Journal of Nonlinear Mathematical Physics Volume 12, Supplement 1 (2005), 106–123 **Birthday Issue**

Hyperelliptic Addition Law

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This article is part of the special issue published in honour of Francesco Calogero on the occasion of his 70th birthday

Abstract

Given a family of genus g algebraic curves, with the equation $f(x, y, \Lambda) = 0$, we consider two fiber-bundles U and X over the space of parameters Λ. A fiber of U is the Jacobi variety of the curve. U is equipped with the natural groupoid structure that induces the canonical addition on a fiber. A fiber of X is the g-th symmetric power of the curve. We describe the algebraic groupoid structure on X using the Weierstrass gap theorem to define the 'addition law' on its fiber. The addition theorems that are the subject of the present study are represented by the formulas, mostly explicit, determining the isomorphism of groupoids $\mathsf{U} \to \mathsf{X}$. At g=1 this gives the classic addition formulas for the elliptic Weierstrass $\wp$ and $\wp'$ functions. To illustrate the efficiency of our approach the hyperelliptic curves of the form $y^2 = x^{2g+1} + \sum_{i=0}^{2g-1} \lambda_{4g+2-2i} x^i$ are considered. We construct the explicit form of the addition law for hyperelliptic Abelian vector functions $\boldsymbol{\wp}$ and $\boldsymbol{\wp}'$ (the functions $\boldsymbol{\wp}$ and $\boldsymbol{\wp}'$ form a basis in the field of hyperelliptic Abelian functions, i.e., any function from the field can be expressed as a rational function of $\boldsymbol{\wp}$ and $\boldsymbol{\wp}'$). Addition formulas for the higher genera zeta-functions are discussed. The genus 2 result is written in a Hirota-like trilinear form for the sigma-function. We propose a conjecture to describe the general formula in these terms.

1 Introduction

During the last 30 years the addition laws of elliptic functions stay in the focus of the studies in the nonlinear equations of Mathematical Physics. A large part of the interest was drawn to the subject by the works of F. Calogero [10, 11, 12], where several important problems were reduced to the elliptic addition laws. The term "Calogero-Moser model" being widely used in literature, the papers caused a large series of publications, where on one hand more advanced problems were posed and on the other hand some advances were made in the theory of functional equations. The "addition theorems" for Weierstrass elliptic functions:

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(v) - \wp(u), \tag{1}$$

$$(\zeta(u) + \zeta(v) + \zeta(w))^2 = \wp(u) + \wp(v) + \wp(w), \quad u + v + w = 0, \tag{2}$$

played the key rôle in the works of the that period. At the same time, the development of the algebro-geometric methods of solution of integrable systems [13, 14] employed in an essential way the addition formulas for theta functions of several variables. The same addition formulas were needed in applications of Hirota method. In [6, 7] the addition theorems for vector Baker-Akhiezer functions of several variables are obtained and a program is put forward to apply the addition theorems to problems of the theory of integrable systems, in particular, to multidimensional analogs of Calogero-Moser type systems.

The fundamental fact of the elliptic functions theory is that any elliptic function can be represented as a rational function of Weierstrass functions $\wp$ and $\wp'$. The corresponding result (see [3, 4]) in the theory of hyperelliptic Abelian functions is formulated as follows: any hyperelliptic function can be represented as a rational function of vector functions $\boldsymbol{\wp} = (\wp_{g1}, \dots, \wp_{gg})$ and $\boldsymbol{\wp}' = (\wp_{gg1}, \dots, \wp_{ggg})$, where g is the genus of the hyperelliptic curve on the Jacobi variety of which the field of Abelian functions is built.

In the present paper we find the explicit formulas for the addition law of the vector functions $\boldsymbol{\wp}$ and $\boldsymbol{\wp}'$. As an application the higher genus analogs of the Frobenius-Stickelberger formula (2) are obtained. In particular, for the genus 2 sigma-function we obtain the following trilinear differential addition theorem

$$\left[2(\partial_{u_1} + \partial_{v_1} + \partial_{w_1}) + (\partial_{u_2} + \partial_{v_2} + \partial_{w_2})^3\right]\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0.$$

Our approach is based on the explicit construction of the groupoid structure that is adequate to describe the algebraic structure of the space of g-th symmetric powers of hyperelliptic curves.

2 Algebraic groupoids

2.1 Topological groupoids

Definition 1. Take a topological space Y.

A space X together with a mapping $p_{\mathsf{X}} : \mathsf{X} \to Y$ is called *a space over* Y. The mapping p_{X} is called “an anchor” in Differential Geometry.

Let two spaces X_1 and X_2 over Y be given. The mapping $f : \mathsf{X}_1 \to \mathsf{X}_2$ is called *a mapping over* Y, if $p_{\mathsf{X}_2} \circ f(x) = p_{\mathsf{X}_1}(x)$ for any point $x \in \mathsf{X}_1$. By *the direct product over* Y of the spaces X_1 and X_2 over Y we call the space $\mathsf{X}_1 \times_{\mathsf{Y}} \mathsf{X}_2 = \{(x_1, x_2) \in \mathsf{X}_1 \times \mathsf{X}_2 \mid p_{\mathsf{X}_1}(x_1) = p_{\mathsf{X}_2}(x_2)\}$ together with the mapping $p_{\mathsf{X}_1 \times_{\mathsf{Y}} \mathsf{X}_2}(x_1, x_2) = p_{\mathsf{X}_1}(x_1)$.

The space Y together with the identity mapping p_{Y} is considered as the space over itself.

Definition 2. A space X together with a mapping $p_{\mathsf{X}} : \mathsf{X} \to \mathsf{Y}$ is called *a groupoid over* Y, if there are defined *structure mappings* over Y

$$\mu : \mathsf{X} \times_{\mathsf{Y}} \mathsf{X} \to \mathsf{X} \quad \text{and} \quad \mathrm{inv} : \mathsf{X} \to \mathsf{X}$$

that satisfy the axioms

1. $\mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))$, provided $p_{\mathsf{X}}(x_1) = p_{\mathsf{X}}(x_2) = p_{\mathsf{X}}(x_3)$.

2. $\mu(\mu(x_1, x_2), \mathrm{inv}(x_2)) = x_1$, provided $p_{\mathsf{X}}(x_1) = p_{\mathsf{X}}(x_2)$.

The mapping μ may not be defined for all pairs x_1 and x_2 from X.

Definition 3. A groupoid structure on X over the space Y is called *commutative*, if $\mu(x_1, x_2) = \mu(x_2, x_1)$, provided $p_{\mathsf{X}}(x_1) = p_{\mathsf{X}}(x_2)$.

A groupoid structure on the algebraic variety X over the algebraic variety Y is called *algebraic*, if the mapping p_{X} as well as the structure mappings μ and inv are algebraic.

2.2 Algebraic groupoids related to plane curves

We take as Y the space $\mathbb{C}^N$ with coordinates $\Lambda = (\lambda_i)$, $i = 1, \ldots, N$. Let $f(x, y, \Lambda)$, where $(x, y) \in \mathbb{C}^2$, be a polynomial in x and y. Define the family of plane curves

$$V = \{(x, y, \Lambda) \in \mathbb{C}^2 \times \mathbb{C}^N \mid f(x, y, \Lambda) = 0\}.$$

We assume that at a generic value of Λ, genus of the curve from V has fixed value g.

Let us take as X the universal fiber-bundle of g-th symmetric powers of the algebraic curves from V. A point in X is represented by the collection of an unordered set of g pairs $(x_i, y_i) \in \mathbb{C}^2$ and an N-dimensional vector Λ that are related by $f(x_i, y_i, \Lambda) = 0$, $i = 1, \ldots, g$.

The mapping p_{X} takes the collection to the point $\Lambda \in \mathbb{C}^N$.

Let $\phi(x, y)$ be an entire rational function on the curve V with the parameters Λ. A zero of the function $\phi(x, y)$ on the curve V is the point $(\xi, \eta) \in \mathbb{C}^2$, such that $\{f(\xi, \eta, \Lambda) = 0, \phi(\xi, \eta) = 0\}$. The total number of zeros of the function $\phi(x, y)$ is called the order of $\phi(x, y)$.

The further construction is based on the following fact.

Lemma 2.1. *Let $\phi(x, y)$ be an order $2g + k$, $k \geqslant 0$, entire rational function on the curve V. Then the function $\phi(x, y)$ is completely defined (up to a constant with respect to (x, y) factor) by any collection of $g + k$ its zeros.*

This fact is a consequence of Weierstrass gap theorem (Lükensatz). In particular, an ordinary univariate polynomial is an entire rational function on the curve of genus $g = 0$ and is completely defined by the collection of all its zeros.

