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Quasi-Exactly Solvable Hamiltonians related to Root Spaces

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Abstract

$sl(2)-$Quasi-Exactly-Solvable (QES) generalization of the rational $A_n, BC_n, G_2, F_4, E_{6,7,8}$ Olshanetsky-Perelomov Hamiltonians including many-body Calogero Hamiltonian is found. This generalization has a form of anharmonic perturbations and it appears naturally when the original rational Hamiltonian is written in a certain Weyl-invariant polynomial variables. It is demonstrated that for the QES Hamiltonian there exists a finite-dimensional invariant subspace in inhomogeneous polynomials. Eigenfunctions and corresponding eigenvalues which belong to this subspace are calculated algebraically.

1 Introduction

Undoubtedly, the exact solutions are of great importance, especially, of the multidimensional Schroedinger equations. Up to now the Hamiltonian Reduction Method which also is called the Projection Method [1, 2] provides a unique opportunity to construct non-trivial multidimensional exactly-solvable and completely integrable multidimensional quantal Hamiltonians. These Hamiltonians are associated with root systems, they are related with the Laplace-Beltrami operators on symmetric spaces. Their eigenfunctions and eigenvalues can be found algebraically, by linear algebra means. In particular, (i) their eigenvalues are known explicitly being a second degree polynomial in quantum numbers, (ii) any eigenfunction has a form of the ground state eigenfunction multiplied by a polynomial in some arguments (factorization property). One of the particular cases of the construction is the so-called rational case which provides the rational Hamiltonians.

In general, the rational Hamiltonian associated with a root system $\mathcal{R}$ of algebra g of rank N has a form

$$\mathcal{H} = \frac{1}{2}\sum_{k=1}^{n}\left[-\frac{\partial^2}{\partial x_k^2}+\omega^2 x_k^2\right] + \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} g_{|\alpha|}|\,\alpha|^2\frac{1}{(\alpha\cdot x)^2}\ , \tag{1.1}$$

where $\alpha \in \mathcal{R}_+$ are positive roots of the system $\mathcal{R}$ which are vectors in $\mathbb{R}^n$, $x = (x_1, \ldots, x_n)$ is a set Cartesian coordinates, $|\alpha|^2 = \sum_1^n \alpha_k^2$, and the scalar product $(\alpha \cdot x) = \sum_1^n \alpha_k x_k$ and ω is a parameter. Coupling constants $g_{|\alpha|}$ are assumed to be equal for roots of the same length. Hence for the A_n case there is a single coupling constant since all roots are of the same length, for the BC_n case there are three coupling constants, since there exist roots of three different length etc. For some algebras (G_2, E_6, E_7) it is convenient to consider the roots (and coordinates) in subspace of the vector space of higher dimension $\mathbb{R}^{N+n}$ ($n = 1$ or 2) with appropriate constraints on coordinates. In general, the Hamiltonians of this type describe a quantum particle in multidimensional space, although for A_n and G_2 cases these Hamiltonians allow another interpretation as well as the Hamiltonians describing many-body systems. These two systems are called Calogero [3] and Wolfes [4] models, respectively.

Soon after a discovery of these Hamiltonians it became clear [2] there exists a straightforward generalization of these Hamiltonians to the case of arbitrary coupling constants without breaking any nice property. It led to a loss of immediate group theoretical interpretation. In particular, a property of exact-solvability remained to be preserved. It gave a hint on existence of a more general formalism where above-mentioned Hamiltonians appear naturally. An idea was to connect solvability with possible existence of an intrinsic hidden algebraic structure [5]. It turned out to be true: for arbitrary coupling constants these Hamiltonians are related with elements of the universal enveloping algebra of some algebras of differential operators acting in the space of invariants of the corresponding root space. Such an algebra was called the *hidden* algebra of the Hamiltonian. It was found that for all A_n, B_n, C_n, D_n, BC_n rational (and trigonometric models) this algebra is the *same* (!) – it is the maximal affine subalgebra of the gl_n-algebra realized by the first order differential operators in $\mathbf{R^N}$ in symmetric representation [6, 7]. Thus, one can state that all these models are nothing but different appearances of a single model characterized by the hidden algebra gl_N. Similar situation held for the SUSY generalizations of above models - all of them turned out to be associated to the hidden superalgebra $gl(N|N-1)$, see [7]. However, one can naturally expect that the situation is drastically different for the Hamiltonians related to the root spaces of the exceptional algebras - each Hamiltonian is characterized by its own hidden algebra which is different for different Hamiltonians. A first indication stemmed from a study of the G_2 rational (and trigonometric) models where the common hidden algebra turned out to be a certain infinite-dimensional, but finitely-generated algebra of the differential operators that was called $g^{(2)} \subset \mathrm{diff}(2, \mathbb{R})$ [8, 9]. Later, it was shown that the similar situation holds for the rational (and trigonometric F_4) models [10]. Both models possess the same hidden algebra which was called $f^{(4)} \subset \mathrm{diff}(4, \mathbb{R})$. Recently, a thorough study was completed for the whole set of the rational models related to the exceptional algebras including $E_{6,7,8}$ [11].

