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q,k-Generalized Gamma and Beta Functions

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Abstract

We introduce the q,k-generalized Pochhammer symbol. We construct $\Gamma_{q,k}$ and $B_{q,k}$, the q,k-generalized gamma and beta functions, and show that they satisfy properties that generalize those satisfied by the classical gamma and beta functions. Moreover, we provide integral representations for $\Gamma_{q,k}$ and $B_{q,k}$.

1 Introduction

The usefulness of the gamma and beta functions can hardly be overstated. However, recent results coming from the combinatorics of annihilation and creation operators [7], and the construction of hypergeometric functions from the point of view of the k-generalized Pochhammer symbol given in [6], have impel upon us the need to introduce subtle, yet deep, generalizations of the all mighty gamma and beta functions. The main goal of this paper is to introduce a two parameter deformation of the classical gamma and beta functions, which we call the q,k-generalized gamma and beta functions and will be denoted by $\Gamma_{q,k}$ and $B_{q,k}$, respectively. $\Gamma_{q,k}$ and $B_{q,k}$ fit into the following commutative diagrams

$$\begin{array}{ccc} \Gamma_{q,k}(t) & \xrightarrow{q\to 1} & \Gamma_k(t) \\ {\scriptstyle k\to 1}\downarrow & & \downarrow{\scriptstyle k\to 1} \\ \Gamma_q(t) & \xrightarrow[q\to 1]{} & \Gamma(t) \end{array} \qquad \begin{array}{ccc} B_{q,k}(t,s) & \xrightarrow{q\to 1} & B_k(t,s) \\ {\scriptstyle k\to 1}\downarrow & & \downarrow{\scriptstyle k\to 1} \\ B_q(t,s) & \xrightarrow[q\to 1]{} & B(t,s) \end{array}$$

Let us explain the notation used in the diagrams above. Recall that the Euler's gamma and beta functions are given by the following Riemann integrals:

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \quad t > 0.$$

$$B(t,s) = \int_0^1 x^{t-1}(1-x)^{s-1}dx = \int_0^\infty \frac{x^{t-1}}{(1+x)^{t+s}}dx, \quad t,s > 0.$$

Our motivation to introduce the q,k-generalized gamma and beta functions is two-fold: on one hand, Díaz and Pariguan in [6] defined a k-deformation, $k > 0$ a real number, of the gamma and beta functions given by the following Riemann integrals

$$\Gamma_k(t) = \int_0^\infty x^{t-1} e^{-\frac{x^k}{k}} dx, \quad t > 0.$$

$$B_k(t,s) = \int_0^\infty x^{t-1}(1+x^k)^{-\frac{t+s}{k}} dx, \quad t,s > 0.$$

On the other hand, De Sole and Kac in [2] introduced a q-deformation, $0 < q < 1$ a real number, of the gamma and beta functions given by the following Jackson integrals

$$\Gamma_q(t) = \int_0^{\infty/(1-q)} x^{t-1} E^{-qx} d_q x, \quad t > 0. \tag{1.1}$$

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x, \quad t,s > 0. \tag{1.2}$$

Although equation (1.2) has been known for about a century, the upper limit in formula (1.1) has only been recently established by Koelink and Koornwinder [8], [9]: the factor $\frac{1}{(1-q)}$ was traditionally omitted yielding a divergent Jackson integral.

Our two-parameter deformations $\Gamma_{q,k}$ and $B_{q,k}$ generalize both constructions above. We show that our function $\Gamma_{q,k}$ is related to the q,k-generalized Pochhammer symbol $[t]_{n,k}$, to be defined in Section 2, in the same way as the classical Γ function and the Pochhammer symbol are related to each other. Our functions $\Gamma_{q,k}$ and $B_{q,k}$ are given by the following formulae

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}, \quad t > 0.$$

$$B_{q,k}(t,s) = \frac{(1-q)(1-q^k)_{q,k}^{\frac{s}{k}-1}}{(1-q^t)_{q,k}^{\frac{s}{k}}}, \qquad \text{for all } s,t > 0.$$

In Section 4, we give Jackson integral representations for the q,k-generalized gamma and beta functions in terms of the q,k-analogue of the exponential function $E^x_{q,k}$. These are given by

$$\Gamma_{q,k}(t) = \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} x^{t-1} E_{q,k}^{-\frac{q^k x^k}{[k]_q}} d_q x, \quad t > 0.$$

$$B_{q,k}(t,s) = [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1} \left(1 - q^k \frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}-1} d_q x, \quad t,s > 0.$$

Furthermore, in Section 5, we give other integral representations for $\Gamma_{q,k}$ and $B_{q,k}$, using the q,k-analogue of the exponential function $e^x_{q,k}$. These integral representations are given by

$$\Gamma_{q,k}(t) = c(a,t)\int_0^{\infty/a(1-q^k)^{\frac{1}{k}}} x^{t-1} e_{q,k}^{-\frac{x^k}{[k]_q}} d_qx, \quad t>0. \tag{1.3}$$

$$B_{q,k}(t,s) = c(a,t)[k]_q^{-\frac{t}{k}} \int_0^{\infty/a} \frac{x^{t-1}}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}} d_qx, \quad t,s>0, \tag{1.4}$$

where $c(a,t) = \dfrac{a^t[k]_q^{\frac{t}{k}}}{1+[k]_q a^k}\left(1+\dfrac{1}{[k]_q a^k}\right)_{q,k}^{\frac{t}{k}}\left(1+[k]_q a^k\right)_{q,k}^{1-\frac{t}{k}}$.

We shall see that formulae (1.3) and (1.4) are deeply related to the classical Jacobi triple product identity and to the famous Ramanujan identity, respectively. We remark that all our results are elementary and require little knowledge beyond a basic introduction to q-calculus.

2 Basic results

In this Section we introduce the q,k-generalized Pochhammer symbol and also some basic definitions that will be used in the rest of the paper. For completeness we review well-known material that might be found, for example, in [1], [3], [4] and [5]. Let us begin by introducing q-derivatives and Jackson integrals, see [10],[11].

