Criteria for nonsingular H -matrices

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Abstract—Nonsingular H – matrices play a very important role in matrix analysis and numerical algebra. Based on the concept and properties of the generalized strictly diagonal dominance matrices, we partition the row and column index set of square matrix, construct a positive diagonal matrix according to the elements and row sum, column sum of the matrix as well, then obtain several new criteria for the nonsingular H – matrices, and extend the criteria of nonsingular H – matrices. Finally, advantages are illustrated by a numerical example.

Keywords- H – matrices, α – diagonally dominant matrices, strictly diagonally dominant matrix.

I. INTRODUCTION

Nonsingular H – matrices are very important matrices. It is widely used in matrix theory, mathematical physics, control theory, mathematical economics, etc. (e.g., see [1, 2]). Some sufficient conditions for H – matrices are investigated in some papers, (see [3-10]), but it is difficult to determine H – matrices in practice. In this paper, we obtain a set of practical criteria of H – matrices. In the end, the advantages of the criteria of nonsingular H – matrices are illustrated by a numerical example.

It is well known that A is a nonsingular H – matrix if and only if A is a generalized strictly diagonally dominant matrix. In this paper, we always assume that all diagonal entries of A are nonzero. Let $N = \{1, 2, \dots, n\}$ and $C^{n \times n}$ be the set of all n by n complex matrices. Let $A = (a_{ij}) \in C^{n \times n}$ and $R_i(A) = \sum_{j \in N} |a_{ij}|$, $C_i(A) = \sum_{j \in N} |a_{ji}|$, $\forall i, j \in N$.

Definition 1. Let $A = (a_{ij}) \in C^{n \times n}$. If $|a_{ii}| > R_i(A)$, $\forall i \in N$, then A is said strictly diagonally dominant matrix and denoted $A \in D$. If there exists a positive diagonal matrix X such that $AX \in D$, then A is said a generalized strictly diagonally dominant matrix (i.e. nonsingular H – matrix) and denoted $A \in D^*$.

Definition 2. Let $A = (a_{ij}) \in C^{n \times n}$. If $\exists \alpha \in [0, 1]$, such that $|a_{ii}| > R_i^\alpha(A) C_i^{1-\alpha}(A)$, $\forall i \in N$, then A is said a strictly α – diagonally dominant matrix and denoted $A \in D(\alpha)$. If there exists a positive diagonal matrix X such that $AX \in D(\alpha)$, then A is said a generalized

strictly α – diagonally dominant matrix and denoted $A \in D^*(\alpha)$.

Definition 3. Let $A = (a_{ij}) \in C^{n \times n}$. If $\exists \alpha \in [0, 1]$, such that $|a_{ii}| \geq R_i^\alpha(A) C_i^{1-\alpha}(A)$, $\forall i \in N$, at least one strict inequality holds and for every i with $|a_{ii}| = R_i^\alpha(A) C_i^{1-\alpha}(A)$, there exists a nonzero elements chain $a_{i j_1} a_{j_1 j_2} \dots a_{j_{k-1} j_k} \neq 0$ such that $|a_{j_k j_k}| > R_{j_k}^\alpha(A) C_{j_k}^{1-\alpha}(A)$. Then A is said an α – diagonally dominant matrix with nonzero elements chain.

Definition 4. Let $A = (a_{ij}) \in C^{n \times n}$ be irreducible. If $|a_{ii}| \geq R_i^\alpha(A) C_i^{1-\alpha}(A)$, $\forall i \in N$ and a strict inequality holds for at least one $i \in N$, then A is said an irreducibly α – diagonally dominant matrix.

Lemma 1^[4]. Let $A = (a_{ij}) \in C^{n \times n}$. Then A is a generalized strictly α – diagonally dominant matrix if and only if $A \in D^*$.

Lemma 2^[5]. Let $A = (a_{ij}) \in C^{n \times n}$. If A is an irreducible α – diagonally dominant matrix, or A is an α – diagonally dominant matrix with nonzero elements chain, then A is a nonsingular H – matrix.