Let us construct the mapping inv.

Let a point $U_1 = \{[(x_i^{(1)}, y_i^{(1)})], \Lambda\} \in \mathsf{X}$ be given. Let $R_{2g}^{(1)}(x, y)$ be the entire rational function of order $2g$ on the curve V defined by the vector Λ, such that $R_{2g}^{(1)}(x, y)$ is zero in U_1, that is $R_{2g}^{(1)}(x_i^{(1)}, y_i^{(1)}) = 0$, $i = 1, \ldots, g$. Denote by $[(x_i^{(2)}, y_i^{(2)})]$ the complement of $[(x_i^{(1)}, y_i^{(1)})]$ in the set of zeros of $R_{2g}^{(1)}(x, y)$. Denote by U_2 the point in X thus obtained and set $\mathrm{inv}(U_1) = U_2$.

So, the set of zeros of the function $R_{2g}^{(1)}(x, y)$, which defines the mapping inv, is the pair of points $\{U_1, \mathrm{inv}(U_1)\}$ from X and $p_{\mathsf{X}}(U_1) = p_{\mathsf{X}}(\mathrm{inv}\, U_1)$.

Lemma 2.2. *The mapping* inv *is an involution, that is* $\mathrm{inv} \circ \mathrm{inv}(U_1) = U_1$.

Let us construct the mapping μ.

Let two points $U_1 = \{[(x_i^{(1)}, y_i^{(1)})], \Lambda\}$ and $U_2 = \{[(x_i^{(2)}, y_i^{(2)})], \Lambda\}$ from X be given. Let $R_{3g}^{(1,2)}(x,y)$ be the entire rational function of order $3g$ on the curve V defined by the vector Λ, such that $R_{3g}^{(1,2)}(x,y)$ is zero in $\mathrm{inv}(U_1)$ and $\mathrm{inv}(U_2)$. Denote by $[(x_i^{(3)}, y_i^{(3)})]$ the complementary g zeros of $R_{3g}^{(1,2)}(x,y)$ on the curve V. Denote by U_3 the point in X thus obtained and set $\mu(U_1, U_2) = U_3$.

So, the set of zeros of the function $R_{3g}^{(1,2)}(x,y)$, which defines the mapping μ, is the triple of points $\{\mathrm{inv}(U_1), \mathrm{inv}(U_2), \mu(U_1,U_2)\}$ from X and $p_{\mathsf{X}}(U_1) = p_{\mathsf{X}}(U_2) = p_{\mathsf{X}}(\mu(U_1,U_2))$.

Theorem 2.3. *The above mappings μ and* inv *define the structure of the commutative algebraic groupoid over $\mathsf{Y} = \mathbb{C}^N$ on the universal fiber-bundle X of g-th symmetric powers of the plane algebraic curves from the family V.*

Proof. The mapping μ is symmetric with respect to U_1 and U_2, and thus defines the commutative operation. By the construction the mappings p_{X}, μ and inv are algebraic.

Lemma 2.4. *The mapping μ is associative.*

Proof. Let three points U_1, U_2 and U_3 be given. Let us assign

$$U_4 = \mu(U_1,U_2), \quad U_5 = \mu(U_4,U_3), \quad U_6 = \mu(U_2,U_3), \quad U_7 = \mu(U_1,U_6).$$

We have to show that $U_5 = U_7$.

Let $R_{3g}^{(i,j)}(x,y)$ be the function defining the point $\mu(U_i,U_j)$, and $R_{2g}^{(i)}(x,y)$ be the function defining the point $\mathrm{inv}(U_i)$.

Consider the product $R_{3g}^{(1,2)}(x,y)R_{3g}^{(4,3)}(x,y)$. It is the entire function of order $6g$ with zeros at $\{\mathrm{inv}(U_1), \mathrm{inv}(U_2), U_4, \mathrm{inv}(U_4), \mathrm{inv}(U_3), U_5\}$. Therefore, the function

$$Q_1(x,y) = \frac{R_{3g}^{(1,2)}(x,y)R_{3g}^{(4,3)}(x,y)}{R_{2g}^{(4)}(x,y)}$$

is the entire function of order $4g$ with the zeros $\{\mathrm{inv}(U_1), \mathrm{inv}(U_2), \mathrm{inv}(U_3), U_5\}$.

Similarly, the product $R_{3g}^{(2,3)}(x,y)R_{3g}^{(1,6)}(x,y)$ is the entire function of order $6g$ with zeros at $\{\mathrm{inv}(U_2), \mathrm{inv}(U_3), U_6, \mathrm{inv}(U_6), \mathrm{inv}(U_1), U_7\}$. Hence we find that

$$Q_2(x,y) = \frac{R_{3g}^{(2,3)}(x,y)R_{3g}^{(1,6)}(x,y)}{R_{2g}^{(6)}(x,y)}$$

is the entire function of order $4g$ with the zeros $\{\mathrm{inv}(U_1), \mathrm{inv}(U_2), \mathrm{inv}(U_3), U_7\}$.

The functions $Q_1(x,y)$ and $Q_2(x,y)$ have order $4g$ and both vanish at the points $\{\mathrm{inv}(U_1), \mathrm{inv}(U_2), \mathrm{inv}(U_3)\}$. Thus by Weierstrass gap theorem $Q_1(x,y) = Q_2(x,y)$ and, therefore, $U_5 = U_7$. ∎

Lemma 2.5. *The mappings μ and* inv *satisfy the axiom* 2.

Proof. Let two points U_1 and U_2 be given. Assign

$$U_3 = \mu(U_1,U_2), \quad U_4 = \mathrm{inv}(U_2), \quad U_5 = \mu(U_3,U_4).$$

We have to show that $U_5 = U_1$.

Consider the product $R_{3g}^{(1,2)}(x,y)R_{3g}^{(3,4)}(x,y)$, which is the function of order $6g$ with the zeros $\{\mathrm{inv}(U_1), \mathrm{inv}(U_2), U_3, \mathrm{inv}(U_3), \mathrm{inv}(U_4), U_5\}$. As $\mathrm{inv}(U_4) = \mathrm{inv}\circ\mathrm{inv}(U_2) = U_2$, the function

$$Q(x,y) = \frac{R_{3g}^{(1,2)}(x,y)R_{3g}^{(3,4)}(x,y)}{R_{2g}^{(3)}(x,y)R_{2g}^{(2)}(x,y)}$$

is the entire function of order $2g$ with zeros at $\{\mathrm{inv}(U_1), U_5\}$, that is $Q(x,y) = R_{2g}^{(5)}(x,y)$. Hence it follows that $U_5 = \mathrm{inv}\circ\mathrm{inv}(U_1) = U_1$. ■

The Theorem is proved. ■

The Lemma below is useful for constructing the addition laws on our groupoids.

Lemma 2.6. *Given $U_1, U_2 \in \mathsf{X}$, let us assign $U_3 = \mu(U_1, U_2)$ and $U_{i+3} = \mathrm{inv}(U_i)$, $i = 1, 2$. Then*

$$R_{2g}^{(3)}(x,y) = \frac{R_{3g}^{(1,2)}(x,y)R_{3g}^{(4,5)}(x,y)}{R_{2g}^{(1)}(x,y)R_{2g}^{(2)}(x,y)}.$$

The formula of Lemma 2.6 is important because its left hand side depends formally on U_3 only, while the right hand side is completely defined by the pair U_1, U_2.

The above general construction becomes effective once we fix a model of the family of curves, that is once the polynomial $f(x,y,\Lambda)$ is given. We are especially interested in the models of the form, cf. for instance [5, 8, 9],

$$f(x,y,\Lambda) = y^n - x^s - \sum \lambda_{ns-in-js}x^i y^j,$$

where $\gcd(n,s) = 1$ and the summation is carried out over the range $0 < i < s-1$, $0 < j < n-1$ under the condition $ns - in - js \geqslant 0$. It is important that a model of the kind (possibly with singular points) exists for an arbitrary curve. At the generic values of Λ a curve in such a family has genus $g = (n-1)(s-1)/2$. In this paper we consider in detail the case $(n,s) = (2, 2g+1)$, that is the families of hyperelliptic curves. However, the method of the further sections is straightforward to generalize to a generic (n,s)-model. We will describe the generalization in our future publications.