Definition 1. Let us denote by $\mathrm{Func}(\mathbb{R},\mathbb{R})$ the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Fix $0<q<1$ and consider the linear operators $I_q, d_q, \partial_q : \mathrm{Func}(\mathbb{R},\mathbb{R}) \longrightarrow \mathrm{Func}(\mathbb{R},\mathbb{R})$, given for all $f \in \mathrm{Func}(\mathbb{R},\mathbb{R})$ by:

- $I_q(f)(x) = f(qx), \quad$ for all $x \in \mathbb{R}$.
- $d_q(f) = I_q(f) - f$.
- $\partial_q(f) = \dfrac{d_q f}{d_q x} = \dfrac{I_q(f)-f}{(q-1)x}$. $\quad \partial_q(f)$ is called the q-derivative of the function f.

Definition 2. 1. The definite q-integral of a function $f \in \mathrm{Func}(\mathbb{R},\mathbb{R})$ from 0 to $b>0$ is given by

$$\int_0^b f(x)d_qx = (1-q)b\sum_{n=0}^{\infty} q^n f(q^n b).$$

2. The improper q-integral of a function $f \in \mathrm{Func}(\mathbb{R},\mathbb{R})$ is given by

$$\int_0^{\infty/a} f(x)d_qx = (1-q)\sum_{n\in\mathbb{Z}} \frac{q^n}{a} f\left(\frac{q^n}{a}\right).$$

Proposition 1. *For any functions $f, g \in \mathrm{Func}(\mathbb{R},\mathbb{R})$, the following properties hold:*

- $\partial_q(f+g) = \partial_q(f) + \partial_q(g)$.
- $\partial_q(fg) = f\partial_q(g) + I_q(g)\partial_q(f)$.
- $\partial_q(f/g) = \dfrac{\partial_q(f)g - f\partial_q(g)}{I_q(g)(g)}$.
- $\partial_q(f(ax^b)) = a[b]_q(\partial_{q^b}f)(ax^b)x^{b-1}$, *for all* $a, b \in \mathbb{R}$.
- $f(b)g(b) - f(a)g(a) = \displaystyle\int_a^b f\partial_q(g)d_qx + \int_a^b I_q(g)\partial_q(f)d_qx$, *for all* $0 \leq a < b \leq +\infty$.

Definition 3. Let $0 < q < 1$ be a fixed real number. Let us denote by $[\ ]_q : \mathbb{R} \to \mathbb{R}$ the map given by $[t]_q = \dfrac{(1-q^t)}{(1-q)}$, for all $t \in \mathbb{R}$.

The map $[\ ]_q : \mathbb{R} \to \mathbb{R}$ it is not an algebra homomorphism. Nevertheless, the following identities are satisfied:

1. $[s+t]_q = [s]_q + q^s[t]_q$, for all $s, t \in \mathbb{R}$.
2. $[st]_q = [s]_{q^t}[t]_q$, for all $s, t \in \mathbb{R}$.
3. $[1]_q = 1$.
4. $[0]_q = 0$.

Next definition is fundamental for the rest of the paper. Indeed, our original motivation for this work was to find integral representations for the q,k-generalized Pochhammer symbol.

Definition 4. 1. Let $t \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. The k-generalized Pochhammer symbol is given by

$$(t)_{n,k} = t(t+k)(t+2k)...(t+(n-1)k) = \prod_{j=0}^{n-1}(t+jk).$$

2. The q,k-generalized Pochhammer symbol is given by

$$[t]_{n,k} = [t]_q[t+k]_q[t+2k]_q...[t+(n-1)k]_q = \prod_{j=0}^{n-1}[t+jk]_q.$$

Notice that $[t]_{n,k} \to (t)_{n,k}$ as $q \to 1$. Let us introduce some notation that will be used throughout the paper.

Definition 5. Let $x, y, t \in \mathbb{R}$ and $n \in \mathbb{Z}^+$

1. $(x+y)^n_{q,k} := \displaystyle\prod_{j=0}^{n-1}(x+q^{jk}y)$.
2. $(1+x)^t_{q,k} := \dfrac{(1+x)^\infty_{q,k}}{(1+q^{kt}x)^\infty_{q,k}}$.

Lemma 1. *Let $x, s, t \in \mathbb{R}$. Then* $(1+x)^{s+t}_{q,k} = (1+x)^s_{q,k}(1+q^{ks}x)^t_{q,k}$.

3 Explicit formulae for $\Gamma_{q,k}$ and $B_{q,k}$

The k-generalization of the gamma function introduced by Díaz and Pariguan in [6], is univocally determined by the following properties:

1. $\Gamma_k(t+k) = t\Gamma_k(t), \quad t > 0.$ 2. $\Gamma_k(k) = 1.$ 3. Γ_k is logarithmically convex.

Properties 1 and 2 imply that $(t)_{n,k} = \dfrac{\Gamma_k(t+nk)}{\Gamma_k(t)}$, for all $t > 0$ and $n \in \mathbb{Z}^+$.

We define the q,k-generalized gamma function $\Gamma_{q,k}$ by demanding it satisfies the q,k-analogue of properties 1 and 2 above. Thus we assume that $\Gamma_{q,k}$ is such that: $\Gamma_{q,k}(t+k) = [t]_q\Gamma_{q,k}(t)$ and $\Gamma_{q,k}(k) = 1$. This implies that,

$$\Gamma_{q,k}(nk) = \prod_{j=1}^{n-1}[jk]_q = \prod_{j=1}^{n-1}\frac{(1-q^{jk})}{(1-q)} = \frac{(1-q^k)_{q,k}^{n-1}}{(1-q)^{n-1}}.$$

After the change of variable t=nk, one is lead to the following

Definition 6. The function $\Gamma_{q,k}$ is given by the formula

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}, \quad t > 0.$$

Lemma 2. *The infinite product expression for the function $\Gamma_{q,k}$ is given by*

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q)^{\frac{t}{k}-1}}, \quad t > 0.$$

Next proposition guarantees that $\Gamma_{q,k}$ indeed satisfies the q,k-analogue of properties 1 and 2 above.

