Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 535–547 Birthday Issue

What an Effective Criterion of Separability says about the Calogero Type Systems

Stefan RAUCH-WOJCIECHOWSKI † *and Claes WAKSJÖ*

† *Matematiska institutionen, Linköpings universitet, SE-581 83 Linköping, Sweden*
E-mail: strau@mai.liu.se

This article is part of the special issue published in honour of Francesco Calogero on the occasion of his 70th birthday

Abstract

In [15] we have proved a 1-1 correspondence between all separable coordinates on R^n (according to Kalnins and Miller [9]) and systems of linear PDEs for separable potentials $V(q)$. These PDEs, after introducing parameters reflecting the freedom of choice of Euclidean reference frame, serve as an effective criterion of separability. This means that any $V(q)$ satisfying these PDEs is separable and that the separation coordinates can be determined explicitly. We apply this criterion to Calogero systems of particles interacting with each other along a line.

1 Introduction

Separation of variables is the most significant method for solving Hamilton's equations for natural Hamiltonians and for solving the stationary Schrödinger equation. Almost all book examples, such as the Kepler problem, the hydrogen atom, the Stark problem, the problem of two centers of gravitation and many others have been solved by separating equations in some appropriate curvilinear coordinates.

A physical problem, characterised by a natural Hamiltonian $H = \frac{1}{2}p^2 + V(q)$, $(q,p) = (q_1,\dots,q_n;p_1,\dots,p_n)$, or by a stationary Schrödinger equation $\big(\sum_{i=1}^n(\partial/\partial q_i)^2+V(q)\big)\Psi(q) = E\Psi(q)$, is usually expressed in terms of Cartesian coordinates q_i. The method of separation of variables consists of finding such curvilinear coordinates $x(q) = (x_1,\dots,x_n)$ so that the transformed Hamiltonian $H = \frac{1}{2}p^2 + V(q) = \frac{1}{2}\sum_{i,j=1}^n g^{ij}(x)\,y_iy_j + V(q(x))$, where y_i denotes momenta conjugate to x_i, admits an additively separated solution of the corresponding Hamilton–Jacobi equation. For the transformed Schrödinger equation $\big(\Delta + V(q(x))\big)\Psi(x) = E\,\Psi(x)$ it is the question of finding a multiplicatively separated solution $\Psi(x) = \psi_1(x_1)\cdots\psi_n(x_n)$. Until now, for each particular problem the variables of separation have been invented through a qualified ansatz based on some transparent or, maybe hidden, geometrical symmetry of the problem.

1.1 The problem

When applying the method of separation of variables to physical problems, the most important problem is *to decide if a given potential $V(q)$ is separable,* i.e., to decide if there exists coordinates $x(q)$ such that the problem can be separated, and *to determine these coordinates explicitly* in order to solve the problem.

In his lectures on dynamics, Jacobi himself considered this problem not to be algorithmically solvable. He writes (see the translation in Arnol'd [1, p. 266]): "The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied."

This statement had a profound influence on the further development of separability theory, which mainly was concerned with the question of characterizing all separable Hamiltonians in terms of separation coordinates. Stäckel [14] formulated a necessary and sufficient condition for separability of natural Hamiltonians. Levi-Civita [10] found a system of equations to be satisfied by any separable Hamiltonian. Eisenhart [6] derived a complete description of separable Riemannian metrics $g_{ij}(x)$ and solved his conditions for spaces of constant curvature. Benenti [2] described completely separable webs for all natural Hamiltonians including those with cyclic coordinates. Kalnins and Miller [9] classified all orthogonal separable coordinates on R^n and S^n.

Even if these results at first seem to be formulated only for natural Hamiltonian systems, it is known since Eisenhart [6] that (in R^n or S^n, or more generally, spaces with diagonal Ricci tensor) the separability of such systems is equivalent to the separability of the stationary Schrödinger equation.

We present here the main steps of the complete solution to the Jacobi problem, which has been given in our paper [15]. The solution has the form of an algorithm, and we will apply it to the Calogero systems of particles interacting with each other in one dimension.

2 Formulation of a criterion of separability

Not all coordinate systems allow for separation of variables. In order to understand the criterion it is necessary to have some knowledge of separable coordinate systems.

2.1 Separable coordinate systems

The classical algebraic condition for an orthogonal coordinate system to be separable is that the metric is in Stäckel form [14]. An equivalent condition is that the metric satisfies a certain system of PDEs given by Levi-Civita [10]. Eisenhart [6] solved these PDEs under the assumption of constant curvature, and found that the metric of a separable coordinate system only can take on a few very special distinct forms. Eisenhart also determined all corresponding coordinate systems in R^3; there are eleven of them, and they are well-known today (see for instance [12]).