It was already many years ago when V.I. Arnold [12] paid attention that the flat metric with upper indices written in terms of the polynomial Weyl invariants of the fixed degree is characterized by polynomial matrix elements. It implied that the coefficient functions in front of the second derivatives in the Laplace-Beltrami operator are polynomials in invariants of any Weyl group. Crucial observations were made in [6, 7, 8, 10, 11]: in the variables which are the Weyl-invariant polynomials of the fixed degree the rational Hamiltonians which are a combination of Laplace-Beltrami operator and a potential – after the gauge (similarity) transformation both the coefficient functions in front of the second and the first derivatives remain polynomials[1]. It implies that each rational Hamiltonian takes an algebraic form and preserves a certain flag of polynomials. Even more if a certain set of the polynomial Weyl invariants is chosen as variables, the Hamiltonian preserves a *minimal* flag of invariant subspaces of polynomials (for definition of the minimal flag see, for instance, [11]). All these Hamiltonians in an algebraic form reveal a property of existence an infinite family of eigenfunctions depending on a single variable [11]. This property leads to a chance to construct a certain Lie-algebraic, Quasi-Exactly-Solvable (QES) generalization of the rational models (see [17]).

By definition a linear differential operator is quasi-exactly-solvable (QES) if it preserves a finite-dimensional functional space with explicitly defined basis. It implies that the operator has a finite-dimensional invariant subspace spanned by known function. Therefore one can indicate explicitly a basis where the operator being written in the matrix form has a block-triangular form. This property leads to reducibility of the original transcendental secular equation: its r.h.s. becomes the product of a polynomial in spectral parameter to a transcendental function. Therefore the QES operator is characterized by the explicit (algebraic) knowledge of a finite number of eigenstates - some eigenstates can be found by the algebraic means. For almost all known examples of the QES operators the finite-dimensional invariant subspace is a space of inhomogeneous polynomials in one or several variables. What is truly remarkable it is a fact that in many cases the space of polynomials can be identified with a finite-dimensional representation space of a Lie algebra of differential operators.

The notion *Quasi-Exact-Solvability* was introduced in [15] and was further developed in [16], where ten multi-parametric one-dimensional QES Schroedinger equations were constructed. In [17]) it was shown all those examples are related with the $sl(2)$-algebra of the first-order differential operators in one variable. Later it was proven that any linear differential operator of finite order which preserves a space of polynomials in one variable is an element of the enveloping algebra of the $sl(2)$-algebra of the first-order differential operators in one variable [18]. A classification and detailed analysis of the QES Schroedinger differential equations in one variable was done in [19]. A generalization of the idea of the quasi-exact-solvability to multidimensional and matrix differential operators was proposed in [20] and further developed in [21, 22]. Finite-difference QES operators were introduced and investigated in [23, 24]. In parallel with a study of QES differential and finite-difference operator it was discovered a connection of QES problems with conformal field theories [25] as well as with the Bethe Anzatz approach to finite magnetic, the spin chains (see [26, 27] and references therein). Perhaps, it is worth mentioning there exist several review articles on the subject [5, 28, 29] and a book by Ushveridze [26] which is mostly dedicated to a so-called *analytic* approach to the quasi-exact-solvability rather than a (Lie)-algebraic one.

For the case of the Calogero (A_n rational) model QES generalization was already found some time ago [30]. The goal of the present article is to demonstrate that one can construct QES generalizations in the explicit form for all remaining rational models - the models related to the BC_n and exceptional algebra root spaces in a unified way.

[1]Similar results we also obtained in above-mentioned works for the trigonometric A_n, BC_n, G_2 and F_4 Hamiltonians when the trigonometric Weyl invariants (which mean the Weyl invariants with periodicity property in each variable with a requirement that all periods are equal) are used as coordinates. Even for elliptic A_1 and BC_n models the elliptic Weyl invariants allow to get the polynomials coefficients in front of derivatives (see for details [13] and [14], correspondingly)

2 Construction

Let us consider the spectral problem for the Hamiltonian $\mathcal{H}$ given by (1.1)

$$\mathcal{H}\Psi(x) \;=\; E\Psi(x) \;. \tag{2.1}$$

Then make a gauge rotation of the Hamiltonian with the ground state eigenfunction Ψ_0 as a gauge factor

$$h = -(\Psi_0(x))^{-1}(\mathcal{H} - E_0)\Psi_0(x) \;, \tag{2.2}$$

where E_0 is the ground state energy. A new spectral problem appears

$$h\varphi(x) \;=\; -\epsilon\varphi(x) \;, \tag{2.3}$$

with a new spectral parameter $\epsilon = E - E_0$. If in (2.1) the boundary condition corresponds to the normalizability of $\Psi(x)$, for (2.3) it is the normalizability of $\varphi(x)\Psi_0(x)$. It implies that instead of the standard scalar product with unit measure, we deal with the Hilbert space with measure $\Psi_0^2(x)$. By construction, the lowest eigenvalue $\epsilon_0 = 0$ and the lowest eigenfunction $\varphi_0 = \text{const}$.

