simplicity: arbitrary initial conditions yield the same results.*

Obviously $X \in \tilde{U}, Y \in \tilde{W}$.

Let us show the above equations in detail:

$$\dot{X}_1 = A_1 X_1 \ , \tag{13}$$

$$\dot{X}_2 = A_1 X_2 + A_2 X_4 \ , \tag{14}$$

$$\dot{X}_4 = A_4 X_4 \ , \tag{15}$$

$$\dot{Y}_3 = Y_3 B_1 + Y_4 B_3 \ ; \tag{16}$$

and

$$\dot{Y}_1 = Y_1 B_1 \ , \tag{17a}$$

$$\dot{Y}_4 = Y_4 B_4 \ . \tag{17b}$$

Given that B_1, B_4 are known matrices , then also Y_1, Y_4 are known (time dependent) matrices.

Now consider the matrix $P(t)$:

$$P = YX \ . \tag{18}$$

Note that

$$P(t=0) = P_0 = I \ . \tag{19}$$

Obviously

$$\dot{P} = \dot{Y}X + Y\dot{X} = YBX + YAX = Y\tilde{L}X = Y\left(\sum_{n=-\infty}^{\infty} c_n L^n\right)X \ . \tag{20}$$

Taking into account that

$$\begin{aligned} YL^nX &= YX\left(X^{-1}LY^{-1}\right)YX\left(X^{-1}LY^{-1}\right)YX....\left(X^{-1}LY^{-1}\right)YX & (21)\\ &= \left(P\left(X^{-1}LY^{-1}\right)\right)^n P \ , & (22)\end{aligned}$$

we have

$$\dot{P} = \left(\sum_{n=-\infty}^{\infty} c_n \left(P\left(X^{-1}LY^{-1}\right)\right)^n\right)P \ . \tag{23}$$

Setting

$$\bar{L} = \left(X^{-1}LY^{-1}\right) \ , \tag{24}$$

we have

$$\dot{\bar{L}} = X^{-1}\left(-AL + \dot{L} - LB\right)Y^{-1} \ . \tag{25}$$

Thus, if

$$\dot{L} = AL + LB \ , \tag{26}$$

then

$$\dot{\bar{L}} = 0, \quad \bar{L} = L(t=0) = L_0 \ . \tag{27}$$

Note that

$$L(t) = X(t)L_0Y(t) \ . \tag{28}$$

Eq. (23) now reads

$$\dot{P} = \left(\sum_{n=-\infty}^{\infty} c_n \left(PL_0\right)^n\right)P \ . \tag{29}$$

Setting

$$\tilde{P} = PL_0 \ , \tag{30}$$

we have

$$\dot{\tilde{P}} = \sum_{n=-\infty}^{\infty} c_n \tilde{P}^{n+1} \ , \tag{31a}$$

with

$$\tilde{P}_0 = L_0 \ . \tag{31b}$$

The first order matrix equation (31a) (with the initial condition (31b)), involves just one matrix, thus, in principle, is solvable.

Then the nonlinear matrix equation (26), is also solvable.

Sketch of the procedure

Aim: solve

$$\dot{L} = AL + LB \ , \tag{32a}$$

with

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \ , \tag{32b}$$

$$A + B = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ B_3 & B_4 \end{pmatrix} = \tilde{L} = \begin{pmatrix} \tilde{L}_1 & \tilde{L}_2 \\ \tilde{L}_3 & \tilde{L}_4 \end{pmatrix} \ , \tag{32c}$$

where

$$\tilde{L} = \sum_{n=-\infty}^{\infty} c_n L^n \ , \tag{32d}$$

and the matrices B_1, B_4 (possibly depending on time) are known matrices.

Steps:

- given the initial data L_0 , solve (31a,31b), finding $P(t)$;
- decompose $P(t)$ according to (18) (unique decomposition!), finding $X(t), Y(t)$;
- according to (28), find the solution $L(t)$ of (32).

4 Examples

- $\tilde{L} = L, \ B_4, B_1$ constant matrices:

$$\dot{L}_1 = (L_1)^2 + 2L_2L_3 + L_1B_1 - B_1L_1 \ , \tag{33}$$

$$\dot{L}_2 = L_1L_2 + L_2L_4 + L_2B_4 - B_1L_2 \ , \tag{34}$$

$$\dot{L}_3 = 2L_4L_3 + L_3B_1 - B_4L_3 \ , \tag{35}$$

$$\dot{L}_4 = (L_4)^2 + L_4 B_4 - B_4 L_4 \ . \tag{36}$$

Reductions and second order equations:
Setting

$$L_4 = 0,\ \ L_3 = C, \tag{37}$$

$$B_1 = B_4 = I \ , \tag{38}$$

we get

$$\dot{L}_1 = (L_1)^2 + 2L_2 C \ , \tag{39}$$

$$\dot{L}_2 = L_1 L_2 \ . \tag{40}$$

This simple system can be cast as second order matrix evolution equation in two ways:

$$\ddot{L}_2 = 2\dot{L}_2 L_2^{-1} \dot{L}_2 + 2L_2 C L_2 \ , \tag{41}$$

$$\ddot{L}_1 = \dot{L}_1 L_1 + 2L_1 \dot{L}_1 - (L_1)^3 \ \ . \tag{42}$$

Setting

$$L_4 = 0\,, \tag{43}$$

and

$$S = L_2 L_3 \ , \tag{44}$$

we get

$$\dot{S} = L_1 S - B_1 S + S B_1 \ , \tag{45}$$

$$\dot{L}_1 = (L_1)^2 + 2S + L_1 B_1 - B_1 L_1 \ . \tag{46}$$

Again this first order system can be cast as second order matrix evolution equation in two ways:

$$\begin{aligned} \ddot{S} \ &= \ 2\dot{S}S^{-1}\dot{S} + 2S^2 + 2\dot{S}S^{-1}B_1 S - 2SB_1 S^{-1}\dot{S} \\ &\quad -2SB_1 S^{-1} B_1 S + (B_1)^2 S + S\,(B_1)^2 \ \ , \end{aligned} \tag{47}$$

$$\begin{aligned} \ddot{L}_1 \ &= \ \dot{L}_1 L_1 + 2L_1 \dot{L}_1 - (L_1)^3 + \\ &\quad +L_1 B_1 L_1 + B_1 (L_1)^2 - 2\,(L_1)^2 B_1 + \\ &\quad +2\dot{L}_1 B_1 - 2B_1 \dot{L}_1 + \\ &\quad - (B_1)^2 L_1 - L_1 (B_1)^2 + B_1 L_1 B_1 \ . \end{aligned} \tag{48}$$

- $\tilde{L} = L^2,\ B_4, B_1$ constant matrices:

$$\dot{L}_1 = (L_1)^3 + 2L_2L_3L_1 + L_1L_2L_3 + 2L_2L_4L_3 + L_1B_1 - B_1L_1 \ , \tag{49}$$

$$\dot{L}_2 = (L_1)^2 L_2 + L_2L_3L_2 + L_1L_2L_4 + L_2 (L_4)^2 + L_2B_4 - B_1L_2 \ , \tag{50}$$

$$\dot{L}_3 = 2 (L_4)^2 L_3 + L_3L_2L_3 + L_4L_3L_1 + L_3B_1 - B_4L_3 \ , \tag{51}$$

$$\dot{L}_4 = (L_4)^3 + L_3L_2L_4 + L_4B_4 - B_4L_4 \ . \tag{52}$$

Reductions and second order equations:

Setting:

$$L_1 = 0,\ L_4 = 0 \ , \tag{53}$$

we get

$$\dot{L}_2 = L_2L_3L_2 + L_2B_4 - B_1L_2 \ , \tag{54}$$

$$\dot{L}_3 = L_3L_2L_3 + L_3B_1 - B_4L_3 \ . \tag{55}$$

The above first order system can be cast as a second order matrix evolution equation:

$$\begin{aligned} \ddot{L}_2 \ &= \ 3\left(\dot{L}_2 - L_2B_4 + B_1L_2\right)(L_2)^{-1}\left(\dot{L}_2 - L_2B_4 + B_1L_2\right) \\ &\quad +2\left(\dot{L}_2 - L_2B_4 + B_1L_2\right)B_4 - 2B_1\left(\dot{L}_2 - L_2B_4 + B_1L_2\right) + \\ &\quad +L_2 (B_4)^2 - 2B_1L_2B_4 + (B_1)^2 L_2 \ . \end{aligned} \tag{56}$$

- $\tilde{L} = L^{-1}, B_4, B_1$ constant matrices:

$$\begin{aligned} \dot{L}_1 \ &= \ \left(L_1 - L_2 (L_4)^{-1} L_3\right)^{-1} L_1 \\ &\quad - (L_1)^{-1} L_2 \left(L_4 - L_3 (L_1)^{-1} L_2\right)^{-1} L_3 + \\ &\quad -L_2 (L_4)^{-1} L_3 \left(L_1 - L_2 (L_4)^{-1} L_3\right)^{-1} \\ &\quad +L_1B_1 - B_1L_1, \end{aligned} \tag{57}$$

$$\begin{aligned} \dot{L}_2 \ &= \ \left(L_1 - L_2 (L_4)^{-1} L_3\right)^{-1} L_2 \\ &\quad - (L_1)^{-1} L_2 \left(L_4 - L_3 (L_1)^{-1} L_2\right)^{-1} L_4 \\ &\quad +L_2B_4 - B_1L_2 \ , \end{aligned} \tag{58}$$

$$\dot{L}_3 = \left(L_4 - L_3\,(L_1)^{-1}\,L_2\right)^{-1} L_3 \qquad (59)$$
$$-L_3\left(L_1 - L_2\,(L_4)^{-1}\,L_3\right)^{-1}$$
$$+L_3B_1 - B_4L_3\ ,$$

$$\dot{L}_4 = \left(L_4 - L_3\,(L_1)^{-1}\,L_2\right)^{-1} L_4$$
$$+L_4B_4 - B_4L_4\ . \qquad (60)$$

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