Proposition 2. *The function $\Gamma_{q,k}$ satisfies the following identities for $t > 0$:*

1. $\Gamma_{q,k}(t+k) = [t]_q\Gamma_{q,k}(t)$.

2. $\Gamma_{q,k}(k) = 1$.

3. $\dfrac{\Gamma_{q,k}(t+nk)}{\Gamma_{q,k}(t)} = [t]_{n,k}$, *for all* $n \in \mathbb{Z}^+$.

Proof. 2. $\Gamma_{q,k}(t+k) = \dfrac{(1-q^t)}{(1-q)}\dfrac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} = [t]_q\Gamma_{q,k}(t)$. 1. Obvious.

3. $$\frac{\Gamma_{q,k}(t+nk)}{\Gamma_{q,k}(t)} = \frac{(1-q^t)_{q,k}^n}{(1-q)^n} = \prod_{j=0}^{n-1}\frac{(1-q^{t+jk})}{(1-q)} = \prod_{j=0}^{n-1}[t+jk]_q = [t]_{n,k}.$$ ■

Definition 7. The function $B_{q,k}(t,s)$ is given by the formula

$$B_{q,k}(t,s) = \frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)}, \quad \text{for all } s,t>0.$$

Which in turns imply the next

Lemma 3. *1.* $B_{q,k}(t,s) = \dfrac{(1-q)(1-q^k)_{q,k}^{\frac{s}{k}-1}}{(1-q^t)_{q,k}^{\frac{s}{k}}}$, *for all* $s,t>0$.

2. $B_{q,k}(t,s) = \dfrac{(1-q)(1-q^k)_{q,k}^{\infty}(1-q^{s+t})_{q,k}^{\infty}}{(1-q^s)_{q,k}^{\infty}(1-q^t)_{q,k}^{\infty}}$, *for all* $s,t>0$.

Proof. Use Definition 6 and Definition 7. ■

Proposition 3. *The function $B_{q,k}$ satisfies the following formulae for $s,t>0$*

1. $B_{q,k}(t,\infty) = (1-q)^{\frac{t}{k}}\Gamma_{q,k}(t)$.

2. $B_{q,k}(t+k,s) = \dfrac{[t]_q}{[s]_q}B_{q,k}(t,s+k)$.

3. $B_{q,k}(t,s+k) = B_{q,k}(t,s) - q^s B_{q,k}(t+k,s)$.

4. $B_{q,k}(t,s+k) = \dfrac{[s]_q}{[s+t]_q}B_{q,k}(t,s)$.

5. $B_{q,k}(t,k) = \dfrac{1}{[t]_q}$.

6. $B_{q,k}(t,nk) = (1-q)\dfrac{(1-q^k)_{q,k}^{n-1}}{(1-q^t)_{q,k}^{n}} = (1-q)\dfrac{(1-q^k)_{q,k}^{n-1}(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q^k)_{q,k}^{\frac{t}{k}+n-1}}$, $n\in\mathbb{Z}^+$.

Proof. 1. $B_{q,k}(t,\infty) = \dfrac{(1-q)(1-q^k)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}} = \dfrac{(1-q)(1-q^k)_{q,k}^{\infty}}{(1-q^{k(\frac{t-k}{k})}q^k)_{q,k}^{\infty}} = (1-q)(1-q^k)_{q,k}^{\frac{t-k}{k}}$.

Then $B_{q,k}(t,\infty) = (1-q)^{\frac{t}{k}}\Gamma_{q,k}(t)$.

2. $\dfrac{B_{q,k}(t+k,s)}{B_{q,k}(t,s+k)} = \dfrac{(1-q^k)_{q,k}^{\frac{s}{k}-1}(1-q^t)_{q,k}^{\frac{s}{k}+1}}{(1-q^k)_{q,k}^{\frac{s}{k}}(1-q^{t+k})_{q,k}^{\frac{s}{k}}} = \dfrac{(1-q^t)}{(1-q^s)} = \dfrac{[t]_q}{[s]_q}$.

3. $$\frac{B_{q,k}(t,s+k)-B_{q,k}(t,s)}{B_{q,k}(t+k,s)} = \frac{(1-q^{t+k})_{q,k}^{\frac{s}{k}}(1-q^k)_{q,k}^{\frac{s}{k}}}{(1-q^t)_{q,k}^{\frac{s}{k}+1}(1-q^k)_{q,k}^{\frac{s}{k}-1}} - \frac{(1-q^{t+k})_{q,k}^{\frac{s}{k}}}{(1-q^t)_{q,k}^{\frac{s}{k}}} = \frac{(1-q^s)}{(1-q^t)} - \frac{(1-q^{t+s})}{(1-q^t)} = -q^s.$$

4. $$\frac{B_{q,k}(t,s+k)}{B_{q,k}(t,s)} = \frac{(1-q^k)_{q,k}^{\frac{s}{k}}(1-q^t)_{q,k}^{\frac{s}{k}}}{(1-q^t)_{q,k}^{\frac{s}{k}+1}(1-q^k)_{q,k}^{\frac{s}{k}-1}} = \frac{(1-q^k)_{q,k}^{\frac{s}{k}}(1-q^t)_{q,k}^{\frac{s}{k}}}{(1-q^t)_{q,k}^{\frac{s}{k}}(1-q^{t+s})(1-q^k)_{q,k}^{\frac{s}{k}}(1-q^s)^{-1}} = \frac{(1-q^s)}{(1-q^{t+s})} = \frac{[t]_q}{[t+s]_q}.$$

5. $B_{q,k}(t,k) = \dfrac{(1-q)}{(1-q^t)} = \dfrac{1}{[t]_q}.$

6. Using items 4 and 5 of this

$$B_{q,k}(t,nk) = \frac{\prod_{j=1}^{n-1}[jk]_q}{\prod_{j=0}^{n-1}[jk+t]_q} = \frac{(1-q)(1-q^k)_{q,k}^{n-1}}{(1-q^t)_{q,k}^{n}} = (1-q)\frac{(1-q^k)_{q,k}^{n-1}(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q^k)_{q,k}^{\frac{t}{k}+n-1}}.$$

Letting $n\to\infty$, we get $B_{q,k}(t,\infty) = (1-q)(1-q^k)_{q,k}^{\frac{t}{k}-1}$.