Take the root space of the algebra g. In this space the Weyl group W_g acts. The algebraically independent invariant polynomials of the *lowest* possible degrees a generate the algebra S^{W_g} of W_g-invariant polynomials. The powers a are the *degrees* of the group W_g. A particular form of these polynomials (denoted below as $t_a^{(\Omega)}$) can be found by averaging elementary polynomials $(\omega, x)^a$ over some group orbit Ω,

$$t_a^{(\Omega)}(x) = \sum_{\omega\in\Omega} (\omega, x)^a \;, \tag{2.4}$$

(see [31]), where x's are some formal variables. We call the variables (2.4) as orbit variables and denote them for simplicity as $t_a \equiv t_a^{(\Omega)}$. Usually, averaging over different orbits gives algebraically related invariants. It is worth to emphasize that for any semi-simple algebra g there exists invariant of the degree two, $t_2^{(\Omega)}$, which does not depend on chosen orbit. The ground state eigenfunction of (1.1) is always related with this invariant and has a form

$$\Psi_0 \;=\; \Delta_g \exp\left(-\frac{\omega}{2} t_2^{(\Omega)}\right) \;, \tag{2.5}$$

where

$$\Delta_g \;=\; \prod_{\mathcal{R}_+} |(\alpha_k, y)|^{\nu_{|\alpha|}}$$

and $\nu_{|\alpha|}$ are defined through $g_{|\alpha|} = \nu_{|\alpha|}(\nu_{|\alpha|} - 1)$ and assumed to be equal for roots of the same length. In the case of the short roots (or if all roots are of the same length as for A_n) we denote $\nu_{|\alpha|} = \nu$, for the long roots $\nu_{|\alpha|} = \mu$.

The variables (2.4) are the variables where the gauge-rotated Hamiltonian (2.2) takes the algebraic form

$$h(\tau) \;=\; \mathcal{A}_{ij}(t)\frac{\partial^2}{\partial t_i \partial t_j} + \mathcal{B}_i(t)\frac{\partial}{\partial t_i} \;, \tag{2.6}$$

where the coefficient functions $\mathcal{A}_{ij}, \mathcal{B}_i$ are polynomials. This property was discovered at first for A_n [6] and then for BC_n [7] as well as for G_2 [8], F_4 [10], $E_{6,7,8}$ [11]. It was found for all these cases a remarkable feature holds: the coefficient functions in front of the second and first derivatives in t_2 depend on the variable t_2 only,

$$\mathcal{A}_{22}(t) = 4t_2 \ , \ \mathcal{B}_2(t) \ = \ -4\omega t_2 + 2\beta_g \ . \tag{2.7}$$

where β_g is a parameter. This parameter β_g depends on the root system and is equal to

- $\beta_{A_n} = n(1 + \nu + \nu n)$
- $\beta_{BC_n} = n(1 - 2\nu + \mu + 2\nu n)$
- $\beta_{G_2} = 2(1 + 3\nu + 3\mu)$
- $\beta_{F_4} = 4(1 + 6\nu + 6\mu)$
- $\beta_{E_6} = 6(1 + 12\nu)$
- $\beta_{E_7} = 7(1 + 18\nu)$
- $\beta_{E_8} = 8(1 + 30\nu)$

Surprisingly, the first factor is always equal to the rank of the algebra or, in different words, it states

$$\beta_g(\nu = 0, \mu = 0) = \text{rank } g \ . \tag{2.8}$$

It can be considered as a feature of the flat Laplacian written in the Weyl-invariant coordinates (2.4).

The above feature (2.7) allows immediately to draw a conclusion that among eigenfunctions of the operator (1.1), which, in general, depend on several variables, there exists *an exceptional family of eigenfunctions depending on the single variable* t_2. These eigenstates are the solutions of the eigenvalue problem

$$-h_2^{(es)}\varphi \equiv -4t_2 \ \frac{\partial^2\varphi}{\partial t_2 \partial t_2} \ + (4\omega t_2 - 2\beta_g) \ \frac{\partial \varphi}{\partial t_2} \ = \ \epsilon\varphi \ , \tag{2.9}$$

they coincide to the Laguerre polynomials

$$\varphi_k(t_2) \ = \ L_k^{(\frac{\beta_g}{2}-1)}(\omega t_2) \ , \ \epsilon_k = 4\omega k \ , \ k = 0, 1, 2, \dots \ . \tag{2.10}$$

It is worth to mention that the operator $h_2^{(es)}$ can be reduced to the self-adjoint form by a change of variables and a gauge transformation. Finally, in the variable $y = \sqrt{t_2}/\omega$ this operator takes a form

$$\mathcal{H}_2^{(es)} = -\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{\omega^2}{2}y^2 + \frac{\beta_g^2 - 1}{8y^2} - \frac{\omega}{2}(1 + 2\nu) \ ,$$

which correspond to the harmonic oscillator on half-line with singular interaction at $y = 0$. Their eigenfunctions are

$$\Psi_2^{(es)} = \varphi_k(y) y^{\frac{\beta_g + 1}{2}} e^{-\frac{\omega}{2}y^2} \ .$$

(see (2.10)).

The operator in the l.h.s. of (2.9) can be rewritten in terms of the generators J_k^0, J^- of the Cartan subalgebra of the algebra $sl(2)$ of the first order differential operators:

$$J_k^+ = t_2^2 \frac{\partial}{\partial t_2} - kt_2 \ , \ J_k^0 = t_2 \frac{\partial}{\partial t_2} - \frac{k}{2} \ , \ J^- = \frac{\partial}{\partial t_2} \ , \tag{2.11}$$

(see e.g. [16]). For integer k the generators (2.11) have a common invariant subspace in polynomials of the degree not higher than k,

$$\mathcal{P}_k = \langle t_2^p \mid 0 \le p \le k \rangle \ , \tag{2.12}$$

where the dimension of the representation $\dim \mathcal{P}_k = (k+1)$. Finally, the operator (2.9) takes the $sl(2)$-Lie-algebraic form

$$h_2^{(es)} = 4J_0^0 J^- - 4\omega J_0^0 + 2\beta_g J^- \ . \tag{2.13}$$

It is easy to check that the operator $h_2^{(es)}$ preserves an infinite flag of the spaces of polynomials (2.12),

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_k \subset \ldots \ , \tag{2.14}$$

and, in particular, any eigenfunction $L_k^{(\frac{\beta_g}{2}-1)}(\omega t_2) \in \mathcal{P}_k$ (see (2.10)) and hence it is an element of the flag.