3 The structure of hyperelliptic groupoid on $\mathbb{C}^{3g}$

A hyperelliptic curve V of genus g is usually defined by a polynomial of the form

$$f(x,y,\lambda_0,\lambda_2,\dots) = y^2 - 4x^{2g+1} - \sum_{i=0}^{2g-1} \lambda_i x^i.$$

In this paper we apply the change of variables

$$(x, y, \lambda_{2g-1}, \lambda_{2g-2}, \dots, \lambda_0) \to (x, 2y, 4\lambda_4, 4\lambda_6, \dots, 4\lambda_{4g+2}),$$

in order to simplify the formulas in the sequel. Below we study the constructions related to the hyperelliptic curves defined by the polynomials of the form

$$f(x,y,\lambda_{4g+2},\lambda_{4g},\dots) = y^2 - x^{2g+1} - \sum_{i=0}^{2g-1} \lambda_{4g+2-2i}x^i. \tag{3}$$

Let us introduce the grading by assigning $\deg x = 2$, $\deg y = 2g+1$ and $\deg \lambda_k = k$. Then the polynomial $f(x,y,\Lambda)$ becomes a homogeneous polynomial of the weight $4g+2$.

An entire function on V has a unique representation as the polynomial $R(x,y) = r_0(x) + r_1(x)y$, where $r_0(x), r_1(x) \in \mathbb{C}[x]$. In such representation we have not more than one monomial x^iy^j of each weight, by definition $\deg x^iy^j = 2i + j(2g+1)$. The order of a function $R(x,y)$ is equal to the maximum of weights of the monomials that occur in $R(x,y)$. In fact, on the set of zeros of the polynomial $R(x,y)$ we have $y = -r_0(x)/r_1(x)$. Therefore, the zeros of $R(x,y)$ that lie on the curve are defined by the roots $x_1,\dots,x_m$ of the equation $r_0(x)^2 - r_1(x)^2(x^{2g+1} + \lambda_4 x^{2g-1} + \dots) = 0$. The total number of the roots is equal $m = \max(2\deg_x r_0(x), 2g+1+2\deg_x r_1(x))$, where $\deg_x r_j(x)$ denotes the degree of the polynomial $r_j(x)$ in x, which is exactly the highest weight of the monomials in $R(x,y)$.

In this case Weierstrass gap theorem asserts that the sequence of nonnegative integers $\{\deg x^iy^j\}$, $j = 0,1$, $i = 0,1,\dots$, in ascending order has precisely g "gaps" in comparison to the sequence of all nonnegative integers. All the gaps are less than $2g$.

Lemma 3.1. *For a given point* $U_1 = \{[(x_i^{(1)}, y_i^{(1)})], \Lambda\} \in \mathsf{X}$ *the entire function* $R_{2g}^{(1)}(x,y)$ *defining the mapping* inv *has the form*

$$R_{2g}^{(1)}(x,y) = (x - x_1^{(1)})\dots(x - x_g^{(1)}).$$

Proof. In fact, as $\deg y > 2g$, any entire function of order $2g$ does not depend on y. ∎

The function $R_{2g}^{(1)}(x,y)$ defines the unique point

$$\mathrm{inv}(U_1) = \{[(x_i^{(1)}, -y_i^{(1)})], \Lambda\},$$

which, obviously, also belongs to X.

Let us construct the functions $R_{3g}^{(i,j)}(x,y)$ that have the properties required in Lemmas 2.4 and 2.5.

Lemma 3.2. *Define the* $(2g+1)$*-dimensional row-vector*

$$m(x,y) = (1, x, \dots, x^{2g-1-\rho}, y, yx, \dots, yx^\rho), \quad \rho = \left[\frac{g-1}{2}\right],$$

which is composed of all monomials x^iy^j *of weight not higher than* $3g$ *(the restriction* $j = 0,1$ *applies). Then, up to a factor constant in* (x,y)*, the function* $R_{3g}^{(1,2)}(x,y)$ *is equal to the determinant of the matrix composed of* $2g+1$ *rows* $m(x,y)$*,* $m(x_i^{(1)}, -y_i^{(1)})$ *and* $m(x_i^{(2)}, -y_i^{(2)})$*,* $i = 1,\dots,g$*.*

Proof. By the construction the function $R_{3g}^{(1,2)}(x,y)$ vanishes at the points $\mathrm{inv}(U_1)$ and $\mathrm{inv}(U_2)$, and is uniquely defined by this property. As $\max(4g-2-2\rho, 2g+1+2\rho) = 3g$, at fixed Λ the function $R_{3g}^{(1,2)}(x,y)$ has $2g$ zeros at the given points of the curve and the collection of zeros at $[(x_i^{(3)}, y_i^{(3)})]$, $i = 1,\dots,g$, which defines the unique point U_3 in X. ∎

Let $\mathrm{Sym}^n(\mathsf{M})$ denote the n-th symmetric power of the space M. A point of the space $\mathrm{Sym}^n(\mathsf{M})$ is an unordered collection $[m_1, \dots, m_n]$, $m_i \in \mathsf{M}$. Lemmas 3.1 and 3.2 define the groupoid over $\mathsf{Y} = \mathbb{C}^{2g}$ structure on the space $\mathsf{X} \subset \mathrm{Sym}^g(\mathbb{C}^2) \times \mathbb{C}^{2g}$, which is related to the family of hyperelliptic curves (3). We want to transfer the hyperelliptic groupoid structure onto the linear space $\mathbb{C}^{3g}$. This is done in two steps.

Consider the space $\mathsf{S} = \mathrm{Sym}^g(\mathbb{C}^2) \times \mathbb{C}^g$. As the parameters Λ enter (3) linearly, one can parameterize the space X by S, and the parametrization can be made a mapping over Y. Namely, let us define the mapping $p_{\mathsf{S}} : \mathsf{S} \to \mathsf{Y} = \mathbb{C}^{2g}$. Take a point $T = \{[(\xi_i, \eta_i)], Z\} \in \mathsf{S}$. Denote by $\mathscr{V}$ the Vandermonde matrix, composed of g rows $(1, \xi_i, \dots, \xi_i^{g-1})$, denote by $\mathscr{X}$ the diagonal matrix $\mathrm{diag}(\xi_1^g, \dots, \xi_g^g)$ and by $\mathscr{Y}$ the vector $(\eta_1^2 - \xi_1^{2g+1}, \dots, \eta_g^2 - \xi_g^{2g+1})^t$. Set

$$p_{\mathsf{S}}(T) = (Z_1, Z_2), \quad \text{where} \quad Z_1 = \mathscr{V}^{-1}\mathscr{Y} - (\mathscr{V}^{-1}\mathscr{X}\mathscr{V})Z \quad \text{and} \quad Z_2 = Z.$$

It is clear that the domain of definition of the mapping p_{S} is the open and everywhere dense subset S_0 in S consisting of the points $\{[(\xi_i, \eta_i)], Z\}$ such that the determinant $\mathscr{V}$ does not vanish. We define the mappings $\gamma : \mathsf{X} \to \mathsf{S}$ and $\delta : \mathsf{S} \to \mathsf{X}$ by the following formulas: let $U \in \mathsf{X}$ and $T \in \mathsf{S}$, then

$$\gamma(U) = \gamma(\{[(x_i, y_i)], \Lambda\}) = \{[(x_i, y_i)], \Lambda_2\}, \quad \text{where } \Lambda_2 = (\lambda_{2(g-i)+4}), \quad i = 1, \dots, g,$$
$$\delta(T) = \delta(\{[(\xi_i, \eta_i)], Z\}) = \{[(\xi_i, \eta_i)], p_{\mathsf{S}}(T)\}.$$

By the construction, the mappings are mappings over Y. The domain of definition of δ coincides with the domain $\mathsf{S}_0 \subset \mathsf{S}$ of definition of the mapping p_{S}. Let $T \in \mathsf{S}_0$, then $\gamma \circ \delta(T) = T$. Let $\gamma(U) \in \mathsf{S}_0$, then $\delta \circ \gamma(U) = U$. Thus we have

Lemma 3.3. *The mappings γ and δ establish the birational equivalence of the spaces X and S over $\mathsf{Y} = \mathbb{C}^{2g}$.*

The assertion of Lemma 3.3 helps to transfer onto the space S the groupoid over Y structure, which is introduced by Theorem 2.3 on the space X. Let $T_1, T_2 \in \mathsf{S}$. The birational equivalence induces the mappings μ_* and inv_* that are defined by the formulas

$$\mu_*(T_1, T_2) = \gamma \circ \mu(\delta(T_1), \delta(T_2)), \quad \mathrm{inv}_*(T_1) = \gamma \circ \mathrm{inv} \circ \delta(T_1).$$

Theorem 3.4. *The mappings μ_* and inv_* define the structure of commutative algebraic groupoid over the space $\mathsf{Y} = \mathbb{C}^{2g}$ on the space S.*

Let us proceed to constructing the structure of algebraic groupoid over $\mathbb{C}^{2g}$ on the space $\mathbb{C}^{3g}$. The classical Viète mapping is the homeomorphism of spaces $\mathrm{Sym}^g(\mathbb{C}) \to \mathbb{C}^g$. Let us use Viète mapping to construct a birational equivalence $\varphi : \mathrm{Sym}^g(\mathbb{C}^2) \to \mathbb{C}^{2g}$.