For $\alpha \in [0, 1]$, we define

$$\begin{aligned} N_1 &= \{i \in N : 0 < |a_{ii}| = R_i^\alpha(A) C_i^{1-\alpha}(A)\}, \\ N_2 &= \{i \in N : 0 < |a_{ii}| < R_i^\alpha(A) C_i^{1-\alpha}(A)\}, \\ N_3 &= \{i \in N : |a_{ii}| > R_i^\alpha(A) C_i^{1-\alpha}(A) \geq 0\}. \end{aligned} \quad (1)$$

Obviously $N = N_1 \oplus N_2 \oplus N_3$. Denote

$$x_i = \frac{R_i^\alpha(A) C_i^{1-\alpha}(A)}{|a_{ii}|}, i \in N.$$

II. MAIN RESULTS

Theorem 1. Let $A = (a_{ij}) \in C^{n \times n}$. If $\exists \alpha \in [0, 1]$, such that

$$\begin{aligned} |a_{ii}| &> \left[\frac{x_i}{x_i - 1} \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| \right) \right. \\ &\quad \left. + \sum_{j \in N_3} x_j |a_{ij}| \right]^\alpha C_i^{1-\alpha}(A), i \in N_2, \end{aligned} \quad (2)$$

then A is a nonsingular H – matrix.

Proof. By (1), we know that

$$0 < \frac{x_i - 1}{x_i} < 1, i \in N_2, 0 < x_i < 1, i \in N_3.$$

Construct a positive diagonal matrix $X = \text{diag}(d_1, d_2, \dots, d_n)$, where

$$d_i = \begin{cases} 1, & i \in N_1 \\ \frac{x_i - 1}{x_i}, & i \in N_2 \\ x_i, & i \in N_3 \end{cases}.$$

Let $B = AX = (b_{ij}) \in C^{n \times n}$. Next we will prove $B \in D(\alpha)$.

For $\forall i \in N_1$,

$$\begin{aligned} R_i^\alpha(B)C_i^{1-\alpha}(B) &= \left(\sum_{j \in N_1, j \neq i} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3} x_j |a_{ij}| \right)^\alpha C_i^{1-\alpha}(A) \\ &< \left(\sum_{j \in N_1, j \neq i} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| + \sum_{j \in N_3} |a_{ij}| \right)^\alpha C_i^{1-\alpha}(A) \\ &= R_i^\alpha(A)C_i^{1-\alpha}(A) \\ &= |a_{ii}| = |b_{ii}|. \end{aligned}$$

For $\forall i \in N_2$, from (2), we have

$$\begin{aligned} R_i^\alpha(B)C_i^{1-\alpha}(B) &= \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| \right. \\ &\quad \left. + \sum_{j \in N_3} x_j |a_{ij}| \right)^\alpha \left(\sum_{j \in N_1, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ji}| \right)^{1-\alpha} \\ &= \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| \right. \\ &\quad \left. + \sum_{j \in N_3} x_j |a_{ij}| \right)^\alpha \left(\sum_{j \in N_1, j \neq i} |a_{ji}| \right)^{1-\alpha} \left(\frac{x_i - 1}{x_i} \right)^{1-\alpha} \\ &< |a_{ii}| \left(\frac{x_i - 1}{x_i} \right) \\ &= |b_{ii}|. \end{aligned}$$

For $\forall i \in N_3$,

$$\begin{aligned} R_i(A) &= \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| + \sum_{j \in N_3, j \neq i} |a_{ij}| \\ &\geq \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3, j \neq i} x_j |a_{ij}|. \end{aligned}$$

Obviously, for $\forall i \in N_3$,

$$R_i^\alpha(A) \geq \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha.$$

So, for $\forall i \in N_3$, we have

$$|b_{ii}| - R_i^\alpha(B)C_i^{1-\alpha}(B)$$

$$\begin{aligned} &= R_i^\alpha(A)C_i^{1-\alpha}(A) - \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| \right. \\ &\quad \left. + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha \left(\sum_{j \in N_1, j \neq i} |a_{ji}| \right)^{1-\alpha} \\ &= R_i^\alpha(A)C_i^{1-\alpha}(A) - x_i^{1-\alpha} \left(\sum_{j \in N_1} |a_{ij}| \right. \\ &\quad \left. + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha C_i^{1-\alpha}(A) \\ &> C_i^{1-\alpha}(A) [R_i^\alpha(A) - \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| \right. \\ &\quad \left. + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha] \\ &\geq 0. \end{aligned}$$