Following a standard strategy for construction of quasi-exactly-solvable problems [16] (for a general discussion see e.g. [5]), let us modify the operator (2.13) by adding a term with raising generator J_k^+,

$$h_2^{(qes)} = 4J_k^0 J^- \ + \ 4aJ_k^+ - \ 4\omega J_k^0 \ + \ 2(k + \beta_g - 2\gamma)J^- \ , \tag{2.15}$$

where $a \ge 0$ and γ are parameters. It is evident that $h_2^{(qes)}$ maps a certain space $\mathcal{P}_k$ to itself but it does not preserve the flag (2.14). By substitution of the explicit expressions of the $sl(2)$-generators (2.11) into (2.15) we get a second order differential operator

$$h_2^{(qes)} = \ 4t_2 \ \frac{\partial^2}{\partial t_2 \partial t_2} \ + 2(2at_2^2 - 2\omega t_2 + \beta_g - 2\gamma) \ \frac{\partial}{\partial t_2} - 4akt_2 + 2\omega k \ , \tag{2.16}$$

which has the space $\mathcal{P}_k$ as a finite-dimensional invariant subspace. Hence (2.16) has $(k+1)$ polynomial eigenfunctions of the form

$$P_j^{(k)}(t_2) = \sum_{i=0}^{k} \gamma_i^{(j)} t_2^i \ , \quad j = 0, 1, \ldots, k \ ,$$

where for any j, without loss of generality, the coefficient in front of the leading degree can be placed equal to one, $\gamma_k^{(j)} = 1$.

Now we are ready to proceed to a construction of a QES generalization of (1.1). We look for QES Hamiltonian in a certain form:

$$\mathcal{H}^{(qes)} = \mathcal{H} + V^{(qes)}(t_2) \ , \tag{2.17}$$

where $V^{(qes)}(t_2)$ is a potential. Let us make a gauge rotation of (2.17) in the form (2.2) with gauge factor (2.3). Then impose a requirement that the resulting operator possesses t_2-depending family of eigenfunctions. It results to the equation

$$-h_2^{(qes)}\varphi \equiv -4t_2\, \frac{\partial^2\varphi}{\partial t_2 \partial t_2} \ + (4\omega t_2 - 2\beta_g)\, \frac{\partial\varphi}{\partial t_2} \ + V^{(qes)}(t_2)\varphi \ = \ \epsilon\varphi\ , \tag{2.18}$$

which describes this family.

Now one can pose a question: under what condition on potential $V^{(qes)}$, the operator $h_2^{(qes)}$ is Lie-algebraic. The problem appeared for the first time in [30], where a QES generalization of the Calogero model was studied. Its solution is given by implementation of the following procedure: we look for a gauge rotation of $h_2^{(qes)}$ which allows (i) to gauge away potential $V^{(qes)}$ (up to constant terms), and (ii) to reduce the resulting operator to the $sl(2)$ Lie-algebraic form. Following the philosophy of quasi-exact-solvability it is not surprising that such a gauge rotation exists. Finally, we arrive at the operator

$$\begin{aligned} h_2^{(sl(2)-qes)}(t_2) &= t_2^{-\gamma}\exp(\frac{a}{4}t_2^2)\ h_2^{(qes)}\ t_2^{\gamma}\exp(-\frac{a}{4}t_2^2) \\ &= 4J_k^0 J^- \ + \ 4aJ_k^+ - \ 4\omega J_k^0 \ + \ 2(k+\beta_g-2\gamma)J^-\ , \end{aligned} \tag{2.19}$$

where $a \geq 0$ and γ are parameters, and $J^{+,0,-}$ are the $sl(2)$-algebra generators (2.11) (cf. (2.15)). The corresponding potential $V^{(qes)}$ (see (2.18)) which assure the existence of the representation (2.19) is of the form

$$V^{(qes)} \ = \ a^2 t_2^3 + 2a\omega t_2^2 - 2a(2k+2\beta_g-\gamma-1)t_2 + \frac{2\gamma(\gamma-2\beta_g+3)}{t_2}\ , \tag{2.20}$$

where the constant terms are dropped off. Now one can give the final expression of the $sl(2)$-quasi-exactly-solvable Hamiltonian associated with the root space g and written in the Weyl invariant form

$$\begin{aligned} \mathcal{H} = &-\frac{1}{2}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2} + \left[\frac{\omega^2}{2} - a(2k+2\beta_g-\gamma-1)\right]\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2 + \\ &\frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} g_{|\alpha|}|\,\alpha|^2\frac{1}{(\alpha\cdot x)^2} + \frac{2\gamma(\gamma-2\beta_g+3)}{\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2} + \\ &2a\omega\bigg(\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2\bigg)^2 + a^2\bigg(\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2\bigg)^3\ , \end{aligned} \tag{2.21}$$

where center-of-mass is added (if necessary). In this Hamiltonian we know $(k+1)$ eigenstates explicitly, they can be calculated by algebraic means. Their eigenfunctions are of the form