■

4 Integral representations for $\Gamma_{q,k}$ and $B_{q,k}$

As promised in the introduction, in this Section we provide Jackson integral representations for our $\Gamma_{q,k}$ and $B_{q,k}$ functions, in terms of the q,k-analogue exponential function $E_{q,k}^x$. Recall that

$$E_{q,k}^x = \sum_{n=0}^{\infty}\frac{q^{kn(n-1)/2}x^n}{[n]_{q^k}!} = (1+(1-q^k)x)_{q,k}^{\infty}.$$

Theorem 1. *1. The function $\Gamma_{q,k}(t)$ is given by the following Jackson integral*

$$\Gamma_{q,k}(t) = \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} x^{t-1}E_{q,k}^{-\frac{q^k x^k}{[k]_q}}\,d_qx, \quad t>0. \tag{4.1}$$

2. *The function $B_{q,k}$ is given by the following Jackson integral*

$$B_{q,k}(t,s) = [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1} \left(1 - q^k \frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}-1} d_q x, \quad t,s > 0. \tag{4.2}$$

In order to prove Theorem 1 we begin by denoting the right hand side of (4.1) by $\overline{\Gamma}_{q,k}$, and the right hand side of (4.2) by $\overline{B}_{q,k}$. Let us check that $\overline{\Gamma}_{q,k}$ and $\overline{B}_{q,k}$ satisfy properties analogue to those stated in Proposition 2 and Proposition 3 for $\Gamma_{q,k}$ and $B_{q,k}$, respectively.

Proposition 4. *The function $\overline{\Gamma}_{q,k}$ satisfies the following identities for $t > 0$*

1. $\overline{\Gamma}_{q,k}(k) = 1$.

2. $\overline{\Gamma}_{q,k}(t+k) = [t]_q \overline{\Gamma}_{q,k}(t)$.

Proof. 1. $\overline{\Gamma}_{q,k}(k) = -\int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} \partial_q \left(E_{q,k}^{\frac{x^k}{[k]_q}}\right) d_q x = 1$, since $E_{q,k}^{-\frac{1}{1-q^k}} = 0$, and $E_{q,k}^{0} = 1$.

2. Using q-integration by parts

$$\begin{aligned}
\overline{\Gamma}_{q,k}(t+k) &= \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} x^{t+k-1} E_{q,k}^{\frac{-q^k x^k}{[k]_q}} d_q x \\
&= -\int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} x^t \partial_q \left(E_{q,k}^{-\frac{x^k}{[k]_q}}\right) d_q x \\
&= [t]_q \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} x^{t-1} E_{q,k}^{-\frac{q^k x^k}{[k]_q}} d_q x.
\end{aligned}$$

■

Proposition 5. *The function $\overline{B}_{q,k}$ satisfies the following formulae for $s, t > 0$*

1. $\overline{B}_{q,k}(t,\infty) = (1-q)^{\frac{t}{k}} \overline{\Gamma}_{q,k}(t)$.

2. $\overline{B}_{q,k}(t+k,s) = \dfrac{[t]_q}{[s]_q} \overline{B}_{q,k}(t,s+k)$.

3. $\overline{B}_{q,k}(t,s+k) = \overline{B}_{q,k}(t,s) - q^s \overline{B}_{q,k}(t+k,s)$.

4. $\overline{B}_{q,k}(t,s+k) = \dfrac{[s]_q}{[s+t]_q} \overline{B}_{q,k}(t,s)$.

5. $\overline{B}_{q,k}(t,k) = \dfrac{1}{[t]_q}$.

6. $\overline{B}_{q,k}(t,nk) = (1-q)\dfrac{(1-q^k)_{q,k}^{n-1}}{(1-q^t)_{q,k}^{n}} = (1-q)\dfrac{(1-q^k)_{q,k}^{n-1}(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q^k)_{q,k}^{\frac{t}{k}+n-1}}, \quad n \in \mathbb{Z}^+.$

Proof. 1. Using the change $x = (1-q^k)^{\frac{1}{k}} y$, (4.4) is obtained from (4.3).

$$\begin{aligned}
\overline{B}_{q,k}(t,\infty) &= [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1} E_{q,k}^{-\frac{q^k x^k}{(1-q^k)[k]_q}} d_q x && (4.3)\\
&= (1-q)^{\frac{t}{k}} \int_0^{\left(\frac{[k]_q}{(1-q^k)}\right)^{\frac{1}{k}}} y^{t-1} E_{q,k}^{-\frac{q^k y^k}{[k]_q}} d_q y && (4.4)\\
&= (1-q)^{\frac{t}{k}} \overline{\Gamma}_{q,k}(t).
\end{aligned}$$

2. Using the formula $\partial_q \left(1 + b\frac{x^k}{[k]_q}\right)_{q,k}^{t} = \frac{[kt]_q}{[k]_q} b x^{k-1} \left(1 + bq^k \frac{x^k}{[k]_q}\right)_{q,k}^{t-1}$ in (4.5) we have (4.6)

$$\begin{aligned}
\overline{B}_{q,k}(t+k,s) &= [k]_q^{-\frac{t}{k}-1} \int_0^{[k]_q^{\frac{1}{k}}} x^{t+k-1} \left(1 - q^k \frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}-1} d_q x && (4.5)\\
&= -\frac{[k]_q^{-\frac{t}{k}}}{[s]_q} \int_0^{[k]_q^{\frac{1}{k}}} x^t \partial_q \left(\left(1 - \frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}}\right) d_q x && (4.6)\\
&= \frac{[t]_q}{[s]_q} \overline{B}_{q,k}(t,s+k).
\end{aligned}$$

3. Using Lemma 1, we get

$$\begin{aligned}
\overline{B}_{q,k}(t,s+k) = {} & [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1} \left(1 - q^k \frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}-1}\\
& - q^s [k]_q^{-\frac{t}{k}-1} \int_0^{[k]_q^{\frac{1}{k}}} x^{t+k-1} \left(1 - q^k \frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}-1} d_q x\\
= {} & \overline{B}_{q,k}(t,s) - q^s \overline{B}_{q,k}(t+k,s).
\end{aligned}$$

4. It is easy to check that $\overline{B}_{q,k}(t,s+k) = \frac{[s]_q}{[s+t]_q} \overline{B}_{q,k}(t,s)$ using the properties 2 and 3 above.

