Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 689–696 **Birthday Issue**

On the Equilibrium Configuration of the BC-type Ruijsenaars-Schneider System

Jan F VAN DIEJEN

Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile
E-mail: diejen@inst-mat.utalca.cl

This article is part of the special issue published in honour of Francesco Calogero on the occasion of his 70th birthday

Abstract

It is shown that the ground-state equilibrium configurations of the trigonometric BC-type Ruijsenaars-Schneider systems are given by the zeros of Askey-Wilson polynomials.

1 Introduction

The Ruijsenaars-Schneider systems [14, 6, 7] are integrable deformations of the celebrated Calogero-Moser n-particle models [12]. It is well-known that the equilibrium configurations of the Calogero-Moser models are described by the zeros of classical orthogonal polynomials such as the Hermite, Laguerre, Chebyshev, and Jacobi polynomials [2, 3, 4, 12, 10, 5]. This connection between the equilibria of one-dimensional integrable particle models and the locations of zeros of the classical hypergeometric orthogonal polynomials, first observed by Calogero, is closely related to a beautiful electrostatic interpretation of the zeros of orthogonal polynomials due to Stieltjes [16]. Recently, it was noticed that the equilibrium configurations of the Ruijsenaars-Schneider systems can also be described in a similar way by means of the zeros orthogonal polynomials [15, 13, 11]; all polynomials that appear in in this context turn out to be classical in the sense that they sit somewhere in Askey's hierarchy of (basic) hypergeometric orthogonal polynomials [1, 9]. The top of this hierarchy is formed by the Askey-Wilson polynomials [1]. (All other (basic) hypergeometric families of classical orthogonal polynomials are special (limiting) cases of the Askey-Wilson polynomials [9].) In this note we show that the zeros of these Askey-Wilson polynomials correspond to the ground-state equilibrium configurations of the trigonometric BC-type Ruijsenaars-Schneider systems introduced in Refs. [6, 7]. In the rational limit, one recovers the characterization of the ground-state equilibrium configurations of the rational BC-type Ruijsenaars-Schneider systems in terms of Wilson polynomials [17] due to Odake and Sasaki [11]

We will need to employ the following standard conventions from the theory of (basic) hypergeometric orthogonal polynomials [1, 9]: q-shifted factorials are denoted by

$$(a;q)_k := \begin{cases} 1 & \text{for } k = 0, \\ (1-a)(1-aq)\cdots(1-aq^{k-1}) & \text{for } k = 1, 2, 3, \ldots, \end{cases}$$

with the convention that $(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$ (for $|q|<1$); products of q-shifted factorials are abbreviated in the usual way via

$$(a_1,\ldots,a_r;q)_k := (a_1;q)_k\cdots(a_r;q)_k.$$

The ${}_{r+1}\Phi_r$ terminating basic hypergeometric series is defined as

$${}_{r+1}\Phi_r\left[\begin{array}{c} q^{-n},a_1\ldots,a_r \\ b_1,\ldots,b_r \end{array} \mid q;z\right] := \sum_{k=0}^{n}\frac{(q^{-n},a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_r;q)_k}z^k$$

(where it is assumed that the parameters are such that denominators do not vanish). The hypergeometric degeneration of this series is given by

$${}_{r+1}F_r\left[\begin{array}{c} -n,a_1\ldots,a_r \\ b_1,\ldots,b_r \end{array} \mid z\right] := \sum_{k=0}^{n}\frac{(-n,a_1,\ldots,a_r)_k}{(1,b_1,\ldots,b_r;q)_k}z^k,$$

where $(a_1,\ldots,a_r)_k := (a_1)_k\cdots(a_r)_k$ with $(a)_k := a(a+1)\cdots(a+k-1)$ (and $(a)_0 := 1$ by convention).