Let $[(\xi_j, \eta_j)] \in \mathrm{Sym}^g(\mathbb{C}^2)$ and $(p_{2g+1}, p_{2g}, \dots, p_2) \in \mathbb{C}^{2g}$. Let us assign

$$P_{\mathrm{od}} = (p_{2g+1}, p_{2g-1}, \dots, p_3)^t \quad \text{and} \quad P_{\mathrm{ev}} = (p_{2g}, p_{2g-2}, \dots, p_2)^t,$$

and let $X = (1, x, \dots, x^{g-1})^t$. Define the mapping φ and its inverse ψ with the help of the relations

$$x^g - \sum_{i=1}^{g} p_{2i} x^{g-i} = x^g - X^t P_{\mathrm{ev}} = \prod_{j=1}^{g} (x - \xi_j),$$

$$\eta_i = P_{\text{od}}^t X|_{x=\xi_i}, \quad i = 1, \ldots, g.$$

Note, that φ is a rational mapping, while ψ is a nonsingular algebraic mapping.

Using the mapping φ we obtain the mapping

$$\varphi_1 = \varphi \times \text{id} : \mathsf{S} = \text{Sym}^g(\mathbb{C}^2) \times \mathbb{C}^g \to \mathbb{C}^{2g} \times \mathbb{C}^g \cong \mathbb{C}^{3g}$$

and its inverse $\psi_1 = \psi \times \text{id}$.

The companion matrix of a polynomial $x^g - X^t P_{\text{ev}}$ is the matrix

$$C = \sum_{i=1}^{g} e_i(e_{i-1} + p_{2(g-i+1)}e_g)^t,$$

where e_i is the i-th basis vector in $\mathbb{C}^g$. Its characteristic polynomial $|x \cdot 1_g - C|$ is $x^g - X^t P_{\text{ev}}$.

Example 1. The companion matrices C of the polynomials $x^g - X^t P_{\text{ev}}$ for $g = 1, 2, 3, 4$, have the form

$$p_2, \quad \begin{pmatrix} 0 & p_4 \\ 1 & p_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & p_6 \\ 1 & 0 & p_4 \\ 0 & 1 & p_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & p_8 \\ 1 & 0 & 0 & p_6 \\ 0 & 1 & 0 & p_4 \\ 0 & 0 & 1 & p_2 \end{pmatrix}.$$

Note, that the companion matrix for $g = k$ is included in the companion matrix for $g > k$ as the lower right $k \times k$ submatrix.

We make use of the following property of a companion matrix.

Lemma 3.5. *Let the polynomial* $p(x) = x^g - \sum_{i=1}^{g} p_{2i}x^{g-i}$ *and one of its roots* ξ *be given. Set* $\Upsilon = (1, \xi, \ldots, \xi^{g-1})^t$. *Then the relations*

$$\xi^k \Upsilon^t A = \Upsilon^t C^k A, \quad k = 0, 1, 2, \ldots,$$

hold for an arbitrary vector $A \in \mathbb{C}^g$.

Proof. Let $A = (a_1, \ldots, a_g)$, then

$$\xi \Upsilon^t A = \xi \sum_{i=1}^{g} a_i \xi^{i-1} = a_g \xi^g + \sum_{i=2}^{g} a_{i-1}\xi^{i-1} = \sum_{i=1}^{g}((1 - \delta_{i,1})a_{i-1} + a_g p_{2(g-i+1)})\xi^{i-1}.$$

Thus the Lemma holds for $k = 1$. One can complete the proof by induction. ■

The mapping $p_{\mathbb{C}^{3g}} : \mathbb{C}^{3g} \to \mathbb{C}^{2g}$, with respect to which φ_1 is a mapping over $\mathbb{C}^{2g}$, is given by the formula

$$p_{\mathbb{C}^{3g}}(P, Z) = (Z_1, Z_2), \quad Z_1 = \Big(\sum_{i=1}^{g} p_{2i+1} C^{g-i}\Big) P_{\text{od}} - C^g(CP_{\text{ev}} + Z), \quad Z_2 = Z. \quad (4)$$

Using Lemma 3.5 one can directly verify the "over" property, that is, that $p_{\mathbb{C}^{3g}} \circ \varphi_1(T) = p_{\mathsf{S}}(T)$ for any $T \in \mathsf{S}_0$.

Let $A_1, A_2 \in \mathbb{C}^{3g}$. The birational equivalence φ_1 induces the mappings μ_{**} and inv_{**} defined by formulas

$$\mu_{**}(A_1, A_2) = \varphi_1 \circ \mu_*(\psi_1(A_1), \psi_1(A_2)), \quad \text{inv}_{**}(A_1) = \varphi_1 \circ \text{inv}_* \circ \psi_1(A_1).$$

Theorem 3.6. *The mappings* μ_{**} *and* inv_{**} *define the structure of commutative algebraic groupoid over the space* $\mathsf{Y} = \mathbb{C}^{2g}$ *on the space* $\mathbb{C}^{3g}$.

4 The addition law of the hyperelliptic groupoid on $\mathbb{C}^{3g}$

In what follows we use the shorthand notation

$$\overline{A} = \mathrm{inv}_{**}(A) \quad \text{and} \quad A_1 \star A_2 = \mu_{**}(A_1, A_2).$$

Lemma 4.1. *Let* $A = (P_{\mathrm{ev}}, P_{\mathrm{od}}, Z) \in \mathbb{C}^{3g}$. *Then*

$$\overline{A} = (P_{\mathrm{ev}}, -P_{\mathrm{od}}, Z).$$

Introduce the $(g \times \infty)$-matrix

$$K(A) = \big(Y, CY, C^2 Y, \dots\big),$$

that is composed of the $(g \times 2)$-matrix $Y = (P_{\mathrm{ev}}, P_{\mathrm{od}})$ with the help of the companion matrix C of the polynomial $x^g - X^t P_{\mathrm{ev}}$. Denote by $L(A)$ the matrix composed of the first g columns of $K(A)$, and denote by $\ell(A)$ the $(g+1)$-st column of $K(A)$.

Theorem 4.2. *Let* $A_1, A_2 \in \mathbb{C}^{3g}$ *and let* $A_3 = A_1 \star A_2$, *then*

$$\operatorname{rank} \begin{pmatrix} 1_g & L(\overline{A}_1) & \ell(\overline{A}_1) \\ 1_g & L(\overline{A}_2) & \ell(\overline{A}_2) \\ 1_g & L(A_3) & \ell(A_3) \end{pmatrix} < 2g + 1.$$

Proof. Suppose the points $U_i = \delta \circ \psi_1(A_i)$, $i = 1, 2$, are defined. We rewrite the function $R_{3g}^{(1,2)}(x, y)$ as a linear combination of monomials

$$R_{3g}^{(1,2)}(x, y) = \sum_{i,j,w(i,j) \geqslant 0} h_{w(i,j)} x^i y^j = r_1(x) y + x^g r_2(x) + r_3(x),$$

where $w(i, j) = 3g - (2g + 1)j - 2i$,

$$r_1(x) = \sum_{i=0}^{\rho} h_{g-2i-1} x^i, \ r_2(x) = \sum_{i=0}^{g-\rho-1} h_{g-2i} x^i, \ r_3(x) = \sum_{i=0}^{g-1} h_{3g-2i} x^i, \quad \rho = \Big[\frac{g-1}{2}\Big].$$

Let us set $h_0 = 1$. We assign weights to the parameters h_k by the formula $\deg h_k = k$. Then $\deg R_{3g}^{(1,2)}(x, y) = 3g$.