5. $\overline{B}_{q,k}(t,k) = [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1} = \frac{1}{[t]_q}$.

6. Use items 4 and 5 of this proposition.

■

Proof. Proof of Theorem 1. By Proposition 3 part 1, $\overline{B}_{q,k}(t,\infty) = (1-q)^{\frac{t}{k}}\overline{\Gamma}_{q,k}(t),\ t > 0$. Also, it follows from Proposition 3 part 6 that $\overline{B}_{q,k}(t,\infty) = (1-q)^{\frac{t}{k}}\Gamma_{q,k}(t),\ t > 0$. Therefore,

$$\overline{\Gamma}_{q,k}(t) = \Gamma_{q,k}(t),\ t > 0.$$

Items 3 and 4 of Proposition 5 imply that

$$\overline{B}_{q,k}(t,s) = B_{q,k}(t,s),\ \text{for all } t > 0 \text{ and } s = nk, \quad \text{with } n \in \mathbb{Z}^+.$$

We would like to prove that

$$\overline{B}_{q,k}(t,s) = \frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)}, \qquad \text{for all } s,t > 0.$$

By Lemma 2 we have that

$$\frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)} = \frac{(1-q)(1-q^k)_{q,k}^{\infty}(1-q^{t+s})_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q^s)_{q,k}^{\infty}}.$$

By the definite Jackson integral and Definition 5 item 2, we obtain

$$\begin{aligned}
\overline{B}_{q,k}(t,s) &= [k]_q^{-\frac{t}{k}} \int_0^{[k]_q^{\frac{1}{k}}} x^{t-1}\left(1 - q^k \frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}-1} d_q x \\
&= (1-q)\sum_{n=0}^{\infty} q^{nt}(1-q^{k(n+1)})_{q,k}^{\frac{s}{k}-1} \\
&= (1-q)\sum_{n=0}^{\infty} q^{nt}\frac{(1-q^{k(n+1)})_{q,k}^{\infty}}{(1-q^{s+nk})_{q,k}^{\infty}}.
\end{aligned}$$

So, we have to show that

$$\frac{(1-q)(1-q^k)_{q,k}^{\infty}(1-q^{t+s})_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q^s)_{q,k}^{\infty}} = (1-q)\sum_{n=0}^{\infty} q^{nt}\frac{(1-q^{k(n+1)})_{q,k}^{\infty}}{(1-q^{s+nk})_{q,k}^{\infty}}.$$

Making the changes $u = q^t$ and $v = q^s$, we reduce our problem to prove that

$$\frac{(1-q)(1-q^k)_{q,k}^{\infty}(1-uv)_{q,k}^{\infty}}{(1-u)_{q,k}^{\infty}(1-v)_{q,k}^{\infty}} = (1-q)\sum_{n=0}^{\infty} b^n\frac{(1-q^{k(n+1)})_{q,k}^{\infty}}{(1-vq^{nk})_{q,k}^{\infty}}. \tag{4.7}$$

Now, both sides of equation (4.7) are formal power series in q with rational coefficients in u and v. Since we already know that they agree for an infinite number of values, namely $u = q^t$ and $v = q^s$, where $t > 0$ and $s = nk$ with $n \in \mathbb{Z}^+$, the desired result holds. ■

5 Other integral representations

In this Section we provided Jackson integral representations for the $\Gamma_{q,k}$ and $B_{q,k}$ using the q,k-analogue of the exponential function $e^x_{q,k}$. We remark that in an earlier version of this paper we took the upper limit in our integrals to be ∞ leading to divergent integrals. The correct upper limit in our definition was clear to us only after reading [2]. Recall that

$$e^x_{q,k} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q^k}!} = \frac{1}{(1-(1-q^k)x)^{\infty}_{q,k}}.$$

Definition 8. The function $\gamma^{(a)}_{q,k}$, $a > 0$, is given by the following Jackson integral

$$\gamma^{(a)}_{q,k}(t) = \int_0^{\infty/a(1-q^k)^{\frac{1}{k}}} x^{t-1} e_{q,k}^{-\frac{x^k}{[k]_q}} d_q x, \quad t > 0.$$

Next proposition shows that $\gamma^{(a)}_{q,k}$ satisfies properties similar to those given in Proposition 2 for the $\Gamma_{q,k}$ function.

Proposition 6. *The function $\gamma^{(a)}_{q,k}$ satisfies the following formulae for $a, t > 0$*

1. $\gamma^{(a)}_{q,k}(k) = 1$.

2. $\gamma^{(a)}_{q,k}(t+k) = q^{-t}[t]_q \gamma^{(a)}_{q,k}(t)$.

3. $\gamma^{(a)}_{q,k}(nk) = q^{-kn(n-1)/2}\Gamma_{q,k}(nk)$, *for every $n \in \mathbb{Z}^+$.*

Proof. 1. $$\gamma^{(a)}_{q,k}(k) = \int_0^{\infty/a(1-q^k)^{\frac{1}{k}}} x^{k-1} e_{q,k}^{-\frac{x^k}{[k]_q}} d_q x = -\int_0^{\infty/a(1-q^k)^{\frac{1}{k}}} \partial_q\left(e_{q,k}^{-\frac{x^k}{[k]_q}}\right) = 1.$$

2. $$\begin{aligned}\gamma^{(a)}_{q,k}(t+k) &= -q^{-t}\int_0^{\infty/a(1-q^k)^{\frac{1}{k}}} (qx)^t \partial_q\left(e_{q,k}^{-\frac{x^k}{k}}\right) d_q x \\ &= [t]_q q^{-t} \int_0^{\infty/a(1-q^k)^{\frac{1}{k}}} x^{t-1} e_{q,k}^{-\frac{x^k}{[k]_q}} d_q x = q^{-t}[t]_q \gamma^{(a)}_{q,k}(t).\end{aligned}$$