2 The trigonometric BC-type Ruijsenaars-Schneider system

The trigonometric BC-type Ruijsenaars-Schneider system is a one-dimensional n-particle model characterized by the Hamiltonian [6, 7]

$$H(\mathbf{p},\mathbf{x}) = \sum_{j=1}^{n}\left(\cosh(p_j)\sqrt{V_j(\mathbf{x})V_j(-\mathbf{x})} - \big(V_j(\mathbf{x})+V_j(-\mathbf{x})\big)/2\right), \tag{2.1a}$$

where

$$V_j(\mathbf{x}) = w(x_j)\prod_{1\le k\le n,\, k\ne j} v(x_j+x_k)\,v(x_j-x_k), \tag{2.1b}$$

$$v(x) = \frac{\sin(x+ig)}{\sin(x)},\quad w(x) = \frac{\sin(x+ig_1)\cos(x+ig_2)\sin(x+ig_3)\cos(x+ig_4)}{\sin^2(x)\cos^2(x)} \tag{2.1c}$$

(and $i := \sqrt{-1}$). Throughout this note we will assume that the coupling parameters g and g_r ($r=1,2,3,4$) are *positive*. This guarantees in particular that the Hamiltonian $H(\mathbf{p},\mathbf{x})$ (2.1a)-(2.1c) constitutes a nonnegative (smooth) function on the phase space

$$\Omega = \{(\mathbf{p},\mathbf{x})\in\mathbb{R}^{2n} \mid 0<x_1<x_2<\cdots<x_{n-1}<x_n<\pi/2\}. \tag{2.2}$$

Indeed, one has that $H(\mathbf{p},\mathbf{x}) \ge H(\mathbf{0},\mathbf{x}) \ge 0$ (since $V_j(-\mathbf{x}) = \overline{V_j(\mathbf{x})}$ and $|V_j(\mathbf{x})| \ge \mathrm{Re}(V_j(\mathbf{x}))$).

3 The ground-state equilibrium configuration

The equilibrium configurations correspond to the critical points of the Hamiltonian $H(\mathbf{p},\mathbf{x})$ and the ground-state equilibrium configurations correspond in turn to the global minima. It is clear that the only way in which the nonnegative BC-type Ruijsenaars-Schneider Hamiltonian $H(\mathbf{p},\mathbf{x})$ (2.1a)-(2.1c) may vanish (thus actually reaching the lower bound zero) is when $\mathbf{p}=\mathbf{0}$ and $\mathbf{x}$ is such that $V_j(\mathbf{x})$ is positive for $j=1,\ldots,n$. This requires in particular that $V_j(\mathbf{x})$ is real-valued, i.e. that

$$V_j(\mathbf{x}) = V_j(-\mathbf{x}), \qquad j=1,\ldots,n, \tag{3.1a}$$

or more explicitely

$$\prod_{1\leq k\leq n,\, k\neq j} \frac{\sin(x_j+x_k+ig)\sin(x_j-x_k+ig)}{\sin(x_j+x_k-ig)\sin(x_j-x_k-ig)} = \tag{3.1b}$$
$$\frac{\sin(x_j-ig_1)\cos(x_j-ig_2)\sin(x_j-ig_3)\cos(x_j-ig_4)}{\sin(x_j+ig_1)\cos(x_j+ig_2)\sin(x_j+ig_3)\cos(x_j+ig_4)}, \qquad j=1,\ldots,n.$$

We will see below that the nonlinear system of algebraic equations in Eq. (3.1b) has a unique solution $0<x_1<x_2<\cdots<x_n<\pi/2$ given by the zeros of the Askey-Wilson polynomial of degree n. It is not difficult to see that for this solution in fact $V_j(\mathbf{x})>0$ for $j=1,\ldots,n$, whence $H(\mathbf{0},\mathbf{x})=0$. Indeed, $V_j(\mathbf{x})$ is real-valued by Eq. (3.1a). Furthermore, for sufficiently small values of the coupling parameters $V_j(\mathbf{x})$ must be positive as the function in question tends to 1 for $g,g_r\to 0$. This positivity remains valid for general positive parameter values g, g_r by a continuity argument revealing that the sign cannnot flip (as none of the factors in $V_j(\mathbf{x})$ (2.1b), (2.1c) becomes zero or singular).