Kalnins and Miller [9] subsequently generalized Eisenhart's analysis to R^n and S^n of arbitrary dimensions. From their results follows that all orthogonal separable coordinate systems in R^n can be viewed as degenerations of the elliptic coordinate system introduced

by Jacobi. This coordinate system is defined by the rational equation

$$1+\sum_{i=1}^{n}\frac{q_i^2}{z-\lambda_i}=\prod_{j=1}^{n}(z-x_j)\Big/\prod_{k=1}^{n}(z-\lambda_k),$$

where $\lambda_1<\cdots<\lambda_n$ are n fixed parameters determining the eccentricity of the coordinate surfaces. This equation allows for determining functions $q_i=q_i(x;\lambda)$ by examining the residues at $z=\lambda_i$.

Elliptic coordinates can be subjected to an improper degeneration by letting λ_n tend to $+\infty$. This results in the subclass of parabolic coordinates: the parabolic coordinate system with parameters $\lambda_1<\cdots<\lambda_{n-1}$ is defined through the equation

$$\sum_{i=1}^{n-1}\frac{q_i^2}{z-\lambda_i}+2q_n-z=-\prod_{j=1}^{n}(z-x_j)\Big/\prod_{k=1}^{n-1}(z-\lambda_k).$$

Elliptic and parabolic coordinates can also be degenerated by letting some parameters coincide; this introduces a spherical symmetry in the subspace corresponding to the coinciding parameters. In these subspaces R^m, a special kind of polar coordinates (r,ω) known as conical coordinates have to be introduced to preserve separability. These are defined by $q=r\omega$ and $\omega_1^2+\cdots+\omega_m^2=1$ (so that $r^2=q_1^2+\cdots+q_m^2$) and by choosing elliptic coordinates on the unit sphere. In general, the elliptic coordinate system on S^{m-1} (embedded in R^m) with parameters $\lambda'_1<\cdots<\lambda'_m$ (which should not be confused with the coinciding lambdas from the definition of degenerated elliptic/parabolic coordinates) is defined through the equation

$$\sum_{i=1}^{m}\frac{\omega_i^2}{z-\lambda'_i}=\prod_{j=1}^{m-1}(z-x_j)\Big/\prod_{k=1}^{m}(z-\lambda'_k).$$

The radii r will then replace the Cartesian coordinates corresponding to the subspaces R^m in the definition of elliptic or parabolic coordinates on R^n. Together with the coordinates on the spheres, this yields a coordinate system on R^n.

The elliptic coordinates on the sphere can also be degenerated by letting parameters coincide, and on the corresponding subspheres, new elliptic coordinates have to be introduced. This process can then be continued in a recursive manner to generate all degenerate elliptic and parabolic coordinate systems.

To get all separable orthogonal coordinate systems, degenerate or non-degenerate elliptic or parabolic coordinate systems can be glued together orthogonally. By this is meant that R^n is considered an orthogonal sum of subspaces, on which the above-mentioned coordinate systems are introduced. For instance, the Cartesian coordinate system is the ultimate variant of this construction, since it can be viewed as a sum of n one-dimensional elliptic coordinate systems. All coordinate systems constructed this way are given with respect to a canonical Euclidean reference frame $\{q\}$. It is however also possible to subject R^n to a Euclidean transformation $q\to Aq+b$, which is the composition of a rotation [multiplication by a $\mathrm{SO}(n)$ matrix A] and a translation (addition of a vector b).

As follows from Kalnins and Miller [9], it is sufficient to consider orthogonal separability since any non-orthogonal separable potential (in R^n or S^n, or more generally, Riemannian spaces of constant curvature) always admits orthogonal separation.

2.2 PDEs that characterize separable potentials

Already Levi-Civita derived a system of $\binom{n}{2}$ PDEs that a separable potential given in separation coordinates has to satisfy. However, a physical potential $V(q)$ is usually given in Cartesian coordinates and appropriate separation coordinates have to be found.

The first attempts to characterize separable potentials $V(q)$ were made by Rauch-Wojciechowski [17], Marshall and Rauch-Wojciechowski [11] and by Benenti [3], who independently and by different means proved the following. (In the former references a larger dependent set of PDEs was given but, as is shown in [15, Proposition A.1], they are equivalent to the PDEs given below.)

Theorem 1 (Elliptic separability). *A potential V is separable in elliptic coordinates with parameters λ_i if and only if it satisfies the system of $\binom{n}{2}$ PDEs*

$$(\lambda_i - \lambda_j)\partial_{ij}V - \mathsf{J}_{ij}(2+\mathsf{R})V = 0, \quad i \neq j, \tag{2.1}$$

where $\mathsf{J}_{ij} = q_i\partial_j - q_j\partial_i$ and $\mathsf{R} = \sum_{i=1}^n q_i\partial_i$.

The partial derivatives are $\partial_i = \partial/\partial q_i$ and $\partial_{ij} = \partial^2/\partial q_i \partial q_j$, so that the PDEs (2.1) are formulated purely through Cartesian coordinates. Since the PDEs also involve the parameters of the elliptic coordinates, they can be used to test if a potential $V(q)$ is separable in an elliptic coordinate system. Namely, if it is possible to find values of the parameters λ_i so that the potential satisfies (2.1), then the potential separates in elliptic coordinates with these parameters.