3. From items 1 and 2 above, we have that

$$\gamma^{(a)}_{q,k}(nk) = q^{-kn(n-1)/2} \prod_{j=1}^{n-1} [jk]_q = q^{-kn(n-1)/2}\Gamma_{q,k}(nk).$$

■

Definition 9. The function $\beta_{q,k}^{(a)}$, $a>0$, is given by the following Jackson integral

$$\beta_{q,k}^{(a)}(t,s)=[k]_q^{-\frac{t}{k}}\int_0^{\infty/a}\frac{x^{t-1}}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}}d_qx,\quad t,s>0.$$

Proposition 7. *The function $\beta_{q,k}^{(a)}$ satisfies the following formulae for $a,t,s>0$*

1. $\beta_{q,k}^{(a)}(t,\infty)=(1-q)^{\frac{t}{k}}\gamma_{q,k}^{(a)}(t)$.
2. $\beta_{q,k}^{(a)}(t+k,s)=q^{-t}\frac{[t]_q}{[t+s]_q}\beta_{q,k}^{(a)}(t,s)$.
3. $\beta_{q,k}^{(a)}(k,s)=\frac{1}{[s]_q}$.
4. $\beta_{q,k}^{(a)}(t,s+k)=\frac{[s]_q}{[t+s]_q}\beta_{q,k}^{(a)}(t,s)$.
5. $\beta_{q,k}^{(a)}(nk,s)=q^{-kn(n-1)/2}B_{q,k}(nk,s)$, *for all* $n\in\mathbb{Z}^+$.

Proof. 1. Using the change $x=(1-q^k)^{\frac{1}{k}}y$, (5.2) is obtained from (5.1).

$$\begin{aligned}\beta_{q,k}^{(a)}(t,\infty) &= [k]_q^{-\frac{t}{k}}\int_0^{\infty/a}x^{t-1}e_{q,k}^{-\frac{x^k}{(1-q^k)[k]_q}}d_qx. &(5.1)\\ &= [k]_q^{-\frac{t}{k}}(1-q^k)^{\frac{t}{k}}\int_0^{\infty/a(1-q^k)^{\frac{1}{k}}}y^{t-1}e_{q,k}^{-\frac{y^k}{[k]_q}}d_qy &(5.2)\\ &= (1-q)^{\frac{t}{k}}\gamma_{q,k}^{(a)}(t).\end{aligned}$$

2. Using the formula $\partial_q\left(\frac{\left(1+a\frac{x^k}{[k]_q}\right)_{q,k}^s}{\left(1+b\frac{x^k}{[k]_q}\right)_{q,k}^t}\right)=\frac{[ks]_qax^{k-1}\left(1+aq^k\frac{x^k}{[k]_q}\right)_{q,k}^{s-1}}{[k]_q\left(1+bq^k\frac{x^k}{[k]_q}\right)_{q,k}^t}-\frac{[kt]_qbx^{k-1}\left(1+a\frac{x^k}{[k]_q}\right)_{q,k}^s}{[k]_q\left(1+b\frac{x^k}{[k]_q}\right)_{q,k}^{t+1}}$.

$$\begin{aligned}\beta_{q,k}^{(a)}(t+k,s) &= -\frac{[k]_q^{-\frac{t}{k}}}{[t+s]_q}q^{-t}\int_0^{\infty/a}(qx)^t\partial_q\left(\frac{1}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}}\right)d_qx\\ &= \frac{[t]_q}{[t+s]_q}q^{-t}[k]_q^{-\frac{t}{k}}\int_0^{\infty/a}\frac{x^{t-1}}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}}d_qx\\ &= \frac{[t]_q}{[t+s]_q}q^{-t}\beta_{q,k}^{(a)}(t,s).\end{aligned}$$

3. Using the identity $\partial_q\left(\frac{1}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}}}\right) = -\frac{x^{k-1}[s]_q}{[k]_q\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}+1}}$, we get

$$\beta_{q,k}^{(a)}(k,s) = -\frac{1}{[s]_q}\int_0^{\infty/a}\partial_q\left(\frac{1}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{s}{k}}}\right)d_qx = \frac{1}{[s]_q}.$$

4. Using $\partial_q\left(\frac{ax^{ks}}{[k]_q^s\left(1+b\frac{x^k}{[k]_q}\right)_{q,k}^t}\right) = \frac{ax^{ks-1}[ks]_q}{[k]_q^s\left(1+b\frac{x^k}{[k]_q}\right)_{q,k}^{t+1}} - b([kt]_q-[ks]_q)\frac{ax^{k(s+1)-1}}{[k]_q^{s+1}\left(1+b\frac{x^k}{[k]_q}\right)_{q,k}^{t+1}}$.

$$\begin{aligned}\beta_{q,k}^{(a)}(t,s+k) &= \frac{[k]_q^{\frac{t+s}{k}}[k]_q^{-\frac{t}{k}}q^s}{[t+s]_q}\int_0^{\infty/a}\frac{1}{(qx)^s}\partial_q\left(\frac{x^{t+s}}{[k]_q^{\frac{t+s}{k}}\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}}\right)d_qx\\ &= -\frac{q^s[k]_q^{-\frac{t}{k}}}{[t+s]_q}\int_0^{\infty/a}\frac{x^{t+s}}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}}\partial_q\left(\frac{1}{x^s}\right)d_qx\\ &= \frac{[s]_q}{[t+s]_q}[k]_q^{-\frac{t}{k}}\int_0^{\infty/a}\frac{x^{t-1}}{(1+\frac{x^k}{[k]_q})_{q,k}^{\frac{t+s}{k}}}d_qx = \frac{[s]_q}{[t+s]_q}\beta_{q,k}^{(a)}(t,s).\end{aligned}$$

5. Using property 2 above recursively

$$\beta_{q,k}^{(a)}(nk,s) = q^{-kn(n-1)/2}\frac{(1-q)(1-q^k)_{q,k}^{n-1}(1-q^{nk})_{q,k}^{\frac{s}{k}-n}}{(1-q^s)_{q,k}^n(1-q^{nk})_{q,k}^{\frac{s}{k}-n}} = q^{-kn(n-1)/2}B_{q,k}(nk,s).$$

■

Lemma 4. *Let $s,t\in\mathbb{R}$ and $n\in\mathbb{Z}^+$, we have the following identities*

1. $(1+q^{ks}x)_{q,k}^t = \frac{(1+x)_{q,k}^{s+t}}{(1+x)_{q,k}^s} = \frac{(1+x)_{q,k}^t(1+q^{kt}x)_{q,k}^s}{(1+x)_{q,k}^s}$.