We thus arrive at the following theorem.

Theorem 1. *The trigonometric BC-type Ruijsenaars-Schneider Hamiltonian $H(\mathbf{p},\mathbf{x})$* (2.1a)-(2.1c) *assumes the global minimum $H=0$ only at the point in the phase space Ω* (2.2) *such that $p_1=p_2=\cdots=p_n=0$ and $0<x_1<x_2<\cdots<x_n<\pi/2$ form a solution of the nonlinear system of algebraic equations in Eq.* (3.1b).

4 Zeros of the Askey-Wilson polynomials

The nonlinear system of algebraic equations in Eq. (3.1b) turns out to be a special case of the Bethe Ansatz equations associated to q-Sturm-Liouville problems studied recently by Ismail *et al* [8]. It follows from the machinary in *loc. cit.* that this algebraic system has a unique solution given by the zeros of the Askey-Wilson polynomial of degree n. Below we will provide an independent direct proof of this fact.

To this end, we first need to recall some basic properties of the Askey-Wilson polynomials taken from Ref. [1]. The (monic) Askey-Wilson polynomials are trigonometric polynomials of the form

$$p_n(x) = \cos(2nx) + \sum_{k=0}^{n-1} a_k\cos(2kx), \qquad n=0,1,2,\ldots, \tag{4.1a}$$

obtained by applying Gram-Schmidt orthogonalization of the standard Fourier cosine basis $1, \cos(2x), \cos(4x), \ldots$ on the interval $(0, \pi/2)$ with repect to the inner product

$$\langle f, g\rangle_\Delta = \int_0^{\pi/2} f(x)\overline{g(x)}\Delta(x)\mathrm{d}x, \tag{4.1b}$$

associated to the weight function

$$\Delta(x) = \frac{1}{c(x)c(-x)}, \qquad c(x) = \frac{(ae^{2ix}, b^{2ix}, ce^{2ix}, de^{2ix}; q)_\infty}{(e^{4ix}; q)_\infty}. \tag{4.1c}$$

Here it is assumed that all parameters are real-valued subject to the constraints $0 < q < 1$ and $0 < |a|, |b|, |c|, |d| < 1$. These parameter restrictions ensure in particular that the weight function $\Delta(x)$ (4.1c) is positive in the interval $(0, \pi/2)$. The Askey-Wilson polynomials admit an explicit representation in terms of the following terminating basic hypergeometric series

$$p_n(x) = \frac{(ab, ac, ad; q)_n}{2\, a^n (abcdq^{n-1}; q)_n} {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{2ix}, ae^{-2ix} \\ ab, ac, ad \end{matrix} \mid q; q \right]. \tag{4.2}$$

The polynomials under consideration are the eigenfunctions of a second-order difference operator. Upon performing the parameter substitution

$$q = e^{-2g}, \quad a = e^{-2g_1}, \quad b = -e^{-2g_2}, \quad c = e^{-2g_3}, \quad d = -e^{-2g_4}, \tag{4.3}$$

the corresponding eigenvalue equation becomes of the form

$$Dp_n(x) = E_n p_n(x), \tag{4.4a}$$

where D denotes the difference operator

$$D = W(x)(T_{ig} - 1) + W(-x)(T_{-ig} - 1) \qquad ((T_{ig}f)(x) := f(x + ig)), \tag{4.4b}$$

with

$$W(x) = \frac{\sin(x + ig_1)\cos(x + ig_2)\sin(x + ig_3)\cos(x + ig_4)}{\sin(2x)\sin(2x + ig)}, \tag{4.4c}$$

and the eigenvalue is given by

$$E_n = \big(\cosh(\hat{g} + 2ng) - \cosh(\hat{g})\big)/2, \qquad \hat{g} = g_1 + g_2 + g_3 + g_4 - g. \tag{4.4d}$$