Equations (2.1) are given in a fixed canonical Euclidean reference frame $\{q\}$, and as such they do not take into account the freedom of choice of a Euclidean reference frame. This freedom has to be introduced by substituting $\tilde{q} = A^T(q-b)$, $A \in SO(n)$, $b \in R^n$, for q, where the orthogonal matrix A obeys the non-linear constraint $A^TA = \mathrm{Id}$. If we rename q into $\tilde{q}$ in equations (2.1) and plug in $\tilde{q} = A^T(q-b)$ then

$$(\lambda_k - \lambda_l)\sum_{r,s} A_{rk}A_{sl}\partial_{rs}V - \sum_{r,s}(A_{rl}A_{sk} - A_{rk}A_{sl})(q_s - b_s)\partial_r\left[2 + \sum_t (q_t - b_t)\partial_t\right]V = 0.$$

By multiplying with $A^T_{ki}A^T_{lj}$ and summing over k and l we get

$$\sum_r (S_{ri}\partial_{rj} - S_{rj}\partial_{ri})V - [(q_i - b_i)\partial_j - (q_j - b_j)\partial_i]\left[2 + \sum_t (q_t - b_t)\partial_t\right]V = 0 \tag{2.2}$$

where $S = (S_{ij}) = A\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)A^T$ is a symmetric matrix with all matrix elements free. By performing differentiations in (2.2) and by renaming coefficients as $\beta = -\alpha b$, $\gamma = \alpha(bb^T - S)$ we get equations (2.3) below. Surprisingly, in the transformed equations (2.3), the constraint $A^TA = \mathrm{Id}$ disappears and all parameters are again free. This makes practical application of these equations as a criterion of separability exceptionally simple.

Lemma 2 ([15, Lemma 4.1]). *A necessary and sufficient condition for a potential V to be separable in an elliptic coordinate system with respect to some Euclidean reference frame, is that it satisfies*

$$\sum_k \big((\alpha q_i q_k + \beta_i q_k + \beta_k q_i + \gamma_{ik})\partial_{kj}V - (\alpha q_j q_k + \beta_j q_k + \beta_k q_j + \gamma_{jk})\partial_{ki}V\big) + 3\big((\alpha q_i + \beta_i)\partial_j V - (\alpha q_j + \beta_j)\partial_i V\big) = 0, \tag{2.3}$$

for some parameters α, β_i, $\gamma_{ij} = \gamma_{ji}$, *which fulfil the condition*

$$\alpha \neq 0 \quad \textit{and} \quad \beta\beta^T - \alpha\gamma \textit{ has simple eigenvalues,} \tag{2.4}$$

where $\beta = (\beta_i) \in R^n$ *and* $\gamma = (\gamma_{ij})$ *is a real symmetric* $n \times n$ *matrix.*

For a trivial set of parameters

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = t\,(\mathrm{Id}), \quad t \in R,$$

Equations (2.3) are satisfied by an arbitrary V. On the other hand, (2.4) is precisely the condition needed to guarantee that V satisfies (2.1) after a Euclidean transformation of V.

We call Equations (2.3) Bertrand–Darboux (BD) equations since they play the same role for detecting separability of $V(q)$ as the two-dimensional BD equation [11, 16].

2.3 A criterion for elliptic separability in a general frame

A direct consequence of the proof of Lemma 2 is the following corollary, which explains how to construct the separation coordinates.

Corollary 3 ([15, Corollary 4.2]). *Suppose that the potential* $V(q)$ *satisfies the hypothesis of Lemma 2. Then it is separable in the elliptic coordinate system* $\{x_i\}$ *defined by* $1 + \sum_{i=1}^n (q_i')^2/(z - \lambda_i) = \prod_{j=1}^n (z - x_j)/\prod_{k=1}^n (z - \lambda_k)$. *The Cartesian coordinates* q_i' *are related to* q_i *through* $q = Aq' + b$ *where* $b = -\alpha^{-1}\beta$. *The parameters* λ_k *are the eigenvalues of the symmetric matrix* $S = bb^T - \alpha^{-1}\gamma$ *sorted in increasing order* $\lambda_1 < \cdots < \lambda_n$. *The orthogonal matrix* A *is given by the relation* $S = A\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)A^T$.

This corollary shows how the BD equations can be used as a criterion of separability to test if potentials given in Cartesian coordinates are separable in elliptic coordinates. A simple algorithm that shows how to proceed can be formulated as follows:

1) Insert $V(q)$ into the BD equations, which has to be satisfied identically with respect to q_i, $i = 1, \ldots, n$. This gives a system of linear homogeneous algebraic equations for the unknown parameters α, β_i, γ_{ij}. If $\alpha = 0$, then $V(q)$ is not separable in elliptic coordinates.

2) If $\alpha \neq 0$, set $b = -\alpha^{-1}\beta$, $S = bb^T - \alpha^{-1}\gamma$, and diagonalize $S = A\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)A^T$. If some eigenvalues λ_i coincide, then $V(q)$ is not separable in elliptic coordinates. Otherwise $V(q)$ is separable in elliptic coordinates with parameters λ_i. The change of coordinates is given in the corollary.