2. $(1+q^{-kn}x)_{q,k}^t = (1+x)_{q,k}^t\frac{(x+q^k)_{q,k}^n}{(q^{kt}x+q^k)_{q,k}^n}$.

Next theorem provides our second integral representation for the functions $\Gamma_{q,k}$ and $B_{q,k}$.

Theorem 2. *For all $a,s,t>0$ we have:*

1. $\Gamma_{q,k}(t) = c(a,t)\gamma_{q,k}^{(a)}(t)$.

2. $B_{q,k}(t,s) = c(a,t)\beta^{(a)}_{q,k}(t,s)$.

Where

$$c(a,t) = \frac{a^t [k]_q^{\frac{t}{k}}}{1+[k]_q a^k}\left(1+\frac{1}{[k]_q a^k}\right)_{q,k}^{\frac{t}{k}}\left(1+[k]_q a^k\right)_{q,k}^{1-\frac{t}{k}}.$$

Proof. Since both $B_{q,k}(t,s+k) = \frac{[s]_q}{[s+t]_q}B_{q,k}(t,s)$ and $\beta^{(a)}_{q,k}(t,s+k) = \frac{[s]_q}{[t+s]_q}\beta^{(a)}_{q,k}(t,s)$. It is clear that if $c(a,t)$ is such that $c(a,t)\beta^{(a)}_{q,k}(t,s) = B_{q,k}(t,s)$, then c(a,t) must be

$$\left(\int_0^{\infty/a}\partial_q\left(\frac{x^t}{[k]_q^{\frac{t}{k}}\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t}{k}}}\right)d_qx\right)^{-1}.$$

We know that $\beta^{(a)}_{q,k}(t,k) = \frac{1}{[t]_q}\int_0^{\infty/a}\partial_q\left(\frac{x^t}{[k]_q^{\frac{t}{k}}\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t}{k}}}\right)d_qx$. Thus by definition of q-derivative and the Jackson integral, we have:

$$\int_0^{\infty/a}\partial_q(F_k)d_qx = \lim_{n\to\infty}F_k\left(\frac{1}{aq^n}\right) - \lim_{n\to\infty}F_k\left(\frac{q^n}{a}\right), \tag{5.3}$$

where the limits are taken over integers.

From (5.3), we obtain: $\beta^{(a)}_{q,k}(t,k) = \frac{1}{[t]_q}\lim_{n\to\infty}\left([k]_q^{\frac{t}{k}}(aq^n)^t\left(1+\frac{1}{[k]_q(aq^n)^k}\right)_{q,k}^{\frac{t}{k}}\right)^{-1}$.

Using Lemma 4 part 2 in equation (5.4) we have

$$\begin{aligned} c(a,t) &= [k]_q^{\frac{t}{k}}a^t\lim_{n\to\infty}q^{nt}\left(1+\frac{q^{-nk}}{[k]_q a^k}\right)_{q,k}^{\frac{t}{k}} \\ &= [k]_q^{\frac{t}{k}}a^t\left(1+\frac{1}{[k]_q a^k}\right)_{q,k}^{\frac{t}{k}}\lim_{n\to\infty}q^{nt}\frac{\left(\frac{1}{[k]_q a^k}+q^k\right)_{q,k}^n}{\left(\frac{q^t}{[k]_q a^k}+q^k\right)_{q,k}^n} \\ &= [k]_q^{\frac{t}{k}}a^t\left(1+\frac{1}{[k]_q a^k}\right)_{q,k}^{\frac{t}{k}}\lim_{n\to\infty}\frac{(1+[k]_q a^k q^k)_{q,k}^n}{(1+[k]_q a^k q^{k-t})_{q,k}^n} \\ &= [k]_q^{\frac{t}{k}}\frac{a^t}{1+[k]_q a^k}\left(1+\frac{1}{[k]_q a^k}\right)_{q,k}^{\frac{t}{k}}\left(1+[k]_q a^k\right)_{q,k}^{1-\frac{t}{k}}. \end{aligned} \tag{5.4}$$

Thus, $B_{q,k}(t,nk) = c(a,t)\beta^{(a)}_{q,k}(t,nk)$, for all $n\in\mathbb{Z}^+$. Moreover, proceeding as in Theorem 1 we can prove that part 2 of Theorem 2, i.e., one can show that both sides of equation

$$B_{q,k}(t,s) = c(a,t)[k]_q^{-\frac{t}{k}} \int_0^{\infty/a} \frac{x^{t-1}}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}} d_q x, \quad t,s>0,$$

after the appropriated changes, are formal power series with rational coefficients in the correct variables. Part 1 follows from part 2 using properties $\beta_{q,k}^{(a)}(t,\infty) = (1-q)^{\frac{t}{k}}\gamma_{q,k}^{(a)}(t)$ and $B_{q,k}^{(a)}(t,\infty) = (1-q)^{\frac{t}{k}}\Gamma_{q,k}^{(a)}(t)$. ■

Below we include an alternative prove of this Theorem 2. Let us first give a proposition with further properties of the function $c(a,t)$.

Proposition 8. *1.* $\lim_{q\to 1} c(a;t) = 1$ *for all* $a>0$ *and* $t\in\mathbb{R}$.

2. $\lim_{q\to 0} c(a,t) = a^t + a^{t-k}$ *for all* $a>0$ *and* $0<t<1$.