After these preliminaries, we are now in the position to prove the main result. It follows from the general fact that the polynomials form an orthogonal system on the interval $(0, \pi/2)$ with respect to a positive weight function that the Askey-Wilson polynomial $p_n(x)$ has n simple zeros inside the interval $(0, \pi/2)$. If we denote these zeros by $x_1, \ldots, x_n$, then it is clear that the Askey-Wilson polynomial factorizes as

$$p_n(x) = 2^{2n-1} \prod_{k=1}^{n} \sin(x_k + x)\sin(x_k - x) \tag{4.5}$$

(since $2\sin(x_k+x)\sin(x_k-x) = \cos(2x)-\cos(2x_k)$). After plugging the factorization of the Askey-Wilson polynomial from Eq. (4.5) into the difference equation in Eqs. (4.4a)-(4.4d), and setting of x equal to the jth root x_j, one arrives at the identity

$$
\begin{aligned}
&W(x_j)\prod_{k=1}^{n}\sin(x_k+x_j+ig)\sin(x_k-x_j-ig)+ \\
&W(-x_j)\prod_{k=1}^{n}\sin(x_k+x_j-ig)\sin(x_k-x_j+ig)=0,
\end{aligned} \tag{4.6}
$$

which amounts to Eq. (3.1b). This shows that the roots of the Askey-Wilson polynomial solve the nonlinear system of algebraic equations in Eq. (3.1b).

To see that this is the only solution (up to permutation), we now assume—reversely—that the points $0 < x_k < \pi/2$, $k = 1, \ldots, n$ are such that they constitute *any* solution to Eq. (3.1b), and show that this implies that the corresponding factorized polynomial of the form $p_n(x)$ (4.5) must be equal to the Askey-Wilson polynomial. To this end it is sufficient to infer that the factorized polynomial in question solves the eigenvalue equation in Eqs. (4.4a)-(4.4d) (since the spectrum of D is nondegenerate as a continuous function of the parameters and thus determines the eigenpolynomials uniquely). It is clear from the fact that the Askey-Wilson polynomials form the corresponding eigenbasis that acting with the operator D (4.4b), (4.4c) on a monic polynomial of the form in Eq. (4.5) produces the eigenvalue E_n (4.4d) times a certain monic polynomial $q_n(x)$ of degree n. Furthermore, we have that $E_n q_n(x_j) = (Dp_n)(x_j) = 0$ for $j = 1, \ldots, n$, because of Eq. (4.6) (which holds since the points $x_1, \ldots, x_n$ solve Eq. (3.1b) by assumption). Hence, the monic polynomial $q_n(x)$ has the same roots as $p_n(x)$ and thus coincides with it. In other words, the factorized polynomial $p_n(x)$ solves the Askey-Wilson difference equation, and is thus equal to the Askey-Wilson polynomial, whence the roots $x_1, \ldots, x_n$ correspond to the roots of the Askey-Wilson polynomial. This gives rise to the following theorem.

Theorem 2. *The unique (up to permutation) solution* $0 < x_k < \pi/2$*,* $k = 1, \ldots, n$ *of the nonlinear system of algebraic equations in Eq.* (3.1b) *is given by the (simple) roots of the Askey-Wilson polynomial* $p_n(x)$ (4.2) *with parameters of the form in Eq.* (4.3)*.*

By combining Theorem 1 and Theorem 2, we end up with the desired charaterization of the ground-state equilibrium configuration in terms of zeros of the Askey-Wilson polynomial.