In a word, we have

$$|b_{ii}| > R_i^\alpha(B)C_i^{1-\alpha}(B), \forall i \in N.$$

By Definition 2, $B = AX \in D(\alpha)$, i.e. $A \in D^*(\alpha)$.

Based on Lemma 1, then A is a nonsingular H -matrix.

Denote

$$\begin{aligned} N(A) &= \{i \in N_2 : |a_{ii}| = \left[\frac{x_i}{x_i - 1} \left(\sum_{j \in N_1} |a_{ij}| \right. \right. \\ &\quad \left. \left. + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3} x_j |a_{ij}| \right) \right]^\alpha C_i^{1-\alpha}(A)\}. \end{aligned} \quad (3)$$

Theorem 2. Let $A = (a_{ij}) \in C^{n \times n}$, if $\exists \alpha \in [0, 1]$, such that

$$\begin{aligned} |a_{ii}| &\geq \left[\frac{x_i}{x_i - 1} \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| \right. \right. \\ &\quad \left. \left. + \sum_{j \in N_3} x_j |a_{ij}| \right) \right]^\alpha C_i^{1-\alpha}(A), \forall i \in N_2, \end{aligned}$$

and A satisfies one of the following conditions,

- 1) A is irreducible,
- 2) For $\forall i \in N(A) \neq \emptyset$, there exists a nonzero elements chain $a_{i j_1} a_{j_1 j_2} \cdots a_{j_{k-1} j_k} \neq 0$ where $j_k \in N_2 \setminus N(A) \neq \emptyset$, then

A is a nonsingular H -matrix.

Proof. By (1), we know that

$$0 < \frac{x_i - 1}{x_i} < 1, i \in N_2, 0 < x_i < 1, i \in N_3.$$

Construct a positive diagonal matrix $X = \text{diag}(d_1, d_2, \dots, d_n)$, where

$$d_i = \begin{cases} 1, & i \in N_1 \\ \frac{x_i - 1}{x_i}, & i \in N_2 \\ x_i, & i \in N_3 \end{cases}.$$

Let $B = AX = (b_{ij}) \in C^{n \times n}$, then

For $\forall i \in N_1$,

$$\begin{aligned}
 & R_i^\alpha(B)C_i^{1-\alpha}(B) \\
 &= \left(\sum_{j \in N_1, j \neq i} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3} x_j |a_{ij}| \right)^\alpha C_i^{1-\alpha}(A) \\
 &< \left(\sum_{j \in N_1, j \neq i} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| + \sum_{j \in N_3} |a_{ij}| \right)^\alpha C_i^{1-\alpha}(A) \\
 &= R_i^\alpha(A)C_i^{1-\alpha}(A) \\
 &= |a_{ii}| = |b_{ii}|.
 \end{aligned}$$

For $\forall i \in N(A) \neq \emptyset$, from (3), we have

$$\begin{aligned}
 & R_i^\alpha(B)C_i^{1-\alpha}(B) \\
 &= \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3} x_j |a_{ij}| \right)^\alpha \\
 &\quad \times \left(\sum_{j \in N, j \neq i} \left(\frac{x_i - 1}{x_i} \right) |a_{ji}| \right)^{1-\alpha} \\
 &= \frac{x_i - 1}{x_i} \left[\left(\frac{x_i}{x_i - 1} \right) \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3} x_j |a_{ij}| \right)^\alpha C_i^{1-\alpha}(A) \right. \\
 &\quad \left. + \sum_{j \in N_3} x_j |a_{ij}| \right]^\alpha C_i^{1-\alpha}(A) \\
 &= \frac{x_i - 1}{x_i} |a_{ii}| \\
 &= |b_{ii}|.
 \end{aligned}$$