Note that this algorithm not only gives definite answers to the question if a given potential is separable in elliptic coordinates; it also gives an explicit construction of these coordinates in the separable case.

2.4 A general criterion of separability

What can be said if condition (2.4) is not fulfilled but the potential $V(q)$ nevertheless satisfies the BD equations with non-trivial parameters? Since elliptic coordinates are the most general separable coordinates, one may expect separability in some of its degenerations. The following theorem clarifies the situation; it gives a classification of the equations that are satisfied.

Theorem 4 ([15, Theorem 4.3]). *The BD equations can always be brought into one of the following three canonical forms by a transformation to the canonical Euclidean reference frame. If $\alpha \neq 0$, the canonical form is the elliptic*

$$(\lambda_i - \lambda_j)\partial_{ij}V - \mathsf{J}_{ij}(2+\mathsf{R})V = 0; \tag{2.5a}$$

if $\alpha = 0$ and $\beta \neq 0$, it is the parabolic

$$(\lambda_i - \lambda_j)\partial_{ij}V + \mathsf{J}_{ij}\partial_n V + (\delta_{in}\partial_j - \delta_{jn}\partial_i)(2+\mathsf{R})V = 0; \tag{2.5b}$$

if $\alpha = 0$ and $\beta = 0$, it is the Cartesian

$$(\lambda_i - \lambda_j)\partial_{ij}V = 0; \tag{2.5c}$$

where $\mathsf{J}_{ij} = q_i\partial_j - q_j\partial_i$ and $\mathsf{R} = \sum_{i=1}^n q_i\partial_i$. In all these systems, indices $i, j = 1, \dots, n$ are distinct.

The parameters λ_i occurring in the theorem, as well as the Euclidean transformation $q \to Aq + b$, can be determined similarly as in Corollary 3. These procedures, which slightly differ in each case, essentially consist of diagonalizing a symmetric matrix that is a linear function of γ. An application of the spectral theorem gives the rotation matrix A and the eigenvalues λ_i. The translation vector b is linear in β.

There are only $n-1$ parameters in the definition of the parabolic coordinate system; in order to incorporate these into the general picture, the convention $\lambda_n = 0$ is therefore used. Equations (2.5b) are thus a shorthand for $(\lambda_i - \lambda_j)\partial_{ij}V + \mathsf{J}_{ij}\partial_n V = 0$ and $\lambda_i\partial_{in}V + \mathsf{J}_{in}\partial_n V - \partial_i(2+\mathsf{R})V = 0$, where $i, j = 1, \dots, n-1$, $i \neq j$.

The significance of Theorem 4 is the following. Suppose that a potential satisfies the BD equations (2.3) for certain values of the parameters. Then, by rewriting it in the canonical Euclidean frame, the transformed potential satisfies one of the three canonical forms of the BD equations (2.5). Since the general solution of these systems is known [15, Theorem 5.3], it only remains to explain what this indicates for the particular potential under study.

2.4.1 Distinct lambdas

If all λ_i are distinct, it follows that the potential is separable in elliptic, parabolic or Cartesian coordinates. Indeed, the elliptic case has already been discussed, and analogously, Equations (2.5b) and (2.5c) are satisfied only by potentials separable in parabolic and Cartesian coordinates respectively. The reason for this is that (2.5b) arise when degenerating the elliptic coordinates in Theorem 1 to parabolic coordinates as discussed earlier by letting λ_n tend to $+\infty$. Further, it is clear that the general solution of (2.5c) is a sum of functions of one variable $V = V_{(1)}(q_1) + \cdots + V_{(n)}(q_n)$ and this is the most general potential that is separable in Cartesian coordinates.

2.4.2 Coinciding lambdas

Suppose now that some lambdas coincide. Then the potential may be separable in a degenerated coordinate system; the type is decided by α and β as in Theorem 4. For

the sake of brevity, assume that the lambdas are sorted in non-decreasing order and that there is only one group of coinciding lambdas: $\lambda_1 < \cdots < \lambda_i = \cdots = \lambda_j < \cdots < \lambda_n$, where $1 \le i < j \le n$. If there are more than one group of coinciding lambdas, the corresponding degenerations have to be taken care of by an obvious generalization of the procedure presented below.

Cartesian case. It is immediate that the potential can be written

$$V = V_{(1)}(q_1) + \cdots + V_{(i-1)}(q_{i-1}) + V_{(i)}(q_i, \dots, q_j) + V_{(j+1)}(q_{j+1}) + \cdots + V_{(n)}(q_n).$$

It is separable if $V_{(i)}(q_i, \dots, q_j)$ is separable regarded as a potential, which can be tested by using a lower-dimensional system of BD equations. The separation coordinates will be of orthogonal sum-type: Cartesian coordinates in $\{q_1, \dots, q_{i-1}, q_{j+1}, \dots, q_n\}$ and appropriate separation coordinates on $\{q_i, \dots, q_j\}$.