$$\begin{aligned} \Psi_0^{(\mathrm{r})}(x) = \Delta_g \cdot &\bigg(\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2\bigg)^{\gamma}\cdot P_k(\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2) \\ \exp&\left[-\frac{\omega}{2}\bigg(\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2\bigg) - \frac{a}{4}\bigg(\sum_{\alpha\in\mathcal{R}_+}(\alpha\cdot x)^2\bigg)^2\right]\ , \end{aligned} \tag{2.22}$$

where P_k is a polynomial of degree n and

$$\Delta_g \;=\; \prod_{\alpha\in\mathcal{R}_+} |(\alpha,x)|^{\nu_{|\alpha|}}\ ,$$

where $\nu_{|\alpha|}$ are defined through $g_{|\alpha|}=\nu_{|\alpha|}(\nu_{|\alpha|}-1)$, they assumed to be equal for roots of the same length. In the case of the short roots (or if all roots are of the same length as for A_n) we denote $\nu_{|\alpha|}=\nu$ and the constant $g_{|\alpha|}$ is denoted as g_s, as for the long roots $\nu_{|\alpha|}=\mu$ and g_l, correspondingly.

3 Concrete cases

The aim of the Section to present concrete formulas for all $A-D-E$ models.

Quasi-exactly-solvable generalization of the A_n rational model

After substitution the data of the A_n into (2.21) we arrive at the quasi-exactly-solvable generalization of the Calogero model, or in other words, quasi-exactly-solvable A_n rational model, which was found in [30]. In Cartesian coordinates the Hamiltonian has a form

$$H^{(2)}_{Cal}=\frac{1}{2}\sum_{i=1}^{n}\left(-\frac{\partial^2}{\partial {x_i}^2}+\omega^2 x_i^2\right)+\sum_{j<i}^{n}\frac{g}{(x_i-x_j)^2}+\frac{2\gamma\left[\gamma-2n(1+\nu+\nu n)+3\right]}{\mathbf{x}^2}$$

$$+\ a^2(\mathbf{x}^2)^3\ +\ 2a\omega(\mathbf{x}^2)^2\ -\ a\left[2k+2n(1+\nu+\nu n)-\gamma-1\right]\mathbf{x}^2, \tag{3.1}$$

where $\mathbf{x}^2=\sum_{i<j}^{n} y_i y_j$. The Perelomov coordinates of the relative motion (see e.g. [2] and references therein) are introduced

$$y_{1,\ldots,n}^{Perelomov}=x_{1,\ldots,n}-\frac{1}{n}X\ ,\quad X=\sum_{1}^{n}x_i\ , \tag{3.2}$$

with a condition $\sum_1^n y_i=0$, where X is the center-of-mass coordinate and $\mathbf{x}^2$ is the second order invariant of the A_n root space.

$$\Psi_0^{(\mathrm{qes})}(x)=(\Delta)^{\nu}(\mathbf{x}^2)^{\gamma}P_k(\mathbf{x}^2)\ \exp\left[-\frac{\omega}{2}\sum_{k=1}^{n}x_i^2-\frac{a}{4}(\mathbf{x}^2)^2\right]\ , \tag{3.3}$$

where P_k is a polynomial of degree k, $g_s=\nu(\nu-1)>-\frac{1}{4}$ and $g_l=3\mu(\mu-1)>-\frac{3}{4}$. Here $\Delta(y)$ is the Vandermonde determinant

$$\Delta(x)=\prod_{\mathcal{R}}|(\alpha_k,y)|=\prod_{i<j}^{n}|y_i-y_j|\ .$$

Quasi-exactly-solvable generalization of the BC_n rational model

After substitution the data of the BC_n into (2.21) we arrive at the quasi-exactly-solvable BC_n rational model. In Cartesian coordinates the Hamiltonian has a form

$$\begin{aligned}\mathcal{H}^{(qes)}_{BC_n} = &\ \frac{1}{2}\sum_{i=1}^{n}\left(-\frac{\partial^2}{\partial {x_i}^2}+\omega^2 x_i^2\right)+g\sum_{i<j}^{n}\left[\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right]\\ &+\frac{g_2}{2}\sum_{i=1}^{n}\frac{1}{x_i^2}+a^2(\mathbf{x}^2)^3+2a\omega(\mathbf{x}^2)^2 \\ &-a\left[2k+2n(1-2\nu+\mu+2\nu n)-\gamma-1\right]\mathbf{x}^2+\frac{2\gamma\left[\gamma-2n(1-2\nu+\mu+2\nu n)+3\right]}{\mathbf{x}^2}\ ,\end{aligned} \tag{3.4}$$

where $\mathbf{x}^2=\sum_i^n {x_i}^2$ is the second invariant in BC_n root space. For this Hamiltonian we know $(k+1)$ eigenstates explicitly (by algebraic means) and their eigenfunctions are of the form

$$\Psi_0=\left[\prod_{i<j}|x_i-x_j|^{\nu}|x_i+x_j|^{\nu}\prod_{i=1}^{n}|x_i|^{\mu}\right](\mathbf{x}^2)^{\gamma}P_k(\mathbf{x}^2)e^{-\frac{\omega}{2}\mathbf{x}^2-\frac{a}{4}(\mathbf{x}^2)^2}\ , \tag{3.5}$$

where $g=\nu(\nu-1), g_l=\mu(\mu-1)$ and where P_k is a polynomial of degree k.

