Some Holey designs and Incomplete designs for the join graph of $K_1$ and $C_4$ with a pendent edge

Xiaoshan Liu $^a$
Department of Mathematics & Physics
Shijiazhuang University of Economics
China, Shijiazhuang
liuxiaoshan80617@163.com

Qi Wang $^b$
Graduate School
Hebei University of Economics & Business
China, Shijiazhuang
wangqi80617@163.com

Abstract-A $G$-design of $\lambda K_v$ is a pair $(X,B)$, where $X$ is the vertex set of $K_v$ and $B$ is a collection of subgraphs of $K_v$, such that each block is isomorphic to $G$ and any two distinct vertices in $K_v$ are joined in exact (at most, at least) $\lambda$ blocks of $B$. In this paper, we will discuss some holey designs and incomplete designs for the join graph of $K_1$ and $C_4$ with a pendent edge for $\lambda = 1$.

Keywords- $G$-packing design, $G$-covering design, Holey $G$-design

I. INTRODUCTION

A complete multigraph of order $v$ and index $\lambda$, denoted by $\lambda K_v$, is an undirected graph with $v$ vertices, where any two distinct vertices $x$ and $y$ are joined by $\lambda$ edges $(x,y)$. Let $G$ be a finite simple graph. A $G$-design $G - GD_\lambda(v)$ ( $G$-packing design $G - PD_\lambda(v)$, $G$-covering design $G - CD_\lambda(v)$) of $\lambda K_v$ is a pair $(X,B)$, where $X$ is the vertex set of $K_v$ and $B$ is a collection of subgraphs of $K_v$, called blocks, such that each block is isomorphic to $G$ and any two distinct vertices in $K_v$ are joined in exact (at most, at least) $\lambda$ blocks of $B$. A packing (covering) design is said to be maximum (minimum) if no other such packing (covering) design of the same order has more (fewer) blocks. The number of blocks in a maximum packing design (minimum covering design), denoted by $p(v,G,\lambda) (c(v,G,\lambda))$, is called the packing number (covering number). Obviously,

$$p(v,G,\lambda) \leq U(v,G,\lambda) = \left\lfloor \frac{\lambda v(v-1)}{2E(G)} \right\rfloor$$

$$\leq \left\lfloor \frac{\lambda v(v-1)}{2E(G)} \right\rfloor = V(v,G,\lambda) \leq c(v,G,\lambda),$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (least) integer $y$ such that $y \leq x$ ($y \geq x$). A $G - PD_\lambda(v)$ ($G - CD_\lambda(v)$) is called optimal and is denoted by $G - OPD_\lambda(v)$ ($G - OCD_\lambda(v)$) if the left (right) equality in above inequality holds. Obviously, there exists a $G - GD_\lambda(v)$ if and only if $p(v,G,\lambda) = c(v,G,\lambda)$. So a $G - GD_\lambda(v)$ can be regarded as a $G - OPD_\lambda(v)$ or a $G - OCD_\lambda(v)$. The leave $L_\lambda(P)$ of a packing design $G - PD_\lambda(v) = (v,P)$ is a subgraph of $\lambda K_v$ and its edges are the supplement of $P$ in $\lambda K_v$. When $P$ is maximum, $|L_\lambda(P)|$ is called leave-edges number and is denoted by $l_\lambda(v)$. Similarly, the repeat-edge graph $R_\lambda(C)$ of a covering design $G - CD_\lambda(v) = (v,C)$ is a subgraph of $\lambda K_v$ and its edges are the supplement of $\lambda K_v$ in $C$. When $C$ is minimum, $|R_\lambda(C)|$ is called repeat-edges number and is denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(P)$ and $l_\lambda(v)$ can be denoted by $L_\lambda$ and $l_\lambda$ briefly, while $R_\lambda(P)$ and $r_\lambda(v)$ can be denoted by $R_\lambda$ and $r_\lambda$ correspondingly.

Let $X = \bigcup_{i=1}^t X_i$ be the vertex set of $K_{n_1,n_2,\cdots,n_t}$, a complete multipartite graph consisting of $t$ parts with size $n_1,n_2,\cdots,n_t$ respectively, where the sets $X_i$...
Theorem 3.2 There exists $G - HD(18^{t+2})$ for $t \geq 1$.

Proof. Give the direct construction of $G - HD(9^4)$ on vertex set $Z_9 \times Z_4$ and blocks are:

$\{0, 0, 0, 2, 2, 2, 1, 0, 0\}, \{0, 3, 1, 0, 2, 6, 2, 3, 1, 8, 2, 4\}, \{0, 3, 2, 3, 3, 5, 4, 3, 2\}, \{0, 1, 7, 2, 1, 2, 0, 3, 7\}, \{0, 5, 3, 1, 2, 6, 3, 7, 2\}, \{4, 3\}, \{0, 0, 0, 4, 1, 1, 6, 1, 7\}$ mod (9, −).

IV. CONSTRUCTIONS FOR ID

Theorem 4.1 There exists $G - ID(9 + \sigma, \sigma)$ for $\sigma = 2, 3, \cdots, 7, 8, 12$.