For $\forall i \in N_2 \setminus N(A) \neq \emptyset$, similar to the above proof, then

$$\begin{aligned}
 & R_i^\alpha(B)C_i^{1-\alpha}(B) \\
 &= \frac{x_i - 1}{x_i} \left[\frac{x_i}{x_i - 1} \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2, j \neq i} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3} x_j |a_{ij}| \right)^\alpha C_i^{1-\alpha}(A) \right. \\
 &\quad \left. + \sum_{j \in N_3} x_j |a_{ij}| \right]^\alpha C_i^{1-\alpha}(A) \\
 &< \frac{x_i - 1}{x_i} |a_{ii}| \\
 &= |b_{ii}|.
 \end{aligned}$$

For $\forall i \in N_3$, obviously

$$R_i^\alpha(A) \geq \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha,$$

and

$$\begin{aligned}
 & |b_{ii}| - R_i^\alpha(B)C_i^{1-\alpha}(B) \\
 &= R_i^\alpha(A)C_i^{1-\alpha}(A) - \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha \left(\sum_{j \in N, j \neq i} \left(\frac{x_i - 1}{x_i} \right) |a_{ji}| \right)^{1-\alpha}
 \end{aligned}$$

$$\begin{aligned}
 &= C_i^{1-\alpha}(A) [R_i^\alpha(A) - x_i^{1-\alpha} \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha] \\
 &> C_i^{1-\alpha}(A) [R_i^\alpha(A) - \left(\sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} \left(\frac{x_j - 1}{x_j} \right) |a_{ij}| + \sum_{j \in N_3, j \neq i} x_j |a_{ij}| \right)^\alpha] \\
 &\geq 0.
 \end{aligned}$$

In a word,

$$|b_{ii}| > R_i^\alpha(B)C_i^{1-\alpha}(B), \quad i \in N.$$

When A satisfies condition 1), we know A is an irreducible matrix and so is $B = AX$, then B is an irreducible α -diagonal matrix. When A satisfies condition 2), $B = AX$ is a α -diagonal matrix with nonzero elements chain. Based on Lemma 1 and Lemma 2, we obtain that B is a nonsingular H -matrix, then A is a nonsingular H -matrix.

III. AN EXAMPLE

Let

$$A = \begin{pmatrix} 10 & 1 & 0 & 1 & 0 \\ 1 & 4 & 1 & 6 & 0 \\ 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & 0.5 & 25 & 3.5 \\ 0 & 2 & 0 & 4 & 8 \end{pmatrix},$$

Taking $\alpha = \frac{4}{5}$, then

$$\begin{aligned}
 & R_1(A) = 2, R_2(A) = 8, R_3(A) = 6, R_4(A) = 4, R_5(A) = 6, \\
 & C_1(A) = 1, C_2(A) = 4, C_3(A) = 1.5, C_4(A) = 16, C_5(A) = 3.5, \\
 & N_1 = \emptyset, N_2 = \{2, 3\}, N_3 = \{1, 4, 5\}, \\
 & x_1 = 0.17411, x_2 = 1.74122, x_3 = 1.136788, \\
 & x_4 = 0.211122, x_5 = 0.673356.
 \end{aligned}$$

Since

$$\begin{aligned}
 & |a_{22}| = 4 \\
 &> \left[\frac{x_2}{x_2 - 1} \left(\frac{x_3 - 1}{x_3} |a_{23}| + x_1 |a_{21}| + x_4 |a_{24}| + x_5 |a_{25}| \right)^\alpha C_2^{1-\alpha} \right] \\
 &= [2.349127 \times (0.120329 \times 1 + 0.17411 \times 1 + 0.211122 \times 6 + 0.673356 \times 0)]^{\frac{4}{5}} \times 4^{\frac{1}{5}} \\
 &\approx 3.9013,
 \end{aligned}$$

we know that A satisfies the conditions of the Theorem 1, so A is a nonsingular H -matrix. But A does not satisfy the corresponding conditions of the theorem in [4].

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