Let $A = (P_{\mathrm{ev}}, P_{\mathrm{od}}, Z) \in \mathbb{C}^{3g}$ be the point defining any of the collections of g zeros of the function $R_{3g}^{(1,2)}(x, y)$. Consider the function $Q(x) = R_{3g}^{(1,2)}(x, X^t P_{\mathrm{od}})$. By the construction $Q(\xi) = 0$, if $(x^g - X^t P_{\mathrm{ev}})|_{x=\xi} = 0$. Let us apply Lemma 3.5. We obtain

$$Q(\xi) = \Upsilon^t \big(r_1(C) P_{\mathrm{od}} + r_2(C) P_{\mathrm{ev}} + H_1\big), \tag{5}$$

where $H_1 = (h_{3g}, h_{3g-2}, \dots, h_{g+2})$. Using the above notation, we come to the relation

$$Q(\xi) = \Upsilon^t \big(H_1 + L(A) H_2 + \ell(A)\big),$$

where $H_2 = (h_g, h_{g-1}, \dots, h_1)$. Suppose the polynomial $x^g - X^t P_{\rm ev}$ has no multiple roots, then from the equalities $Q(\xi_j) = 0$, $j = 1, \dots, g$ one can conclude that

$$H_1 + L(A)H_2 + \ell(A) = 0. \tag{6}$$

By substituting the points $\overline{A}_1, \overline{A}_2$ and A_3 to (6), we obtain the system of $3g$ linear equations, which is satisfied by the coefficients H_1, H_2 of the entire function $R_{3g}^{(1,2)}(x,y)$. The assertion of the Theorem is the compatibility condition of the system of linear equations obtained. ■

Corollary 4.3. *The vectors H_1, H_2 of coefficients of the entire function $R_{3g}^{(1,2)}(x,y)$ are expressed by the formulas*

$$H_2(A_1, A_2) = -\big[L(\overline{A}_1) - L(\overline{A}_2)\big]^{-1}(\ell(\overline{A}_1) - \ell(\overline{A}_2)),$$
$$H_1(A_1, A_2) = -\frac{1}{2}\big[(\ell(\overline{A}_1) + \ell(\overline{A}_2)) - (L(\overline{A}_1) + L(\overline{A}_2))H_2(A_1, A_2)\big].$$

as vector functions of the points A_1 and A_2 from $\mathbb{C}^{3g}$.

Now, we know the coefficients H_1, H_2 of $R_{3g}^{(1,2)}(x,y)$ and we can give the expression of $P_{\rm od}^{(3)}$ as a function of A_1, A_2 and $P_{\rm ev}^{(3)}$. It follows from (5) that the following assertion holds.

Lemma 4.4.

$$P_{\rm od}^{(3)} = -\big[r_1(C^{(3)})\big]^{-1}(H_1 + r_2(C^{(3)})P_{\rm ev}^{(3)}), \tag{7}$$

where $C^{(3)}$ is the companion matrix of the polynomial $x^g - X^t P_{\rm ev}^{(3)}$.

Let us find the explicit formula for the function $R_{3g}^{(1,2)}(x,y)$ as a function of the points A_1 and A_2. We introduce the $((2g+1)\times\infty)$-matrix

$$F(x,y;A_1,A_2) = \begin{pmatrix} X^t & \mathscr{K}(x,y) \\ 1_g & K(A_1) \\ 1_g & K(A_2) \end{pmatrix}, \quad \mathscr{K}(x,y) = (x^g, y, \dots, x^{g+k}, yx^k, \dots).$$

Denote by $G(x,y;A_1,A_2)$ the matrix composed of the first $2g+1$ columns of the matrix $F(x,y;A_1,A_2)$.

Theorem 4.5. *The entire rational function $R_{3g}^{(1,2)}(x,y)$ defining the operation $A_1 \star A_2$ has the form*

$$R_{3g}^{(1,2)}(x,y) = \frac{|G(x,y;\overline{A}_1,\overline{A}_2)|}{|L(\overline{A}_2) - L(\overline{A}_1)|}. \tag{8}$$

By a use of the formula (8) and Lemma 2.6 we can find $P_{\rm ev}^{(3)}$. Similar to the condition of Lemma 2.6 denote $A_4 = \overline{A}_1$ and $A_5 = \overline{A}_2$. One can easily show that $R_{3g}^{(4,5)}(x,y) = R_{3g}^{(1,2)}(x,-y)$. Thus, the product $R_{3g}^{(1,2)}(x,y)\,R_{3g}^{(4,5)}(x,y)$ is an even function in y. Set

$$\Phi(x,y^2) = R_{3g}^{(1,2)}(x,y)\,R_{3g}^{(4,5)}(x,y) = (-1)^g \frac{|G(x,y;A_1,A_2)|}{|L(A_2) - L(A_1)|}\frac{|G(x,-y;A_1,A_2)|}{|L(A_2) - L(A_1)|}.$$

Therefore, $\Phi(x, y^2)$, as a function on the curve V, is the polynomial in x and the parameters Z_1 and Z_2. The values of Z_1 and Z_2 are defined by the mapping $p_{\mathbb{C}^{3g}}$ according to (4), and $p_{\mathbb{C}^{3g}}(A_1) = p_{\mathbb{C}^{3g}}(A_2) = (Z_1, Z_2)$. Namely, we have

$$R_{3g}^{(1,2)}(x,y)\, R_{3g}^{(4,5)}(x,y) = \Phi(x, x^{2g+1} + x^g X^t Z_2 + X^t Z_1).$$

Lemma 2.6 asserts that dividing the polynomial $\Phi(x, x^{2g+1} + x^g X^t Z_2 + X^t Z_1)$ by the polynomial $(x^g - X^t P_{\mathrm{ev}}^{(1)})(x^g - X^t P_{\mathrm{ev}}^{(2)})$ gives the zero remainder and the quotient equal $x^g - X^t P_{\mathrm{ev}}^{(3)}$. Thus, the calculation is reduced to the classical algorithm of polynomial division.

Theorem 4.6. *Consider the space $\mathbb{C}^{3g}$ together with the mapping $p_{\mathbb{C}^{3g}}$ defined by* (4) *as a groupoid over $\mathbb{C}^{2g}$. Let $A_1 = (P_{\mathrm{ev}}^{(1)}, P_{\mathrm{od}}^{(1)}, Z)$ and $A_2 = (P_{\mathrm{ev}}^{(2)}, P_{\mathrm{od}}^{(2)}, Z)$ be the points from $\mathbb{C}^{3g}$ such that $p_{\mathbb{C}^{3g}}(A_1) = p_{\mathbb{C}^{3g}}(A_2) = (Z_1, Z_2) \in \mathbb{C}^{2g}$.*

Then the addition law has the form $A_1 \star A_2 = A_3$, where the coordinates of the point $A_3 = (P_{\mathrm{ev}}^{(3)}, P_{\mathrm{od}}^{(3)}, Z)$ are given by the formulas

$$x^g - X^t P_{\mathrm{ev}}^{(3)} = \frac{\Phi(x, x^{2g+1} + x^g X^t Z_2 + X^t Z_1)}{(x^g - X^t P_{\mathrm{ev}}^{(1)})(x^g - X^t P_{\mathrm{ev}}^{(2)})},$$
$$P_{\mathrm{od}}^{(3)} = -\big[r_1(C^{(3)})\big]^{-1}(H_1 + r_2(C^{(3)}) P_{\mathrm{ev}}^{(3)}).$$

Example 2. Let $g = 1$. The family of curves V is defined by the polynomial

$$f(x, y, \Lambda) = y^2 - x^3 - \lambda_4 x - \lambda_6.$$

In the coordinates (λ_6, λ_4) on $\mathbb{C}^2$ and (p_2, p_3, z_4) on $\mathbb{C}^3$ the mapping $p_{\mathbb{C}^3}$ is given by the formula

$$(\lambda_6, \lambda_4) = (p_3^2 - p_2(p_2^2 + z_4), z_4)$$

Let us write down the addition formulas for the points on the groupoid $\mathbb{C}^3$ over $\mathbb{C}^2$. Set $A_1 = (u_2, u_3, \lambda_4)$, $A_2 = (v_2, v_3, \lambda_4)$ and suppose $p_{\mathbb{C}^3}(A_1) = p_{\mathbb{C}^3}(A_2) = (\lambda_6, \lambda_4)$.

Let $A_1 \star A_2 = A_3 = (w_2, w_3, \lambda_4)$.