Quasi-exactly-solvable generalization of the G_2 rational model

After substitution the data of the G_2 into (2.21) we arrive at the quasi-exactly-solvable G_2 rational Hamiltonian. In Cartesian coordinates it has a form

$$\begin{aligned}\mathcal{H}^{(qes)}_{\mathrm{G}_2} = &\ \frac{1}{2}\sum_{i=1}^{3}\left[-\frac{\partial^2}{\partial x_i^2}+\omega^2 x_i^2\right]+g_s\sum_{i<l}^{3}\frac{1}{(x_i-x_l)^2}+g_l\sum_{i<l\ ,\ i,l\neq m}^{3}\frac{1}{(x_i+x_l-2x_m)^2}\\ &+a^2(\mathbf{x}^2)^3+2a\omega(\mathbf{x}^2)^2+2a[2k-1-2\gamma-(3\mu+3\nu+1)]\mathbf{x}^2\\ &+\frac{4\gamma(\gamma+3\mu+3\nu)}{\mathbf{x}^2}\ ,\end{aligned} \tag{3.6}$$

where $\mathbf{x}^2=\sum_{i<j}^{3}(y_i-y_j)^2$ and the Perelomov coordinates of the relative motion (see e.g. [2] and references therein) are introduced

$$y^{Perelomov}_{1,2,3}=x_{1,2,3}-\frac{1}{3}X\ ,\ X=x_1+x_2+x_3\ , \tag{3.7}$$

with a condition $y_1+y_2+y_3=0$, where X is the center-of-mass coordinate and $\mathbf{x}^2$ is the second order invariant of the G_2 root space. For this Hamiltonian we know $(k+1)$ eigenstates explicitly (by algebraic means) and their eigenfunctions are of the form

$$\Psi^{(\mathrm{qes})}_0(x)=(\Delta_s)^{\nu}(\Delta_l)^{\mu}(\mathbf{x}^2)^{\gamma}P_k(\mathbf{x}^2)\ \exp\left[-\frac{\omega}{2}\sum_{i=1}^{3}x_i^2-\frac{a}{4}(\mathbf{x}^2)^2\right]\ , \tag{3.8}$$

where P_k is a polynomial of degree k, $g_s = \nu(\nu-1) > -\frac{1}{4}$ and $g_l = 3\mu(\mu-1) > -\frac{3}{4}$. Here $\Delta_s(y)$ and $\Delta_l(y)$ are Vandermonde determinants

$$\Delta_s(x) = \prod_{\mathcal{R}_{short}} |(\alpha_p, y)| = \prod_{i<j}^{3} |y_i - y_j| \ ,$$

$$\Delta_l(x) = \prod_{\mathcal{R}_{long}} |(\alpha_p, y)| = \prod_{i<j;\ i,j\neq p}^{3} |y_i + y_j - 2y_p| \ .$$

Hence, we constructed the $sl(2)$ QES deformation of the rational G_2 model. All attempts to construct other QES deformation of the G_2 rational model fault so far.

Quasi-exactly-solvable generalization of the F_4 rational model

After substitution the data of the F_4 into (2.21) we arrive at the quasi-exactly-solvable F_4 rational Hamiltonian. In Cartesian coordinates it has a form

$$\begin{aligned}
\mathcal{H}^{(qes)}_{\mathrm{F}_4} = {} & \frac{1}{2}\sum_{i=1}^{4}\left(-\frac{\partial^2}{\partial x_i^2} + 4\omega^2 x_i^2\right) + 2g_l \sum_{j>i}\left(\frac{1}{(x_i-x_j)^2} + \frac{1}{(x_i+x_j)^2}\right) \\
& + \frac{g_s}{2}\sum_{i=1}^{4}\frac{1}{{x_i}^2} + 2g_s \sum_{\nu' s=0,1} \frac{1}{[x_1 + (-1)^{\nu_2}x_2 + (-1)^{\nu_3}x_3 + (-1)^{\nu_4}x_4]^2} \\
& + a^2(\mathbf{x}^2)^3 + 2a\omega(\mathbf{x}^2)^2 + 2a[2k - \gamma + 3(4\mu + 4\nu + 1)]\mathbf{x}^2 \\
& + \frac{2\gamma[\gamma - 12\mu - 12\nu - 1)]}{\mathbf{x}^2} \ , \quad \mathbf{x}^2 = \sum_{i=1}^{4} {x_i}^2 \ ,
\end{aligned} \tag{3.9}$$

where $g_l = \nu(\nu-1)$, $g_s = \mu(\mu-1)$ are coupling constants related to sets of long and short roots. In this Hamiltonian we know $(k+1)$ eigenstates explicitly, they can be calculated by algebraic means. Their eigenfunctions are of the form

$$\Psi_0^{(\mathrm{qes})}(x) = (\Delta_-\Delta_+)^\nu\,(\Delta_0\Delta)^\mu\,(\mathbf{x}^2)^\gamma P_k(\mathbf{x}^2)\ \exp\left[-\omega\mathbf{x}^2 - \frac{a}{4}(\mathbf{x}^2)^2\right] \ , \tag{3.10}$$

where P_k is a polynomial of degree k and

$$\begin{aligned}
\Delta_+\Delta_- &= \prod_{\mathcal{R}_{long}} |(\alpha_k, x)| = \prod_{j<i}^{4}(x_i + x_j)\prod_{j<i}^{4}(x_i - x_j) \ , \\
\Delta_0\Delta &= \prod_{\mathcal{R}_{short}} |(\alpha_k, x)| \\
&= \prod_{i=1}^{4} x_i \prod_{\nu' s=0,1}\left|\frac{x_1 + (-1)^{\nu_2}x_2 + (-1)^{\nu_3}x_3 + (-1)^{\nu_4}x_4}{2}\right| \ .
\end{aligned} \tag{3.11}$$

