Gershgorin Variations of Complex Matrices

Haifeng Sang  
Department of Mathematics  
Beihua University  
Jilin, China  
e-mail: sanghaifeng2008@163.com

Qingchun Li  
Department of Mathematics  
Beihua University  
Jilin, China  
Corresponding author : liqingchun01@163.com

Abstract—We present various “additive” sufficient conditions for the nonsingularity of a complex partitioned matrix and an irreducible partitioned matrix. As the application, a new criteria of the positive stable matrix is given. The obtained results generalize some corresponding results.

Keywords- Nonsingularity; Gerschgorin disc; Stable matrix

I. INTRODUCTION

Matrix theory which is a fundamental tool in Mathematics has been an important tool in some technologic fields. And estimate of eigenvalue which is a significant part in Matrix theory which is a fundamental tool in Mathematics has been an important tool in some technologic fields. In this paper, first of all, we discussed the problem of nonsingularity of complex partitioned matrices, gave the criteria of the positive stable matrix is given. The obtained results generalize some corresponding results.

II. NOTATIONS

Some basic definitions can be found in [5,6,7]. We assume < m = {1,2,...,m} and | S | denotes the number of elements in S which is the subset of < m > . And A = (a ij ) ∈ C<oo> is partitioned as follows:

\[ A = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ A_2 & A_2 & \cdots & A_m \\ \vdots & \vdots & \ddots & \vdots \\ A_m & A_m & \cdots & A_m \end{pmatrix} \]  

(1)

Let \( \mathcal{R}(A) = \sum_{j=1}^{m} \| A_j^{-1} \|_2 \) , \( \mathcal{C}(A) = \sum_{j=1}^{m} \| A_j^{-1} \|_2 \) , and 
\[ \mathcal{C}_s(A) = \max_{j \neq i} \sum_{j=1}^{m} \| A_j^{-1} \|_2 \| A_j \|_2 , i \in < m > . \]

III. MAIN RESULTS

Theorem 1. Let A = (a ij ) ∈ C<oo> be partitioned as (1), in which A i is nonsingular for i ∈ < m > . If there exists k ∈ < m > such that

\[ |S_k| = k \text{ implies } \sum_{i \in S_k} (1 - \mathcal{R}(A)) > 0 \text{ for all } S_k \subseteq \subset m > \]  

and

\[ |S| > \min \left\{ \sum_{i \in S} \mathcal{R}(A), \sum_{i \in S} \mathcal{C}(A) \right\} \text{ for all } S \subseteq \subset m > . \]  

Then A is nonsingular.

Proof. Assume the hypotheses of the theorem, and A is singular. Then there exists a nonzero vector Z such that AZ = 0 , in which the partitioned vector of Z is uniform with A. Without loss of generality, we assume that \( X_j = \| Z_j \| \) for \( j \in < m > \) and

\[ X_1 \geq X_2 \geq \cdots \geq X_m \geq 0 . \]  

(1.3)

Let

\[ P = \begin{pmatrix} A_1^{-1} & A_2^{-1} & \cdots & A_m^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_m^{-1} & A_m^{-1} & \cdots & A_m^{-1} \end{pmatrix} . \]

Then (PA)Z = 0 and \( Z_i = -\sum_{j \neq i} A_i^{-1} A_j Z_j \) for \( i \in < m > \).

Further,

\[ \| Z \| \leq \sum_{j \neq i} \| A_j^{-1} \| \| A_j \| \| Z \| \text{ for } i \in < m > . \]

(1.4)

Define

\[ \mu(P) h = \begin{pmatrix} 1 & -\| A_1 \| \| A_1 \| & \cdots & -\| A_m \| \| A_1 \| \\ \vdots & \ddots & \vdots & \vdots \\ -\| A_1 \| \| A_1 \| & \cdots & -\| A_1 \| \| A_m \| & 1 \end{pmatrix} = K(k) . \]

(1.5)

Then (1.4) becomes

\[ \mu(P) h X \leq 0 , \]

where \( X = (X_1, X_2, \cdots, X_m) \in \subset \subset R > \setminus \{0\} \). Our aim is to show that the hypotheses of the theorem contradict (1.5). To do that, it is sufficient to show that, for some \( l \in < m > , \)

\[ \sum_{j \neq l} k_j X_j > 0 . \]