We have: $R_2^{(1)}(x, y) = x - u_2$, $L(A_1) = u_2$, $\ell(A_1) = u_3$, and so on.

$$F(x, y, A_1, A_2) = \begin{pmatrix} 1 & x & y & x^2 & yx & \dots \\ 1 & u_2 & u_3 & u_2^2 & \dots\dots\dots & \\ 1 & v_2 & v_3 & \dots\dots\dots & \dots & \end{pmatrix}.$$

Thus, the function defining the operation $A_1 \star A_2$ has the expression

$$R_3^{(1,2)}(x, y) = y + \frac{v_3 - u_3}{v_2 - u_2} x - \frac{u_2 v_3 - u_3 v_2}{v_2 - u_2}.$$

Hence, we find: $r_1(x) = 1$, $r_2(x) = \dfrac{v_3 - u_3}{v_2 - u_2}$, $H_1 = -\dfrac{u_2 v_3 - u_3 v_2}{v_2 - u_2}$, and, by (7),

$$w_3 = \frac{u_2 v_3 - u_3 v_2}{v_2 - u_2} - \frac{v_3 - u_3}{v_2 - u_2} w_2.$$

Further, $\Phi(x, x^3+\lambda_4 x+\lambda_6) = x^3+\lambda_4 x+\lambda_6 - \Big(x\dfrac{v_3-u_3}{v_2-u_2} - \dfrac{u_2v_3-u_3v_2}{v_2-u_2}\Big)^2$. Upon dividing the polynomial $\Phi(x, x^3+\lambda_4 x+\lambda_6)$ by the polynomial $(x-u_2)(x-v_2)$ we find

$$\Phi(x, x^3+\lambda_4 x+\lambda_6) = \Big(x+u_2+v_2-\Big(\frac{v_3-u_3}{v_2-u_2}\Big)^2\Big)(x^2-(u_2+v_2)x+v_2u_2)+\dots.$$

And, finally, we obtain the addition law of the elliptic groupoid in the following form

$$\begin{aligned} w_2 &= -(u_2+v_2)+h^2, \\ w_3 &= -\frac{1}{2}(u_3+v_3)+\frac{3}{2}(u_2+v_2)h-h^3, \quad \text{where} \quad h=\Big(\frac{v_3-u_3}{v_2-u_2}\Big). \end{aligned}$$

One may check directly that $p_{\mathbb{C}^3}(A_3) = (\lambda_6, \lambda_4)$.

Let $g=2$. The family of curves V is defined by the polynomial

$$f(x,y,\Lambda) = y^2 - x^5 - \lambda_4 x^3 - \lambda_6 x^2 - \lambda_8 x - \lambda_{10}$$

In the coordinates $(\Lambda_1, \Lambda_2) = ((\lambda_{10}, \lambda_8)^t, (\lambda_6, \lambda_4)^t)$ on $\mathbb{C}^4$ and $(P_{\text{ev}}, P_{\text{od}}, Z)$ on $\mathbb{C}^6$, where $P_{\text{ev}} = (p_4, p_2)^t$, $P_{\text{od}} = (p_5, p_3)^t$, and $Z = (z_6, z_4)^t$, the mapping $p_{\mathbb{C}^6}$ is given by the formula

$$(\Lambda_1, \Lambda_2) = \left(\begin{pmatrix} p_5^2+p_3^2p_4-p_2p_4(p_2^2+p_4+z_4)-p_4(p_2p_4+z_6) \\ 2p_3p_5+p_2p_3^2-(p_2^2+p_4)(p_2^2+p_4+z_4)-p_2(p_2p_4+z_6) \end{pmatrix}, Z\right)$$

Let us write down the addition formulas for the points on the groupoid $\mathbb{C}^6$ over $\mathbb{C}^4$. Set $A_1 = ((u_4, u_2)^t, (u_5, u_3)^t, (\lambda_4), \lambda_6)^t)$, $A_2 = ((v_4, v_2)^t, (v_5, v_3)^t, (\lambda_4, \lambda_6)^t)$ and suppose $p_{\mathbb{C}^6}(A_1) = p_{\mathbb{C}^6}(A_2) = ((\lambda_{10}, \lambda_8)^t, (\lambda_6, \lambda_4)^t)$.

Let $A_3 = A_1 \star A_2$, $A_3 = ((w_4, w_2)^t, (w_5, w_3)^t, (\lambda_6, \lambda_4)^t)$.

We omit the calculation, which is carried out by the same scheme as for $g=1$, and pass to the result. Set $h = h_1$. We have

$$h = -\frac{\begin{vmatrix} v_4-u_4 & v_2v_4-u_2u_4 \\ v_2-u_2 & v_4+v_2^2-(u_4+u_2^2) \end{vmatrix}}{\begin{vmatrix} v_4-u_4 & v_5-u_5 \\ v_2-u_2 & v_3-u_3 \end{vmatrix}}.$$

To shorten the formulas it is convenient to employ the linear differential operator

$$\mathscr{L} = \frac{1}{2}\{(u_3-v_3)(\partial_{u_2}-\partial_{v_2})+(u_5-v_5)(\partial_{u_4}-\partial_{v_4})\},$$

It is important to note that $\mathscr{L}$ adds unity to the weight, $\deg \mathscr{L} = 1$, and that it is *tangent* to the singular set where the addition is not defined:

$$\mathscr{L}\{(u_2-v_2)(u_5-v_5)-(u_3-v_3)(u_4-v_4)\} = 0.$$

Let $h' = \mathscr{L}(h)$ and $h'' = \mathscr{L}(h')$. Note, that $\mathscr{L}(h'') = 0$. Using this notation the addition formulas are written down as follows

$$w_2 = \frac{1}{2}(u_2+v_2)+2h'+h^2,$$

$$w_3 = \frac{1}{2}(u_3+v_3) + \frac{5}{4}(u_2+v_2)h + 2h'' + 3h'h + h^3,$$
$$w_4 = -\frac{1}{2}(u_4+v_4) - u_2v_2 + \frac{1}{8}(u_2+v_2)^2 + (u_3+v_3)h - \frac{1}{2}(u_2+v_2)(h'-h^2) - 2hh'',$$
$$w_5 = -\frac{1}{2}(u_5+v_5) - \frac{1}{2}(u_2u_3+v_2v_3) - \Big\{\frac{1}{2}(u_4+v_4) + u_2v_2 + \frac{1}{8}(u_2+v_2)^2\Big\}h$$
$$+(u_3+v_3)(h'+h^2) - \frac{1}{2}(u_2+v_2)(h''-hh'-2h^3) - 2(h'+h^2)h''.$$

5 Addition theorems for hyperelliptic functions

For each curve V from the family (3) consider the Jacobi variety $\mathrm{Jac}(V)$. The set of all the Jacobi varieties is *the universal space* U *of the Jacobi varieties of the genus g hyperelliptic curves*. The points of U are pairs (u,Λ), where the vector $u=(u_1,\ldots,u_g)$ belongs to the Jacobi variety of the curve with parameters Λ. The mapping $p_{\mathsf{U}} : \mathsf{U} \to \mathbb{C}^{2g}$ that acts as $p_{\mathsf{U}}(u,\Lambda)=\Lambda$ makes U the space over $\mathbb{C}^{2g}$. There is a natural groupoid over $\mathbb{C}^{2g}$ structure on U. Evidently, the mappings $\mu((u,\Lambda),(v,\Lambda))=(u+v,\Lambda)$ and $\mathrm{inv}(u,\Lambda)=(-u,\Lambda)$ satisfy the groupoid over $\mathbb{C}^{2g}$ axioms.

5.1 Addition theorems for the hyperelliptic $\wp$-functions

Let us define the mapping $\pi : \mathsf{U} \to \mathbb{C}^{3g}$ over $\mathbb{C}^{2g}$ by putting into correspondence a point $(u,\Lambda)\in\mathsf{U}$ and the point $(\boldsymbol{\wp}(u), \boldsymbol{\wp}'(u)/2, \Lambda_2) \in \mathbb{C}^{3g}$, where

$$\boldsymbol{\wp}(u) = (\wp_{g,j}(u))^t, \quad \boldsymbol{\wp}'(u) = (\wp_{g,g,j}(u))^t, \quad \Lambda_2 = (\lambda_{2(g-i+2)}), \quad i=1,\ldots,g.$$

Here

$$\wp_{i,j}(u) = -\partial_{u_i}\partial_{u_j}\log\sigma(u) \quad \text{and} \quad \wp_{i,j,k}(u) = -\partial_{u_i}\partial_{u_j}\partial_{u_k}\log\sigma(u)$$

and $\sigma(u)$ is hyperelliptic sigma-function [1, 2, 3, 4].