Elliptic/Parabolic cases. The coinciding lambdas indicate that a spherical symmetry is present in the potential and that polar coordinates should be introduced in the corresponding subspace. Set $(q_i, \dots, q_j) = r \cdot (\omega_i, \dots, \omega_j)$, where $\omega_i, \dots, \omega_j$ are Cartesian coordinates on the unit sphere in the subspace $\{q_i, \dots, q_j\}$. It turns out that the potential can be written as

$$V = f(q_1, \dots, q_{i-1}, r, q_{j+1}, \dots, q_n) + r^{-2}\, g(\omega_i, \dots, \omega_j),$$

and that the function f, regarded as a potential, is separable in elliptic/parabolic coordinates with parameters $\lambda_1, \dots, \lambda_{i-1}, \lambda_i, \lambda_{j+1}, \dots, \lambda_n$. This means that the spherical part g can be separated off and that V is separable if this part is also separable regarded as a potential on the unit sphere. However, if $j = i + 1$, it is always possible to set $(q_i, q_j) = r \cdot (\cos\varphi, \sin\varphi)$, and g is trivial from the point of view of separability since it only depends on φ.

For disclosing separability of the spherical part, there is a test similar to the previous one. It uses cyclic Bertrand–Darboux (CBD) equations:

Theorem 5 ([15, Theorem 4.4]). *The CBD equations*

$$\sum_{\ell} \big((\gamma_{i\ell} q_j - \gamma_{j\ell} q_i)\partial_{k\ell} V + (\gamma_{j\ell} q_k - \gamma_{k\ell} q_j)\partial_{i\ell} V + (\gamma_{k\ell} q_i - \gamma_{i\ell} q_k)\partial_{j\ell} V \big) = 0, \tag{2.6}$$

where $\gamma_{ij} = \gamma_{ji}$, can always be brought into the canonical cyclic form

$$\lambda_i' \partial_i \mathsf{J}_{jk} V + \lambda_j' \partial_j \mathsf{J}_{ki} V + \lambda_k' \partial_k \mathsf{J}_{ij} V = 0 \tag{2.7}$$

by a transformation to the canonical Euclidean reference frame.

Both systems (2.6) and (2.7) are linearly dependent; a basis is formed by fixing one index as below. The CBD equations were first published in [3], but they also follow from the large set of linearly dependent equations given in [11].

Also in this case the transformation $q \to Aq$ and the lambdas follow from the diagonalization of $\gamma = A\,\mathrm{diag}(\lambda_1', \dots, \lambda_n')A^T$. Notice that there is no translation vector b here since the center of Euclidean reference frame is fixed now. The meaning of the CBD equations can be stated as follows:

Theorem 6 (Conical separability). *A potential V is separable in conical coordinates with parameters λ_i' if and only if it satisfies the system of $\binom{n}{2}$ PDEs*

$$\mathsf{J}_{1i}(2+\mathsf{R})V = 0, \quad i = 2, \dots, n, \tag{2.8a}$$

$$\lambda_1'\partial_1\mathsf{J}_{ij}V + \lambda_j'\partial_j\mathsf{J}_{1i}V + \lambda_i'\partial_i\mathsf{J}_{j1}V = 0, \quad i \neq j, \quad i,j = 2, \dots, n. \tag{2.8b}$$

As formulated here, there are two types of equations that characterize potentials separable in conical coordinates. Equations (2.8a) ensure that the form of the potential is $f(r) + r^{-2}g(\omega)$ in polar coordinates, while Equations (2.8b) ensure that g is separable in elliptic coordinates on the sphere.

It should be noted that the system of elliptic canonical equations (2.1) with all coinciding parameters $\lambda_1 = \cdots = \lambda_n$, due to linear dependence of the operators J_{ij} is equivalent to (2.8a). This implies that if a potential satisfies the BD equations with $\alpha \neq 0$ and all lambdas coincide, then it is separable in conical coordinates if it also satisfies the CBD equations with distinct (primed) lambdas. This is in agreement with the general procedure outlined above.

3 Examples

Since any separable coordinate system is a possibly degenerated elliptic or parabolic coordinate system, or is made up of an orthogonal sum of such systems, the considerations in the previous section explain why the BD equations are a natural starting point in the investigation of separability of any given potential.

3.1 Three-dimensional Coulomb anisotropic harmonic oscillator

The Coulomb anisotropic harmonic oscillator potential [7] can be written

$$V(q) = (q_1^2 + q_2^2 + q_3^2)^{-1/2} + q_1^2 + q_2^2 + 4q_3^2 + 2\sigma\, q_3,$$

where σ is a constant. The potential satisfies the BD equations with

$$\alpha = 0, \quad \beta = (0,0,w)^T, \quad \gamma = t\,(\mathrm{Id}), \qquad w, t \in R,$$

which corresponds to rotational parabolic coordinates defined by

$$q^T = \big(r(x_1,x_2)\cdot\cos x_3, r(x_1,x_2)\cdot\sin x_3, q_3(x_1,x_2)\big) - \tfrac{1}{2}t(0,0,1).$$

Here (x_1,x_2) are parabolic coordinates in $\{r,q_3\}$ with parameter $\lambda_1 = t$, and x_3 is a coordinate on the unit sphere in $\{q_1,q_2\}$. The parabolic coordinates are given by $r^2/(z-t) + 2q_3 - z = -(z-x_1)(z-x_2)/(z-t)$, which yields $r = [-(t-x_1)(t-x_2)]^{1/2}$ and $q_3 = \frac{1}{2}(x_1+x_2) - \frac{1}{2}t$.