Corollary. *The trigonometric* BC*-type Ruijsenaars-Schneider Hamiltonian* $H(\mathbf{p}, \mathbf{x})$ *in Eqs.* (2.1a)-(2.1c) *assumes the global minimum* $H = 0$ *only at the point in the phase space* Ω (2.2) *such that* $p_1 = p_2 = \cdots = p_n = 0$ *and* $0 < x_1 < x_2 < \cdots < x_n < \pi/2$ *are given by the (simple) roots of the Askey-Wilson polynomial* $p_n(x)$ (4.2) *with parameters of the form in Eq.* (4.3)*.*

5 Rational degeneration

By working the way down the Askey hierarchy of (basic) hypergeometric orthogonal polynomials, starting from the Askey-Wilson polynomials corresponding to the trigonometric

BC-type Ruijsenaars-Schneider systems, one arrives at the equilibrium configurations associated to the degenerate Ruijsenaars-Schneider systems considered by Sasaki *et al* [13, 11] and at the equilibrium configurations associated to the Calogero-Moser systems considered by Calogero *et al* [2, 3, 4, 12]. As an example, we will wrap up by detailing the important case of the rational BC-type Ruijsenaars-Schneider system [6]. In this case the ground-state equilibrium turns out to be given by the zeros of the Wilson polynomials [11].

The Hamiltonian $H(\mathbf{p},\mathbf{x})$ of the rational BC-type Ruijsenaars-Schneider system is given by Eqs. (2.1a), (2.1b) with potentials of the form [6]

$$v(x) = (x+ig)/x, \quad w(x) = (x+ig_1)(x+ig_2)(x+ig_3)(x+ig_4)/x^2. \tag{5.1}$$

The phase space becomes in this situation

$$\Omega = \{(\mathbf{p},\mathbf{x}) \in \mathbb{R}^{2n} \mid 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n\}. \tag{5.2}$$

The following theorem characterizes the ground-state equilibrium configuration of the rational BC-type Ruijsenaars-Schneider Hamiltonian in terms of the zeros of the Wilson polynomials [17, 9]

$$\begin{aligned} p_n(x) = {} & \frac{(-1)^n (a+b, a+c, a+d)_n}{(n+a+b+c+d-1)_n} \\ & \times {}_4F_3\left[\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} \mid 1 \right], \end{aligned} \tag{5.3a}$$

which satisfy the orthogonality relations

$$\int_0^\infty p_n(x)\overline{p_m(x)}\Delta(x)\mathrm{d}x = 0, \qquad n \neq m, \tag{5.3b}$$

associated to the positive weight function

$$\Delta(x) = \frac{1}{c(x), c(-x)}, \quad c(x) = \frac{\Gamma(2ix)}{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}, \tag{5.3c}$$

where $a, b, c, d > 0$ (and $\Gamma(\cdot)$ refers to the gamma function).

Theorem 3. *The rational BC-type Ruijsenaar-Schneider Hamiltonian $H(\mathbf{p},\mathbf{x})$ from Eqs.* (2.1a),(2.1b)*, with potentials of the form in Eq.* (5.1)*, has a unique global minimum $H = 0$ in the phase space Ω* (5.2) *at $p_1 = p_2 = \cdots = p_n = 0$ and $0 < x_1 < x_2 < \cdots < x_n$ given by the (simple) roots of the rescaled Wilson polynomial $p_n(x/g)$* (5.3a) *with rescaled parameters $a = g_1/g$, $b = g_2/g$, $c = g_3/g$, $d = g_4/g$; these roots in turn constitute the unique positive solution to the nonlinear system of algebraic equations*

$$\prod_{1\le k\le n,\, k\neq j} \frac{(x_j+x_k+ig)(x_j-x_k+ig)}{(x_j+x_k-ig)(x_j-x_k-ig)} = \frac{(x_j-ig_1)(x_j-ig_2)(x_j-ig_3)(x_j-ig_4)}{(x_j+ig_1)(x_j+ig_2)(x_j+ig_3)(x_j+ig_4)},$$

$j = 1, \ldots, n$.

It is clear from the theorem that varying the value of the coupling parameter g gives rise to a linear rescaling of the equilibrium positions. More specifically, starting from the $g = 1$ configuration corresponding to the zeros $x_1, \ldots, x_n$ of the Wilson polynomials $p_n(x)$ (5.3a) with $a = g_1$, $b = g_2$, $c = g_3$, $d = g_4$, one passes to the equilibrium configuration for general positive g via the rescaling $x_j \to gx_j$, $j = 1, \ldots, n$.