Theorem 5.1. *The mapping $\pi : \mathsf{U} \to \mathbb{C}^{3g}$ over $\mathbb{C}^{2g}$ is a birational isomorphism of groupoids:*

$$\pi(u+v,\Lambda) = \pi(u,\Lambda)\star\pi(v,\Lambda), \quad \pi(-u,\Lambda) = \overline{\pi(u,\Lambda)}.$$

Proof. First, by Abel theorem any triple of points $(u,v,w)\in(\mathrm{Jac}(V))^3$ that satisfies the condition $u+v+w=0$ corresponds to the set of zeros (x_i,y_i), $i=1,\ldots,3g$, of an entire rational function of order $3g$ on the curve V. Namely, Let $X=(1,x,\ldots,x^{g-1})^t$, then

$$u = \sum_{i=1}^{g}\int_\infty^{x_i} X\,\frac{\mathrm{d}x}{2y}, \quad v = \sum_{i=1}^{g}\int_\infty^{x_{i+g}} X\,\frac{\mathrm{d}x}{2y}, \quad w = \sum_{i=1}^{g}\int_\infty^{x_{i+2g}} X\,\frac{\mathrm{d}x}{2y}. \tag{9}$$

(For shortness, instead of indicating the end point of integration explicitly, we give only the first coordinate.)

Second, for the given value $u \in \mathrm{Jac}(V)$ the system of g equations

$$u - \sum_{i=1}^{g} \int_{\infty}^{x_i} X \frac{\mathrm{d}x}{2y} = 0$$

with respect to the unknowns $(x_i, y_i) \in V$ is equivalent to the system of algebraic equations

$$x^g - \sum_{k=1}^{g} \wp_{g,k}(u) x^{k-1} = 0, \quad 2y - \sum_{k=1}^{g} \wp_{g,g,k}(u) x^{k-1} = 0,$$

the roots of which are the required points $(x_i, y_i) \in V$ (see, for instance, [1, 3, 4]).

The combination of the two facts implies that the construction of the preceding sections provides the isomorphism. ■

Above all, note that $2g$ hyperelliptic functions $\boldsymbol{\wp}(u) = (\wp_{g,1}(u), \ldots, \wp_{g,g}(u))^t$, $\boldsymbol{\wp}'(u) = (\wp_{g,g,1}(u), \ldots, \wp_{g,g,g}(u))^t$ form a basis of the field of hyperelliptic Abelian functions, i.e., any function of the field can be expressed as a rational function in $\boldsymbol{\wp}(u)$ and $\boldsymbol{\wp}'(u)$. The assertion of Theorem 5.1 written down in the coordinates of $\mathbb{C}^{3g}$ takes the form of the addition theorem for the basis functions $\boldsymbol{\wp}(u)$ and $\boldsymbol{\wp}'(u)$.

Corollary 5.2. *The basis hyperelliptic Abelian functions*

$$\boldsymbol{\wp}(u) = (\wp_{g,1}(u), \ldots, \wp_{g,g}(u))^t \quad \textit{and} \quad \boldsymbol{\wp}'(u) = (\wp_{g,g,1}(u), \ldots, \wp_{g,g,g}(u))^t$$

respect the addition law

$$(\boldsymbol{\wp}(u+v), \boldsymbol{\wp}'(u+v)/2, \Lambda_2) = (\boldsymbol{\wp}(u), \boldsymbol{\wp}'(u)/2, \Lambda_2) \star (\boldsymbol{\wp}(v), \boldsymbol{\wp}'(v)/2, \Lambda_2),$$

the formula of which is given in Theorem 4.6.

Thus we have obtained a solution of the problem to construct an explicit and effectively computable formula of the addition law in the fields of hyperelliptic Abelian functions.

5.2 Addition theorems for the hyperelliptic ζ-functions

One has g functions $\zeta_i(u) = \partial_{u_i} \log \sigma(u)$ and the functions are not Abelian. However, by an application of Abel theorem for the second kind integrals (see [1]) one obtains the addition theorems for ζ-functions as well. On one hand, any ζ-function can be represented as the sum of g second kind integrals and an Abelian function. On the other hand, an Abelian sum of the second kind integrals with the end points at the set of zeros of an entire rational function $R(x, y)$ is expressed rationally in terms of the coefficients of $R(x, y)$. We employ the function (8) computed in the variables indicated in Corollary 5.2.

Theorem 5.3. *Let* $(u, v, w) \in (\mathrm{Jac}(V))^3$ *and* $u + v + w = 0$. *Then*

$$\zeta_g(u) + \zeta_g(v) + \zeta_g(w) = -h_1,$$

where h_1 *is the rational function in* $\boldsymbol{\wp}(u), \boldsymbol{\wp}'(u)$ *and* $\boldsymbol{\wp}(v), \boldsymbol{\wp}'(v)$ *equal to the coefficient of the monomial of the weight* $3g-1$ *in the function* (8) *computed in the variables indicated in Corollary 5.2.*

Proof. We have the identity (see [1],[4, p. 41])

$$\zeta_g(u)+\sum_{i=1}^{g}\int_{\infty}^{x_i} x^g\,\frac{\mathrm{d}x}{2y}=0,\quad \zeta_g(v)+\sum_{i=1}^{g}\int_{\infty}^{x_{i+g}} x^g\,\frac{\mathrm{d}x}{2y}=0,\quad \zeta_g(w)+\sum_{i=1}^{g}\int_{\infty}^{x_{i+2g}} x^g\,\frac{\mathrm{d}x}{2y}=0.$$

Suppose that the closed path γ encloses all zeros $(x_1,y_1),\dots,(x_{3g},y_{3g})$ of the function $R_{3g}(x,y)$. Then we have

$$\sum_{k=1}^{3g}\int_{\infty}^{x_k} x^g\,\frac{\mathrm{d}x}{2y}=\frac{1}{2\pi\imath}\oint_{\gamma}\mathrm{d}\big(\log R_{3g}(x,y)\big)\int_{\infty}^{x} x^g\,\frac{\mathrm{d}x}{2y}.$$

Because $\mathrm{d}\log R_{3g}(x,y)/\mathrm{d}x$ is a rational function on the curve and, hence, a uniform function, the total residue of $\mathrm{d}\big(\log R_{3g}(x,y)\big)\int_{\infty}^{x} x^g\mathrm{d}x/(2y)$ on the Riemann surface of the curve V is zero. To write down this fact explicitly consider the parametrization

$$(x(\xi),y(\xi))=(\xi^{-2},\xi^{-2g-1}\rho(\xi)),\qquad \rho(\xi)=1+\frac{\lambda_4}{2}\xi^4+\frac{\lambda_6}{2}\xi^6+O(\xi^8),$$

of the curve V near the point at infinity and denote $R_{3g}(\xi)=R_{3g}(x(\xi),y(\xi))$. We obtain

$$-\mathrm{Res}_{\xi}\left[\frac{R_{3g}'(\xi)}{R_{3g}(\xi)}\int_{\infty}^{x(\xi)} x^g\,\frac{\mathrm{d}x}{2y}\right]+\sum_{i=1}^{3g}\mathrm{Res}_{x=x_i}\left[\mathrm{d}\big(\log R_{3g}(x,y)\big)\int_{\infty}^{x} x^g\,\frac{\mathrm{d}x}{2y}\right]=0,$$

which is in fact a particular case of Abel theorem. Thus, the final expression is

$$\zeta_g(u)+\zeta_g(v)+\zeta_g(w)=-\mathrm{Res}_{\xi}\left[\frac{R_{3g}'(\xi)}{R_{3g}(\xi)}\int_{\infty}^{x(\xi)} x^g\,\frac{\mathrm{d}x}{2y}\right].$$

It remains to use the expansions

$$\int_{\infty}^{x(\xi)} x^g\,\frac{\mathrm{d}x}{2y}=\frac{1}{\xi}+\frac{\lambda_4}{6}\xi^3+O(\xi^5),\quad R_{3g}(\xi)=\xi^{-3g}(1+h_1\xi+h_2\xi^2+h_3\xi^3+O(\xi^4))$$

to compute the residue. ■

A similar argument leads from the identity (see [1],[4, p. 41])

$$\zeta_{g-1}(u)+\sum_{i=1}^{g}\int_{\infty}^{x_i}(3x^{g+1}+\lambda_4x^{g-1})\frac{\mathrm{d}x}{2y}=\frac{1}{2}\wp_{g,g,g}(u),$$

to the following assertion.