Since the potential only involves q_1 and q_2 in the combination $q_1^2 + q_2^2 = r^2$, it is independent of x_3, which therefore is a cyclic coordinate. In the coordinates (x_1,x_2,x_3), the potential takes the form

$$V = \frac{f(x_1) - f(x_2) + 2}{x_1 - x_2}, \quad f(x) = x^3 + (\sigma - 3t)x^2 + (3t^2 - 2\sigma t)x,$$

which admits separation of variables. The parameter t remains free and can be chosen arbitrarily.

3.2 Four-dimensional Calogero inverse-square potential

The Calogero system consists of n particles interacting with each other along a line under the action of an inverse square potential depending on the distance $q_i - q_j$ between particles. This potential for $n = 3$ was considered already by Jacobi [8]. It attracted again attention of Calogero in 1969 [5] who rediscovered its separability and also studied the quantum version of this problem when an external, homogeneous, harmonic potential $\frac{1}{2}\omega^2(q_1^2 + \cdots + q_n^2)$ is added. Since then this system attracted enormous attention as a key example for illustrating the use of isospectral methods in classical mechanics. It has many known generalisations related to Lie algebras, to periodic boundary conditions and it admits adding of diverse external potentials (like the harmonic one) while preserving integrability of the system [13].

In three dimensions, the Calogero inverse-square potential is separable in five different coordinate systems [5, 4, 15]. The four-dimensional variant

$$V(q) = (q_1 - q_2)^{-2} + (q_1 - q_3)^{-2} + (q_1 - q_4)^{-2} + (q_2 - q_3)^{-2} + (q_2 - q_4)^{-2} + (q_3 - q_4)^{-2}$$

satisfies the BD equations with

$$\alpha = v, \quad \beta = w \cdot N, \quad \gamma = t\,(\mathrm{Id}) + s\,(NN^T - \mathrm{Id}), \qquad v, w, t, s \in R,$$

where $N = (1, 1, 1, 1)^T$. Since v and w can be chosen independently, all three cases of Theorem 4 occur.

Elliptic case. The parameters are $\lambda_1 = \lambda_2 = \lambda_3 = (s - t)/v$ and $\lambda_4 = (4r^2 - 3sv - tv)/v$. These can further coincide, giving rise to two different subcases: lambdas of multiplicity 4 and lambdas of multiplicity 3 and 1. In the latter case, the rotation matrix A can be taken as

$$A = \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{12} & -1/2 \\ -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{12} & -1/2 \\ 0 & 0 & 3/\sqrt{12} & -1/2 \\ 2/\sqrt{6} & 0 & -1/\sqrt{12} & -1/2 \end{pmatrix}.$$

Multiplicity 4. The radius in R^4 can be separated off and the remaining part decides about further separability. This is tested by the four-dimensional CBD equations, giving

$$\gamma = t\,(\mathrm{Id}) + s\,(NN^T - \mathrm{Id}), \quad t, s \in R.$$

This matrix has the same eigenvalues λ_i' as the parameters lambda above: $\lambda_i' = \lambda_i$, $i = 1, 2, 3, 4$. Suppose here that t and s are chosen so as to give multiplicity 3 and 1, in order to avoid trivial parameters corresponding to non-separability. On the eigenspace of dimension three, V is again tested using the three-dimensional CBD equations. This yields $\gamma = t\,(\mathrm{Id})$, which means that V is not separable since it has a non-separable part. Nevertheless, the criterion shows that V is partially separable in the coordinates defined by

$$q = r(x_1) \cdot A\big(\omega_1(x_2)\,\tilde{\omega}_1(x_3, x_4), \omega_1(x_2)\,\tilde{\omega}_2(x_3, x_4), \omega_1(x_2)\,\tilde{\omega}_3(x_3, x_4), \omega_2(x_2)\big)^T.$$

Here x_1 is a coordinate on the positive half-axis corresponding to the radius, x_2 is a coordinate on the unit sphere $\{\omega_1, \omega_2\}$ and (x_3, x_4) are some coordinates on the unit sphere $\{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3\}$. A possible choice is $r = x_1$ and $\omega_1(x_2) = \cos x_2$ and $\omega_2(x_2) = \sin x_2$. In these coordinates, the potential attains the form $V = r^{-2}\,\omega_1^{-2}\,h(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$, where h is non-separable. It can thus be concluded that the potential is indeed partially separable since

$$f(r) + r^{-2}[g(\omega_1, \omega_2) + \omega_1^{-2}\,h(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)]$$

is the most general potential that is separable in these coordinates provided that h is separable regarded as a potential on the two-dimensional unit sphere $\{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3\}$. In V, it is possible to separate off x_1 and x_2.