Hence, we constructed the $sl(2)$ QES deformation of the F_4 rational model. If in (3.9) the parameter g_s (and, hence, μ) vanishes, we arrive at the $sl(2)$ QES generalization of the D_4 rational model. The latter differs from the $sl(5)$ QES deformation of D_4 which was found in [32].

Quasi-exactly-solvable generalization of the E_6 rational model

After substitution the data of the E_6 into (2.21) we arrive at the quasi-exactly-solvable E_6 rational Hamiltonian. Similar to what was done in [11] for E_6 rational model it is convenient to represent the Hamiltonian writing it in an 8−dimensional space $\{x_1, x_2, \ldots x_8\}$ with imposing of two constraints $x_7 = x_6$, $x_8 = -x_6$,

$$\mathcal{H}^{(qes)}_{\mathrm{E}_6} = -\frac{1}{2}\Delta^{(8)} + \frac{\omega^2}{2}\sum_{i=1}^{8} x_i^2 + g\sum_{j<i=1}^{5}\left[\frac{1}{(x_i+x_j)^2} + \frac{1}{(x_i-x_j)^2}\right]$$
$$+\ g\sum_{\nu_j}\frac{1}{\left[\frac{1}{2}\left(-x_8+x_7+x_6-\sum_{j=1}^{5}(-1)^{\nu_j}x_j\right)\right]^2}$$
$$+\,a^2(\mathbf{x}^2)^3 + a\omega(\mathbf{x}^2)^2 - 4a(k+\gamma+18\nu+2)\mathbf{x}^2$$
$$+\,\frac{4\gamma(\gamma+36\nu+2)}{\mathbf{x}^2}\ , \tag{3.12}$$

where $\nu_j = 0, 1$ and $\sum_{j=1}^{5}\nu_j =$ even (see [11]), $\mathbf{x}^2 = 2(\sum_{i=1}^{5} y_i^2 + 1/3 y_6^2)$ with y's defined as

$$\begin{aligned} y_i &= x_i\ , \quad i = 1\ldots 5 \\ y_6 &= x_6 + x_7 - x_8\ , \qquad \text{(using the constraints } y_6 = 3x_6) \\ y_7 &= x_6 - x_7\ , \qquad \text{(using the constraints } y_7 = 0) \\ y_8 &= x_6 + x_8\ , \qquad \text{(using the constraints } y_8 = 0)\ . \end{aligned} \tag{3.13}$$

In these coordinates the Laplacian becomes

$$\Delta^{(8)} = \Delta^{(5)}_y + 3\frac{\partial^2}{\partial y_6^2} + 2\left[\frac{\partial^2}{\partial y_7^2} + \frac{\partial^2}{\partial y_8^2} + \frac{\partial^2}{\partial y_7 \partial y_8}\ ,\right] \tag{3.14}$$

In the Hamiltonian (3.12) we know $(k+1)$ eigenstates explicitly (by algebraic means). Their eigenfunctions are of the form

$$\Psi^{(\mathrm{r})}_0(x) = (\Delta^{(5)}_+\Delta^{(5)}_-)^\nu(\Delta_{E_6})^\nu(\mathbf{x}^2)^\gamma P_k(\mathbf{x}^2)\mathrm{e}^{-\frac{1}{4}\omega\mathbf{x}^2-\frac{a}{4}(\mathbf{x}^2)^2}\ , \tag{3.15}$$

where P_k is a polynomial of degree k and where

$$\begin{aligned} \Delta^{(5)}_\pm &= \prod_{j<i=1}^{5}(y_i \pm y_j) \\ \Delta_{E_6} &= \prod_{\{\nu_j\}}\left(y_6 + \sum_{j=1}^{5}(-1)^{\nu_j}y_j\right) \end{aligned}$$

with $g = \nu(\nu-1)$.

Hence, we constructed the $sl(2)$ QES deformation of the rational E_6 model. All attempts to construct other QES deformations of the E_6 rational model fault so far.

Quasi-exactly-solvable generalization of the E_7 rational model

After substitution the data of the E_7 into (2.21) we arrive at the quasi-exactly-solvable E_7 rational Hamiltonian. Similar to what was done in [11] for E_7 rational model it is convenient to represent the Hamiltonian writing it in an 8−dimensional space $\{x_1, x_2, \dots x_8\}$ with imposing of two constraints $x_8 = -x_7$,

$$\begin{aligned}\mathcal{H}^{(qes)}_{\mathrm{E}_7} = & -\frac{1}{2}\Delta^{(8)} + \frac{\omega^2}{2}\sum_{i=1}^{8} x_i^2 + g\sum_{j<i=1}^{6}\left[\frac{1}{(x_i+x_j)^2} + \frac{1}{(x_i-x_j)^2}\right] \\ & + g\frac{1}{(x_7-x_8)^2} + g\sum_{\nu_j}\frac{1}{\left[\frac{1}{2}\left(-x_8+x_7-\sum_{j=1}^{6}(-1)^{\nu_j}x_j\right)\right]^2} \\ & + 2a^2(\mathbf{x}^2)^3 + 2a\omega(\mathbf{x}^2)^2 - 4a(2k+2\gamma+72\nu+5)\mathbf{x}^2 \\ & + \frac{8\gamma(\gamma+72\nu+3)}{\mathbf{x}^2}\ ,\end{aligned} \tag{3.16}$$