(1.6)
Let \( y_i = X_1 - X_2, y_2 = X_2 - X_3, \ldots, y_{m-1} = X_{m-1} - X_m, y_m = X_m \).
By (1.3),
\[
y \geq 0 \quad \text{(and, since \( Z \neq 0, y \neq 0 \)).} \quad (1.7)
\]
Define \( B(r,s) = \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} k_{ij} \). Then (1.6) becomes
\[
\sum B(i,j)y_j > 0. \quad (1.8)
\]
By (1.1), there exists \( l \leq k \) such that
\[
B(r,m) \leq 0 \quad \text{for \( r < l \),} \quad (1.9)
\]
But
\[
B(l,m) > 0. \quad (1.10)
\]
From (1.2) and (1.9), we infer \( B(l,j) > 0 \) for \( j = 1,2,\ldots,l-1 \). And from (1.10), we infer \( B(l, j) > 0 \) for \( j = l, l+1, \ldots, m \).
In fact, if \( l = 1 \), then \( B(1,j) \geq B(1,2) \cdots \geq B(1,m) > 0 \).
If \( l > 1 \), by (1.10), then \( B(l,m) = \sum_{i=1}^{l-1} (1 - \check{R}_i(A)) > 0 \).
Further, \( B(l,j) \geq B(l+1,j) \cdots \geq B(l,m) > 0 \).
Since
\[
B(l,l-1) = \sum_{i=1}^{l-1} \sum_{j=1}^{l-1} k_{ij} = \sum_{i=1}^{l-1} k_{ij} = \sum_{i=1}^{l-1} \sum_{j=1}^{l-1} k_{ij}
\]
\[
= \sum_{j=1}^{l-1} \sum_{i=1}^{l-1} k_{ij} \geq (1 - \check{C}_j) (A), \quad (1.11)
\]
By (1.9), then \( B(l-1,m) = \sum_{i=1}^{l-1} \sum_{j=1}^{l-1} k_{ij} \geq (1 - \check{R}_j(A)) \leq 0 \).
From (1.2), then \( \sum_{i=1}^{l-1} (1 - \check{C}_j) (A) > 0 \). Further,
\[
B(l, l-1) > 0. \quad \text{Hence}
\]
\[
B(l,l-2) > 0 \cdots B(l,2) > 0, B(l,1) > 0. \quad (1.12)
\]
Further, (1.11), (1.12) and (1.7) imply (1.8), therefore (1.6), contradicting (1.5). Hence, \( A \) is nonsingular.
If we assume that
\[
1 < \tilde{R}_j (A) \quad \text{for at least one \( j \),} \quad (2.1)
\]
then Theorem 1 can be improved.

**Theorem 2.** Let \( A = (a_{ij}) \in C^{m \times m} \) be partitioned as (1) and satisfy (2.1), in which \( A_{ij} \) is nonsingular for \( i < m \).
If there exists \( k \in < m \) such that
\[
|S_k| = k \quad \text{implies} \quad \sum_{S_k} (1 - \tilde{R}_j (A)) \geq 0 \quad \text{for all} \quad S_k \subseteq < m \quad \text{and} \quad (1.6) \quad \text{holds. Then} \quad A \quad \text{is nonsingular.} \quad (2.2)
\]

**Proof.** We will call the index \( t \) a skip if \( y_t > 0 \), and \( SK \) the set of skips (which we know to be nonempty).
Our object is to show that, for suitable \( t \in < m \),
\[
B(l,t) \geq 0 \quad \text{for all} \quad t \in SK, \quad (2.3)
\]
and
\[
B(l,t) > 0 \quad \text{for at least one} \quad t \in SK. \quad (2.4)
\]
This will prove (1.8) and establish a contradiction of the supposition that \( A \) is singular.
Observe first that, by (2.2),
\[
B(r,m) > 0 \quad \text{if} \quad r \geq k. \quad (2.5)
\]
Next, let \( r \) be the smallest index in \( SK \). Firstly, we show the case \( r \geq k \).
If \( B(r,m) > 0 \), then
\[
B(r,r) \geq B(r,r+1) \cdots \geq B(r,m) > 0. \quad (2.6)
\]
Let \( l = r \), then \( B(l,t) = B(r,t) > 0 \) for all \( t \in SK \).
Hence (2.3) and (2.4) hold, and we are done. So suppose \( B(r,m) = 0 \).
Because \( r \) is the first skip, we know
\[
X_i = X_{i+1} = \cdots = X_m = \max \{ X_j \}. \quad (2.7)
\]
Hence \( 1 \leq R(A) \) for all \( i < r \). Since (2.1) and \( r \geq k \), there exists \( S_k \) such that
\[
\sum_{S_k} (1 - R(A)) < 0, \quad \text{contradicting} \quad (2.2).
\]
Secondly we show the case \( r < k \). Let \( l \) be the smallest skip such that \( B(l,m) \) is nonnegative.
If there exists \( l \), by (2.2), then \( l < k \). Further \( t \leq l \) and
\[
B(t,m) \leq 0 \quad \text{for} \quad t \in SK. \quad \text{So} \quad t \leq \sum_{i=1}^{l} R_i(A). \quad (2.8)
\]
By (1.2), then
\[
B(t,l) = \sum_{i=1}^{l} \sum_{j=1}^{l} k_{ij} = \sum_{j=1}^{l} \sum_{i=1}^{l} k_{ij} \geq \sum_{i=1}^{l} \sum_{j=1}^{l} (1 - \tilde{C}_i (A)) > 0.
\]
Hence (2.4) holds. Since
\[
B(l,l) \geq B(l,l+1) \cdots \geq B(l,m) \geq 0, \quad (2.3)
\]
holds. If \( l \) does not exist, then \( B(t,m) < 0 \) for all \( t \leq k \) and \( t \in SK \). Further \( t < \sum_{i=1}^{l} R_i(A) \). Let \( l = k \), by (2.2),
then \( B(k,m) \geq 0 \). Hence \( B(k,t) \geq B(k,m) \geq 0 \). For all \( t \geq k \) and \( t \in SK \), by (2.1), then
\[
B(k,t) = \sum_{i=1}^{l} \sum_{j=1}^{l} k_{ij} = \sum_{j=1}^{l} \sum_{i=1}^{l} k_{ij} \geq \sum_{i=1}^{l} \sum_{j=1}^{l} (1 - \tilde{C}_i (A)) > 0.
\]
Hence (2.3) and (2.4) hold. The theorem is true.