Proof. There are $\sigma + 4$ blocks in each $G - ID(9 + \sigma, \sigma)$.

$\sigma = 2 : Z_9 \times Z_3 \cup \{x_1, x_2\} \mod (3, -) \cup \{0, 1, 0, 2, 0, 1, 2, 1\}$.

$\sigma = 3 : Z_9 \cup \{x_1, x_2, x_3\}$.

$\sigma = 4 : Z_9 \cup \{x_1, x_2, x_3\}$.

$\sigma = 5 : Z_9 \times Z_3 \cup \{x_1, x_2, \cdots, x_9\}$.

$\sigma = 6 : Z_9 \cup \{x_1, x_2, \cdots, x_9\}$. 

Figure 1. Graph $G$
\( (6, x_1, 3, x_2, 4, 8), (5, x_1, 7, x_2, 0, 8), (1, x_1, 8, x_2, 2, 5), (5, x_3, 6, x_4, 8, 2), (0, x_1, 1, x_2, 2, 4), (2, x_1, 7, x_2, 3, 4), (4, x_1, 0, x_4, 1, 5), (8, x_3, 7, x_6, 6, 2), (5, x_4, 3, x_6, 4, 7), (7, 0, 6, 1, 3, 8) \)

\( \mathcal{G} = 7 : Z_9 \cup \{ x_1, x_2, \ldots, x_7 \} \)

\( (0, x_1, 6, x_2, 1, 5), (4, x_1, 5, x_2, 7, 6), (8, x_1, 3, x_2, 2, 6), (2, x_3, 3, x_4, 1, 6), (5, x_5, x_6, 0, 8), (4, x_2, x_3, 3, 5), (7, x_3, 5, x_4, 0, 2), (x_7, 1, 4, 0, 3, 6), (2, x_3, 8, x_4, 6, x_7), (1, x_5, 7, x_6, 8, x_7), (x_7, 2, 5, 8, 7, 3) \)

\( \mathcal{G} = 8 : Z_3 \times Z_3 \cup \{ x_1, x_2, \ldots, x_9 \} \)

\( (0, x_1, 0, x_2, 0, 1, 10), (1, x_1, 0, x_2, 1, 4, 2), (1, x_2, 0, x_3, 0, 1, 1), (2, x_2, 0, x_4, 0, 1, 1) \mod (3, -) \).

\( \mathcal{G} = 12 : Z_6 \cup \{ x_1, x_2, \ldots, x_3 \} \)

\( (6, x_3, 2, x_4, 4, 3), (x_7, x_1, 0, x_6, 4, x_1), (0, x_1, x_1, 1, x_2, 2, 8), (0, x_1, x_4, 2, x_3, 3), (1, x_1, 2, x_2, 5, x_11), (7, x_1, x_8, 2, x_6, 6, x_1), (x_5, x_2, 6, x_3, 3, x_1), (1, x_7, x_6, x_4, 4, x_1), (2, x_3, 3, x_4, 7, x_11), (5, x_3, 0, x_4, 8, x_11), (x_3, x_7, 1, x_8, 7, x_12), (7, x_9, 3, x_9, 1, x_{10}, 5, 4), (6, x_9, 0, x_5, x_12), (4, x_7, x_2, x_8, x_12), (8, x_9, 0, x_{10}, 3, 6) \)

Theorem 4.2: There exists \( G - \text{ID}(18 + \mathcal{G}, \mathcal{G}) \) for \( \mathcal{G} = 2, 4, 5, 6, 7, 8, 9 \).

Proof. There are 2\( \mathcal{G} + 17 \) blocks in each \( G - \text{ID}(18 + \mathcal{G}, \mathcal{G}) \).

\( \mathcal{G} = 2 : Z_3 \times Z_6 \cup \{ x_1, x_2 \} \)

\( (0, x_1, 0, x_2, 0, 1, 1), (0, x_1, 1, x_2, 1, 0), (0, x_1, 1, x_2, 0, 0) \mod (3, -) \).

\( \mathcal{G} = 4 : Z_6 \cup \{ x_1, x_2, x_3 \} \)

\( (0, x_1, 0, x_2, 0, 0), (0, x_1, 1, x_2, 1, 0), (0, x_1, 0, x_2, 1, 0) \mod (3, -) \).

\( \mathcal{G} = 8 : Z_3 \times Z_6 \cup \{ x_1, x_2, \ldots, x_8 \} \)

\( (0, x_1, 0, x_2, 1, 1, 0), (0, x_1, 0, x_2, 1, 0, 1), (0, x_1, 0, x_2, 1, 0) \mod (3, -) \).

V. Acknowledgment

"This author was supported by the funds from Education Department Foundation of Hebei Province under fund number Z2012057 and Foundation of Shijiazhuang University of Economics under fund number ZK201103. Tel.: (+86)13463955288 E-mail address: luxiaoshan80617@163.com."
This author was supported by the funds from Nature Science Foundation of Hebei Province under fund number A2011207003 and Outstanding Youth Fund Project of Scientific Research in Colleges and Universities of Hebei Department of Education under fund number Y2011115.

Tel: (+86)13463955388; E-mail address: stwangqi@heuet.edu.cn

REFERENCES