Multiplicity 3 and 1. As in the previous case, the three-dimensional CBD equations are applied to the three-dimensional eigenspace. It yields partially separable coordinates

$$q = A\big(r_1(x_1, x_2)\,\omega_{1,1}(x_3, x_4), r_1(x_1, x_2)\,\omega_{1,2}(x_3, x_4), r_1(x_1, x_2)\,\omega_{1,3}(x_3, x_4), r_2(x_1, x_2)\big)^T,$$

where (x_1, x_2) are elliptic coordinates in $\{r_1, r_2\}$ and (x_3, x_4) are some coordinates on the unit sphere $\{\omega_{1,1}, \omega_{1,2}, \omega_{1,3}\}$.

Parabolic case. This gives results similar to the second elliptic case above, except that (x_1, x_2) now are parabolic coordinates.

Cartesian case. Here the three-dimensional BD equations are applied to the three-dimensional eigenspace. This gives

$$\alpha = s, \quad \beta = 0, \quad \gamma = t\,(\mathrm{Id}), \qquad t, s \in R,$$

which indicates that polar coordinates should be introduced on this subspace. In order to decide appropriate coordinates on the unit sphere, the three-dimensional CBD are employed as above. The result is the same, giving partially separable coordinates

$$q = A\big(r(x_1)\,\omega_{1,1}(x_2, x_3), r(x_1)\,\omega_{1,2}(x_2, x_3), r(x_1)\,\omega_{1,3}(x_2, x_3), q_4'\big)^T.$$

Summary of the analysis. The four-dimensional Calogero potential studied here is not fully separable in any coordinate system, but partially separable in four different types of coordinate systems. In all cases two variables can be separated off, leaving a non-separable function on S^2. For the three-dimensional Calogero potential, the corresponding function lives on S^1 and is thus trivially separable, resulting in full separation. The two variables that can be separated off reflect the invariance of the Calogero Hamiltonian with respect to translations generated by $\sum p_i$ and dilatations generated by $\sum q_i p_i$. It is expected that for the n-dimensional Calogero potential, again two variables can be separated off, leaving a non-separable part on S^{n-2}.

3.3 Three-dimensional generalized Calogero inverse-square potential

A detailed account for an analysis of the Calogero inverse-square potential using the criterion is given in [15, § 7.2]. Here we outline the main features of a similar analysis of the generalized potential

$$V(q) = c_1(m_1 q_1 - m_2 q_2)^{-2} + c_2(m_2 q_2 - m_3 q_3)^{-2} + c_3(m_3 q_3 - m_1 q_1)^{-2} \tag{3.1}$$

with arbitrary $m_i > 0$ and coupling constants $c_i \neq 0$. This potential follows from the Calogero system of three particles with different masses μ_1, μ_2, μ_3 and with different coupling constants γ_1, γ_2, γ_3 defined by

$$H = \tfrac{1}{2}\left(\frac{p_1^2}{\mu_1} + \frac{p_2^2}{\mu_2} + \frac{p_3^2}{\mu_3}\right) + \frac{\gamma_1}{(q_1-q_2)^2} + \frac{\gamma_2}{(q_2-q_3)^2} + \frac{\gamma_3}{(q_3-q_1)^2}.$$

By defining new momenta as $p_k/\sqrt{\mu_k}$ and new positions as $\sqrt{\mu_k}\,q_k$ we obtain potential (3.1). It satisfies the BD equations with

$$\alpha = v, \quad \beta = w \cdot M, \quad \gamma = t\,(\mathrm{Id}) + s \cdot MM^T, \qquad v, w, t, s \in R,$$

where $M = (m_1^{-1}, m_2^{-1}, m_3^{-1})^T$. The most striking result of this analysis is its independence of the coupling constants, which means that they do not have any impact on the separability issue.

We note that the parameters become the same as in the original case considered in [15] by simply setting all masses equal to unity: $M = (1,1,1)^T$. If one goes through all different cases corresponding to different choices of v, w, t, s (as in [15] and as in § 3.2 above), one will find that the generalized potential is separable in exactly the same coordinate systems as the original potential; the only difference being in the choice of Euclidean reference frame. Thus, the extra "freedom" in the choice of parameters obtained in the generalisation makes no essential difference.

The criterion selects the rotation matrix

$$A_* = \begin{pmatrix} -m_1 & -m_1 & m_1^{-1} \\ m_2 & 0 & m_2^{-1} \\ 0 & m_3 & m_3^{-1} \end{pmatrix} \begin{pmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{pmatrix}$$

(N_1, N_2, N_3 are normalisation factors) for the family of changes of Euclidean reference frame $q = A_* q' + uM$ parametrized by $u \in R$. In any such frame, the potential becomes independent of q_3'. Together with the fact that the potential is homogeneous of degree -2, this is responsible for separability in five different rotationally symmetric coordinate systems, which are all detected by the criterion:

Oblate and prolate spheroidal coordinates as well as rotational parabolic coordinates can all be written

$$q' = \big(r_1(x_1,x_2)\,\omega_{1,1}(x_3), r_1(x_1,x_2)\,\omega_{1,2}(x_3), r_2(x_1,x_2)\big)^T$$

with the most general separable potential $V = f(r_1,r_2) + r_1^{-2}\,g(x_3)$.