The proof of the above theorem runs along the same lines of Sections 3 and 4, and hinges on the second-order difference equation for the rescaled Wilson polynomials of the form in Eqs. (4.4a), (4.4b) with

$$W(x) = \frac{(x + ig_1)(x + ig_2)(x + ig_3)(x + ig_4)}{2x(2x + ig)}, \qquad E_n = -ng(ng + \hat{g}) \tag{5.4}$$

(cf. e.g. [9]). Indeed, it is immediate that the minimization condition $V_j(\mathbf{x}) = V_j(-\mathbf{x})$ now gives rise to the algebraic equations for the equilibrium points stated in the theorem; furthermore, the solution of this system readily follows upon substitution of the factorization $p_n(x) = \prod_{k=1}^{n}(x + x_k)(x - x_k)$ into the difference equation for the rescaled Wilson polynomials.

Acknowledgements

This work was supported in part by the Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Grant No. # 1010217. Thanks are due to Professor Ryu Sasaki for pointing out Ref. [10].

References

[1] Askey R and Wilson J, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* **54** (1985), No. 319.

[2] Calogero F, On the zeros of the classical polynomials, *Lett. Nuovo Cimento* **19** (1977), 505–508.

[3] Calogero F, Equilibrium configuration of the one-dimensional n-body problem with quadratic and inversely quadratic pair potentials, *Lett. Nuovo Cimento (2)* **20** (1977), 251–253.

[4] Calogero F and Perelomov A M, Properties of certain matrices related to the equilibrium configuration of the one-dimensional many-body problems with the pair potentials $V_1(x) = -\log \sin x$ and $V_2(x) = 1/\sin^2 x$, *Comm. Math. Phys.* **59** (1978), 109–116.

[5] Corrigan E and Sasaki R, Quantum versus classical integrability in Calogero-Moser systems, *J. Phys. A: Math. Gen.* **35** (2002), 7017–7061.

[6] van Diejen J F, Integrability of difference Calogero-Moser systems, *J. Math. Phys.* **35** (1994), 2983–3004.

[7] van Diejen J F Deformations of Calogero-Moser systems and finite Toda chains, *Theoret. and Math. Phys.* **99** (1994), 549–554.

[8] Ismail M E H, Lin S, and Roan S, Bethe Ansatz equations of XXZ model and q-Sturm-Liouville problems, *arXiv:math-ph/0407033* (2004).

[9] Koekoek R and Swarttouw R F, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, *arXiv:math.CA/9602214* (1996).

[10] Macdonald I G, Some conjectures for root systems, *SIAM J. Math. Anal.* **13** (1982), 988–1007.

[11] Odake S and Sasaki R, Equilibria of 'discrete' integrable systems and deformation of classical orthogonal polynomials, *arXiv:hep-th/0407155* (2004).

[12] Olshanetsky M A and Perelomov M A, Classical integrable finite-dimensional systems related to Lie algebras, *Phys. Rep.* **71** (1981), 313–400.

[13] Ragnisco O and Sasaki R, Quantum vs classical integrability in Ruijsenaars-Schneider systems, *J. Phys. A: Math. Gen.* **37** (2004), 469–479.

[14] Ruijsenaars S N M and Schneider H, A new class of integrable systems and its relation to solitons, *Ann. Phys. (NY)* **170** (1986), 370–405.

[15] Ruijsenaars S N M, Action-angle maps and scattering theory for some finite-dimensional integrable systems. III. Sutherland type systems and their duals, *Publ. Res. Inst. Math. Sci.* **31** (1995), 247–353.

[16] Szegö G, Orthogonal Polynomials, Fourth Edition, Amer. Math. Soc., Providence, R.I., 1975.

[17] Wilson J A, Some hypergeometric orthogonal polynomials, *SIAM J. Math. Anal.* **11** (1980), 690–701.