Theorem 5.4. *In the conditions of Theorem 5.3 we have*

$$\zeta_{g-1}(u)+\zeta_{g-1}(v)+\zeta_{g-1}(w)-\frac{1}{2}(\wp_{g,g,g}(u)+\wp_{g,g,g}(v)+\wp_{g,g,g}(w))=-h_1^3+3h_1h_2-3h_3,$$

where h_2 and h_3 are the coefficients of the monomials of weight $3g-2$ and $3g-3$ in the function indicated in Theorem 5.3.

Example 3. Let $g = 1$. The function $R_3(x, y)$ has the form $y + h_1 x + h_3$, where $2h_1 = (\wp'(u) - \wp'(v))/(\wp(u) - \wp(v))$, cf. Example 2. Thus, Theorem 5.3 gives the classic formula

$$\zeta(u) + \zeta(v) - \zeta(u + v) = -\frac{1}{2}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right).$$

As $h_2 = 0$ and $2h_3 = (\wp'(v)\wp(u) - \wp'(u)\wp(v))/(\wp(u) - \wp(v))$, cf. Example 2, Theorem 5.4 yields the relation

$$-\wp'(u) - \wp'(v) + \wp'(u + v) = -\frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^3 - 3\frac{\wp'(v)\wp(u) - \wp'(u)\wp(v)}{\wp(u) - \wp(v)},$$

which is the addition formula for Weierstrass $\wp'$-function.

The fact below follows directly from Lemma 2.6.

Lemma 5.5. $\wp_{g,g}(u) + \wp_{g,g}(v) + \wp_{g,g}(u + v) = h_1^2 - 2h_2$.

Combining Lemma 5.5 with Theorem 5.3 we find

$$\big(\zeta_g(u) + \zeta_g(v) + \zeta_g(w)\big)^2 = \wp_{g,g}(u) + \wp_{g,g}(v) + \wp_{g,g}(u + v) + 2h_2 \tag{10}$$

In the case $g = 1$ due to the fact that $h_2 = 0$ formula (10) gives the famous relation

$$(\zeta(u) + \zeta(v) - \zeta(u + v))^2 = \wp(u) + \wp(v) + \wp(u + v). \tag{11}$$

discovered by Frobenius and Stickelberger.

Example 4. Let us pass to the case $g = 2$. We have $R_6(x, y) = x^2 + h_1 y + h_2 x^2 + h_4 x + h_6$. Note that $h_3 = 0$. The coefficient h_1 is expressed as follows, cf. Example 2,

$$h_1 = -2\frac{\begin{vmatrix} \wp_{2,1}(v) - \wp_{2,1}(u) & \wp_{2,2}(u)\wp_{2,1}(v) - \wp_{2,2}(v)\wp_{2,1}(u) \\ \wp_{2,2}(v) - \wp_{2,2}(u) & \wp_{2,1}(v) - \wp_{2,1}(u) \end{vmatrix}}{\begin{vmatrix} \wp_{2,1}(v) - \wp_{2,1}(u) & \wp_{2,2,1}(v) - \wp_{2,2,1}(u) \\ \wp_{2,2}(v) - \wp_{2,2}(u) & \wp_{2,2,2}(v) - \wp_{2,2,2}(u) \end{vmatrix}}.$$

And the coefficient h_2, respectively,

$$h_2 = \frac{\begin{vmatrix} \wp_{2,2}(v)\wp_{2,1}(v) - \wp_{2,2}(u)\wp_{2,1}(u) & \wp_{2,2,1}(v) - \wp_{2,2,1}(u) \\ \wp_{2,1}(v) + \wp_{2,2}(v)^2 - \wp_{2,1}(u) - \wp_{2,2}(u)^2 & \wp_{2,2,2}(v) - \wp_{2,2,2}(u) \end{vmatrix}}{\begin{vmatrix} \wp_{2,1}(v) - \wp_{2,1}(u) & \wp_{2,2,1}(v) - \wp_{2,2,1}(u) \\ \wp_{2,2}(v) - \wp_{2,2}(u) & \wp_{2,2,2}(v) - \wp_{2,2,2}(u) \end{vmatrix}}.$$

We come to the relations

$$\begin{aligned} &\zeta_2(u) + \zeta_2(v) - \zeta_2(u + v) = -h_1, \\ &\wp_{2,2}(u) + \wp_{2,2}(v) + \wp_{2,2}(u + v) = h_1^2 - 2h_2, \\ &\zeta_1(u) + \zeta_1(v) - \zeta_1(u + v) - \frac{1}{2}(\wp_{2,2,2}(u) + \wp_{2,2,2}(v) - \wp_{2,2,2}(u + v)) = -h_1^3 + 3h_1 h_2. \end{aligned}$$

Hence, by eliminating h_1 and h_2, we obtain the identity

$$2\mathfrak{z}_1 - \mathfrak{p}_{2,2,2} - 3\mathfrak{p}_{2,2}\mathfrak{z}_2 + \mathfrak{z}_2^3 = 0, \tag{12}$$

where $\mathfrak{z}_i = \zeta_i(u) + \zeta_i(v) + \zeta_i(w)$ and $\mathfrak{p}_{i,j,\ldots} = \wp_{i,j,\ldots}(u) + \wp_{i,j,\ldots}(v) + \wp_{i,j,\ldots}(w)$, provided $u + v + w = 0$.

5.3 Trilinear addition theorems for hyperelliptic σ-functions

Formula (12) leads to an important corollary.

Theorem 5.6. *The genus* 2 *sigma-function respects the trilinear addition law*

$$\big[2D_1 + D_2^3\big]\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0,$$

where $D_j = \partial_{u_j} + \partial_{v_j} + \partial_{w_j}$.

Proof. Let us multiply the left hand side of (12) by the product $\sigma(u)\sigma(v)\sigma(w)$, then (12) becomes the trilinear relation

$$\big[2(\partial_{u_1} + \partial_{v_1} + \partial_{w_1}) + (\partial_{u_2} + \partial_{v_2} + \partial_{w_2})^3\big]\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0,$$

which is satisfied by the genus 2 sigma-function. ■

It is important to notice that the elliptic identity (11) is equivalent to the trilinear addition law

$$\big[(\partial_u + \partial_v + \partial_w)^2\big]\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0,$$

which is satisfied by Weierstrass sigma-function. Let us denote $D = (\partial_u + \partial_v + \partial_w)$ and $\psi = \sigma(u)\sigma(v)\sigma(w)$. The functions

$$(D + h_1)\psi, \quad (D^3 + 6h_3)\psi, \quad (D^4 - 6\lambda_4)\psi, \quad (D^5 + 18\lambda_4 D)\psi, \quad (D^6 - 6^3\lambda_6)\psi,$$

where h_1 and h_2 are given in Example 3, vanish on the plane $u + v + w = 0$. Moreover, one can show that for any $k > 3$ there exist unique polynomials $q_0, q_1, q_3 \in \mathbb{Q}[\lambda_4, \lambda_6]$ such that

$$(D^k + q_3 D^3 + q_1 D + q_0)\psi\big|_{u+v+w=0} = 0,$$

and at least one of the polynomials q_0, q_1, q_3 is nontrivial.

For hyperelliptic sigma-function of an arbitrary genus g we propose the following hypothesis. Let $\mathcal{P} = \mathbb{Q}[\Lambda]$. Consider the ring $\mathcal{Q} = \mathcal{P}[D_1, \dots, D_g]$ as a graded ring of linear differential operators. We conjecture that there exists a collection of $3g$ linear operators $Q_i \in \mathcal{Q}$, $\deg Q_i = i$, where $i = 1, \dots, 3g$, such that

$$\Big\{\sum_{i=0}^{3g} Q_i \xi^{3g-i} + R_{3g}(\xi^2, \xi^{2g+1})\Big\}\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0,$$

where $Q_0 = 1$ and $R_{3g}(x, y)$ is the function (8) computed in the variables indicated in Corollary 5.2. Thus, g operators Q_{g+2i-1}, $i = 1, \dots, g$, define the trilinear relations

$$Q_{g+2i-1}\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0, \quad i = 1, \dots, g.$$

Note, that the assertions of Theorem 5.3, Lemma 5.5, and Theorem 5.4 imply the relations

$$(D_g + h_1)\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0, \quad (D_g^2 - 2h_2)\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0,$$
$$(2D_{g-1} + D_g^3 + 6h_3)\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0} = 0.$$

We shall return to the problem of explicit description of the trilinear addition theorems for hyperelliptic sigma-function in our future publications.

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