3. c(a,t) satisfies the following recursive formula: $c(a,t+k) = q^t c(a,t)$*, for all* $a>0$ *and* $t\in\mathbb{R}$.

4. For $a>0$ *and* $n\in\mathbb{Z}^+$*, we have that* $c(a,nk) = q^{kn(n-1)/2}$.

5. $\partial_q c(a,t) = 0$*, for all* $a>0$ *and* $t\in\mathbb{R}$.

Proof. 1. Obvious.

2. In the limit $q\to 0$, c(a,t) goes to $\frac{a^t}{1+a^k}\left(1+\frac{1}{a^k}\right)(1+a^k) = a^t + a^{t-k}$, for all $a>0$ and $t\in\mathbb{R}$.

3. $$\frac{c(a,t+k)}{c(a,t)} = [k]_q \frac{a^k\left(1+\frac{q^t}{[k]_q a^k}\right)}{(1+[k]_q q^{-t}a^k)} = q^t.$$

4. Immediate from item 3 and the fact that $c(a,0)=c(a,k)=1$.

5. To show that $\partial_q c(a,t)=0$, it is enough to check that c(qa,t)=c(a,t).

$$\begin{aligned} c(qa,t) &= \frac{[k]_q^{\frac{t}{k}} q^t a^t}{1+[k]_q q^k a^k}\left(1+\frac{1}{[k]_q q^k a^k}\right)_{q,k}^{\frac{t}{k}} (1+[k]_q q^k a^k)_{q,k}^{1-\frac{t}{k}} \\ &= \frac{[k]_q^{\frac{t}{k}} q^t a^t \left(1+\frac{1}{[k]_q a^k}\right)_{q,k}^{\frac{t}{k}}\left(\frac{1}{[k]_q q^k a^k}+1\right)(1+[k]_q a^k)_{q,k}^{1-\frac{t}{k}}(1+[k]_q q^{k-t}a^k)}{(1+[k]_q a^k)(1+[k]_q q^k a^k)\left(\frac{q^t}{[k]_q q^k a^k}+1\right)}. \end{aligned}$$

It is easy to check that $q^t \dfrac{\left(\frac{1}{[k]_q q^k a^k}+1\right)(1+[k]_q q^{k-t}a^k)}{(1+[k]_q q^k a^k)\left(\frac{q^t}{[k]_q q^k a^k}+1\right)} = 1$, concluding that c(qa,t)=c(a,t).

Moreover $\partial_q c(a,t) = 0$, for all $t \in \mathbb{R}$.

■

Theorem 2 may also be deduced from the following chain of arguments. First, notice that using the Jackson integral, Definition 5 item 2 and the infinite product expression for the function $B_{q,k}$ given in Lema 3 part 2, one can show that Theorem 2 part 2, that is,

$$B_{q,k}(t,s) = c(a,t)[k]_q^{-\frac{t}{k}} \int_0^{\infty/a} \frac{x^{t-1}}{\left(1+\frac{x^k}{[k]_q}\right)_{q,k}^{\frac{t+s}{k}}} d_q x, \quad t,s>0,$$

is equivalent to the following relation

$$\sum_{n\in\mathbb{Z}} \frac{q^{nt}\left(1+\frac{1}{[k]_q a^k}\right)_{q,k}^n}{\left(1+\frac{q^{t+s}}{[k]_q a^k}\right)_{q,k}^n} = \frac{(1-q^k)_{q,k}^\infty (1-q^{t+s})_{q,k}^\infty \left(1+\frac{q^t}{[k]_q a^k}\right)_{q,k}^\infty \left(1+\frac{[k]_q q^k a^k}{q^t}\right)_{q,k}^\infty}{\left(1+\frac{q^{t+s}}{[k]_q a^k}\right)_{q,k}^\infty (1+[k]_q q^k a^k)_{q,k}^\infty (1-q^s)_{q,k}^\infty (1-q^t)_{q,k}^\infty}.$$

Making the changes $u=-1/[k]_q a^k, v=-\frac{q^{t+s}}{[k]_q a^k}$ and $x=q^t$, we see that Theorem 2 part 2 is equivalent to the famous Ramanujan identity.

$$\sum_{n\in\mathbb{Z}} \frac{x^n (1-u)_{q,k}^n}{(1-v)_{q,k}^n} = \frac{(1-q^k)_{q,k}^\infty (1-v/u)_{q,k}^\infty (1-ux)_{q,k}^\infty \left(1-q^k/ux\right)_{q,k}^\infty}{(1-v)_{q,k}^\infty (1-q^k/u)_{q,k}^\infty (1-x)_{q,k}^\infty (1-v/ux)_{q,k}^\infty}.$$

Similarly, using the definition of Jackson integral, Definition 5 part 2, the infinite product expression for $e_{q,k}^{-\frac{x^k}{[k]_q}}$ and Lemma 2, one can show that Theorem 2 part 1, that is,

$$\Gamma_{q,k}(t) = c(a,t)\int_0^{\infty/a(1-q^k)^{\frac{1}{k}}} x^{t-1} e_{q,k}^{-\frac{x^k}{[k]_q}} d_q x, \quad t>0,$$

is equivalent to the following triple identity

$$(1-q^k)_{q,k}^\infty \left(1+\frac{q^t}{[k]_q a^k}\right)_{q,k}^\infty \left(1+\frac{[k]_q q^k a^k}{q^t}\right)_{q,k}^\infty = (1-q^t)_{q,k}^\infty (1+[k]_q q^k a^k)_{q,k}^\infty \sum_{n\in\mathbb{Z}} q^n t \left(1+\frac{1}{[k]_q a^k}\right)_{q,k}^n. \tag{5.5}$$

Making the change $x=-\frac{q^t}{a^k}$ and letting $a\to 0$ in (5.5), we obtain

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{kn(n-1)/2}[k]_q^{-n(n-1)/2} x^n = (1-q^k)_{q,k}^\infty \left(1-\frac{x}{[k]_q}\right)_{q,k}^\infty \left(1-\frac{q^k[k]_q}{x}\right)_{q,k}^\infty$$

which is the Jacobi triple product identity.

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