Spherical coordinates can be written

$$q' = \big(r\,\omega_1(x_2)\,\tilde{\omega}_1(x_3), r\,\omega_1(x_2)\,\tilde{\omega}_2(x_3), r\,\omega_2(x_2)\big)^T$$

with the most general separable potential $V = f(r) + r^{-2}\big(g(x_2) + \omega_1(x_2)^{-2}h(x_3)\big)$.

Cylindrical coordinates can be written

$$q' = \big(r\omega_1(x_2), r\omega_2(x_2), q_3'\big)^T$$

with the most general separable potential $V = f(r) + r^{-2}\,g(x_2) + h(q_3')$.

It is easily seen that the generalized Calogero potential studied here takes all of the above forms with vanishing functions f and h. This also explains why the separability is independent of the coupling constants.

4 Conclusions

We have explained here the main points of our criterion of separability for natural Hamiltonian systems. Due to the 1-1 relationship between all separable coordinate systems on R^n and S^n (as classified by Kalnins and Miller [9]) and second order PDEs satisfied by separable $V(q)$ (the BD equations), any given potential can be tested for separability. For separable potentials $V(q)$, the separation coordinates can be determined explicitly and the corresponding Hamilton–Jacobi equation can be solved by an additive ansatz. The examples of the three-dimensional Coulomb anisotropic harmonic oscillator and of the three- and four-dimensional Calogero inverse-square potentials demonstrate practical use of the BD equations as an effective criterion of separability.

References

[1] V. I. Arnol'd. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York, second edition, 1989.

[2] S. Benenti. Separability structures on Riemannian manifolds. In *Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979)*, pages 512–538. Springer, Berlin, 1980.

[3] S. Benenti. Orthogonal separable dynamical systems. In O. Kowalski and D. Krupka, editors, *Differential Geometry and Its Applications (Opava, 1992)*, pages 163–184, Opava, 1993. Silesian Univ. Opava. Available at http://www.emis.de/proceedings/.

[4] S. Benenti, C. Chanu, and G. Rastelli. The super-separability of the three-body inverse-square Calogero system. *J. Math. Phys.*, 41(7):4654–4678, July 2000.

[5] F. Calogero. Solution of a three-body problem in one dimension. *J. Math. Phys.*, 10(12):2191–2196, December 1969.

[6] L. P. Eisenhart. Separable systems of Stäckel. *Ann. of Math.*, 35(2):284–305, April 1934.

[7] L. L. Foldy and S. E. Stansfield. Dynamics of a particle in a superposition of a coulomb potential and certain harmonic-oscillator potentials. *Phys. Rev. A*, 35(3):1415–1418, 1987.

[8] C. Jacobi. *Problema trium corporummutuis acctractionibus cubis distantiarum inverse proportionalibus recta linea se moventium*, in Gesammelte Werke, vol. 4, Berlin, 1866, pp. 533–539.

[9] E. G. Kalnins and W. Miller, Jr. Separation of variables on n-dimensional Riemannian manifolds. I. The n-sphere S_n and Euclidean n-space R_n. *J. Math. Phys.*, 27(7):1721–1736, July 1986.

[10] T. Levi-Civita. Sulla integrazione della equazione di Hamilton–Jacobi per separazione di variabili. *Math. Ann.*, 59:383–397, February 1904.

[11] I. Marshall and S. Wojciechowski. When is a Hamiltonian system separable? *J. Math. Phys.*, 29(6):1338–1346, June 1988.

[12] P. M. Morse and H. Feshbach. *Methods of Theoretical Physics.* McGraw-Hill, New York, 1953.

[13] M. A. Olshanetsky and A. M. Perelomov. Classical finite-dimensional systems related to Lie algebras, *Phys. Rep.*, 71:313–400, 1981.

[14] P. Stäckel. *Ueber die Integration der Hamilton–Jacobi'schen Differentialgleichung mittelst Separation der Variabeln.* Habilitationsschrift. Halle, 1891.

[15] C. Waksjö and S. Rauch-Wojciechowski. How to find separation coordinates for the Hamilton–Jacobi equation: a criterion of separability for natural Hamiltonian systems. *Math. Phys. Anal. Geom.*, 6(4):301–348, 2003.

[16] E. T. Whittaker. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.* Cambridge University Press, fourth edition, 1988.

[17] S. Wojciechowski. *Review of the recent results on integrability of natural Hamiltonian systems*, in Systèmes Dynamiques Non Linéaires: Intégrabilité et Comportement Qualitatif, Montreal, Que., 1986, Presses Univ. Montréal, pp. 294–327.