where $\nu_j = 0, 1$ and $\sum_{j=1}^{6}\nu_j = \text{odd}$ (see [11]), $\mathbf{x}^2 = 1/3(\sum_{i=1}^{6} y_i^2 + 1/2 y_7^2)$ with y's defined as

$$\begin{aligned} y_i &= x_i\ , \quad i = 1\dots 6 \\ y_7 &= x_7 - x_8\ , \qquad \text{(using the constraints } y_7 = 2x_7) \\ Y &= \frac{1}{2}(x_7+x_8)\ , \qquad \text{(using the constraint } Y = 0) \end{aligned}$$

In these variables the Laplacian becomes

$$\Delta^{(8)} = \Delta^{(6)}_y + 2\frac{\partial^2}{\partial y_7^2} + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\ , \tag{3.17}$$

In the Hamiltonian (3.12) we know $(k+1)$ eigenstates explicitly (by algebraic means). Their eigenfunctions are of the form

$$\Psi_0^{(\mathrm{r})}(x) = (\Delta_+^{(6)})^\nu(\Delta_-^{(6)})^\nu y_7^\nu(\Delta_{E_7})^\nu(\mathbf{x}^2)^\gamma P_k(\mathbf{x}^2)\mathrm{e}^{-\frac{3}{2}\omega\mathbf{x}^2-\frac{a}{4}(\mathbf{x}^2)^2}\ , \tag{3.18}$$

where P_k is a polynomial of degree k and

$$\begin{aligned}\Delta_\pm^{(6)} &= \prod_{j<i=1}^{6}(y_i \pm y_j)\ , \\ \Delta_{E_7} &= \prod_{\{\nu_j\}}\left(y_7 + \sum_{j=1}^{6}(-1)^{\nu_j}y_j\right)\ ,\end{aligned}$$

where $g = \nu(\nu-1) > -\frac{1}{4}$.

Hence, we constructed the $sl(2)$ QES deformation of the rational E_7 model. All attempts to construct other QES deformations of the E_7 rational model fault so far.

Quasi-exactly-solvable generalization of the E_8 rational model

After substitution the data of the E_8 into (2.21) we arrive at the quasi-exactly-solvable E_8 rational Hamiltonian written in the Cartesian coordinates

$$\mathcal{H}^{(qes)}_{\mathrm{E}_8} = -\frac{1}{2}\Delta^{(8)} + \frac{\omega^2}{2}\sum_{i=1}^{8} x_i^2 + g\sum_{j<i=1}^{8}\left[\frac{1}{(x_i+x_j)^2} + \frac{1}{(x_i-x_j)^2}\right]$$

$$+ g\sum_{\nu_j}\frac{1}{\left[\frac{1}{2}\left(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\right]^2}$$

$$+ a^2(\mathbf{x}^2)^3 + 2a\omega(\mathbf{x}^2)^2 - 2a(2k+2\gamma+240\nu+9)\mathbf{x}^2$$

$$+ \frac{4\gamma(\gamma+120\nu+3)}{\mathbf{x}^2}\ , \tag{3.19}$$

where $\nu_j = 0, 1$ and $\sum_{j=1}^{7}\nu_j =$ even (see [11]), $\mathbf{x}^2 = (\sum_{i=1}^{8} x_i^2)$, for which we know $(k+1)$ eigenstates explicitly (by algebraic means). Their eigenfunctions are of the form

$$\Psi_0^{(\mathrm{r})}(x) = (\Delta_+^{(8)}\Delta_-^{(8)})^\nu \Delta_{E_8}^\nu (\mathbf{x}^2)^\gamma P_k(\mathbf{x}^2) \mathrm{e}^{-\frac{\omega}{2}\mathbf{x}^2 - \frac{a}{4}(\mathbf{x}^2)^2}\ , \tag{3.20}$$

where P_k is a polynomial of degree k, $g = \nu(\nu-1) > -\frac{1}{4}$ and where

$$\Delta_\pm^{(8)} = \prod_{j<i=1}^{8}(x_i \pm x_j)\ ,$$

$$\Delta_{E_8} = \prod_{\{\nu_j=0,1\}}\left(x_8 + \sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\ ,$$

Hence, we constructed the $sl(2)$ QES deformation of the rational E_8 model. All attempts to construct other QES deformations of the E_8 rational model fault so far.

4 Conclusion

We have found in an unified way the $sl(2)$-quasi-exactly-solvable generalization of all rational integrable models associated with root systems of classical and exceptional Lie algebras. It is a great challenge to find other QES generalizations of these models. The only known particular example of such a type is a QES generalization of the Calogero model found by Hou and Shifman [32].

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This work is dedicated to 70th birthday of Francesco Calogero. I always thought the discovery of an exactly-solvable and integrable many-body quantal model made by F. Calogero, which later was called the *Calogero model*, is one of the most beautiful discoveries in mathematical physics in the second half of the 20th century. This work is supported in part by DGAPA grant No. *IN124202*.

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