**IV. APPLICATIONS**

**Corollary 3.** Let \( A = (a_{ij}) \in C^{m \times m} \) be partitioned as (1), in which \( A_{ij} \) is nonsingular for \( i < m \). If \( A \) and \( k \) satisfy (1.2) or satisfy (2.1) and (2.2), and for some \( \alpha, 0 < \alpha < 1, \)
Corollary 4. Let $A = (a_{ij}) \in C^{m \times m}$ be partitioned as (1), in which $A_{ij}$ is nonsingular $M$-matrix for $i \in < m >$. Take the vector norm to be Euclidean norm. If $A$ and $k$ satisfy (1.1) and (1.2), then $A$ is positive stable matrix.

Proof. For any $\lambda \in \sigma(A)$, there exists a nonzero $X = (X_{1}^T, X_{2}^T, \cdots , X_{m}^T)$ such that $AX = \lambda X$, in which the partitioned vector of $X$ is uniform with $A$.

If $\text{Re } \lambda \leq 0$, then $\lambda \in \bigcup_{j = 1}^{m} \sigma(A_{ij})$ (see [6]). Further,

$$\left\| (A_{ij} - \lambda I_{A_{ij}})^{-1} \right\| \leq \left\| A_{ij}^{-1} \right\| \left\| y \right\|$$

for all $y \in C^{n} \setminus \{0\}$ (see [6]).

Hence $\left\| (A_{ij} - \lambda I_{A_{ij}})^{-1} \right\| \leq \left\| A_{ij}^{-1} \right\|$. For $A_{ij} = A - \lambda I$, then $\tilde{R}(A) \geq \tilde{R}(A_{ij}), \tilde{C}_{ik}(A) \geq \tilde{C}_{ik}(A_{ij})$. Therefore,

$$\sum_{i \in S_{k}} (1 - \tilde{R}(A_{ij})) \geq \sum_{i \in S_{k}} (1 - \tilde{R}(A)) > 0.$$

$|S| > \min \{ \sum_{i \in S} \tilde{R}(A), \sum_{i \in S} \tilde{C}_{ik}(A) \} \geq \min \{ \sum_{i \in S} \tilde{R}(A_{ij}), \sum_{i \in S} \tilde{C}_{ik}(A_{ij}) \}$. By Theorem 1, $A_{ij} = A - \lambda I$ is nonsingular, contradicting $A_{ij}$ to be singular. Hence the theorem is true.

Similarly, we can get the following corollary.

Corollary 5. Let $A = (a_{ij}) \in C^{m \times m}$ be partitioned as (1), in which $A_{ij}$ is nonsingular $M$-matrix for $i \in < m >$. Take the vector norm to be Euclidean norm. If $A$ and $k$ satisfy (1.6), (2.1) and (2.2), then $A$ is the positive stable matrix.

Clearly, the above results have generalized the corresponding results of [1]-[4].

ACKNOWLEDGMENT

We thank the reviewers for their valuable comments and suggestions on this paper.

REFERENCES