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A Note on the Degasperis-Procesi Equation

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Abstract

We prove that smooth solutions of the Degasperis-Procesi equation have infinite propagation speed: they loose instantly the property of having compact support.

1 Introduction

The Degasperis-Procesi equation [13]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \qquad x \in \mathbb{R},\, t \geq 0, \tag{1.1}$$

was derived recently as a shallow water approximation to the Euler equation [14]. The equation (1.1) presents some similarities to the Camassa-Holm equation [1, 17]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad x \in \mathbb{R},\, t \geq 0. \tag{1.2}$$

Indeed [1, 13, 15], both equations are bi-Hamiltonian and have an associated isospectral problem. Therefore they are both formally integrable (the integrability of (1.2) by means of the scattering/inverse scattering approach is discussed in [5, 9, 19]). Also, both equations admit exact peaked solitary wave solutions which have to be understood as weak solutions [10, 8, 22]. Moreover, using Kato's semigroup theory for quasilinear equations of evolution [18], the local existence of solutions to (1.1) in $H^s(\mathbb{R})$, with $s > \frac{3}{2}$ can be established provided the initial data is in $H^s(\mathbb{R})$ (see [23]), and finite-time blowup is possible only if the slope of the solution becomes unbounded in finite time [22]. The last feature parallels the fact that for (1.2) an initial data with the same regularity will either develop into a wave that exists for all times or wave breaking ocurs [2, 3, 4, 12].

Despite these similarities, the equations (1.1) and (1.2) are truly different. For example, for (1.1) the isospectral problem is of third order, whereas in the case of (1.2) we encounter a second order isospectral problem [22]. Moreover, (1.2) is a re-expression of geodesic flow on the group of diffeomorphisms of the line [20] (see also [4, 6, 7]) whereas (1.1) does not have a geometric derivation of this type.

A different inquiry is the aim of the present letter. We will prove that the infinite propagation speed of smooth solutions, observed in [11] for the Camassa-Holm equation, is also a feature of the model (1.1).

2 Main results

Denoting $m = u - u_{xx}$, we can write (1.1) in the form

$$m_t + um_x + 3u_x m = 0, \qquad x \in \mathbb{R},\, t \geq 0. \tag{2.1}$$

Theorem 1. *Let $m_0 : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support. If the solution $m(x,t)$ of equation (1.1) has $[0,T)$ as maximal interval of existence in the future, then, at every moment $t \in [0,T)$, the smooth function $m(.,t)$ has compact support.*

Proof. We shall employ a device from [4]. Let us introduce the initial value problem

$$\begin{cases} \frac{d\psi}{dt} = u(\psi,t),\ t \in [0,T), \\ \qquad \psi(0) = x. \end{cases} \tag{2.2}$$

Since $u_0 = p * m_0$, where $u(x,0) = u_0(x)$ for $x \in \mathbb{R}$, the local regularity theory developed in [23, 22] for (1.1) implies that $u(t,\cdot)$ is smooth on $[0,T)$. Therefore, via standard qualitative theory for ordinary differential equations [16], the existence in $[0,T)$, uniqueness and smooth dependence of the data x for the solution $\psi = \psi(t;x)$ of (2.2) is ensured. By integration in $[0,t)$, with $t < T$, we get

$$\psi(t;x) = x + \int_0^t u(\psi(s;x),s)ds,$$

which yields

$$\psi_x(t;x) = 1 + \int_0^t u_x(\psi(s;x),s)\psi_x(s;x)ds, \qquad t \in [0,T), \tag{2.3}$$

and respectively

$$\frac{d\eta}{dt} = u_x(\psi(t;x),t)\eta,$$

where we have denoted by η the left-hand member of (2.3). Finally, since $\eta(0) = 1$, we obtain

$$\psi_x(t;x) = \eta(t) = \exp\left(\int_0^t u_x(\psi(s;x),s)ds\right), \qquad t \in [0,T). \tag{2.4}$$

By introducing the function $\varphi(x,t) = \psi(t;x)$, we have obtained an element of $C^1(\mathbb{R} \times [0,T), \mathbb{R})$. Furthermore, since, due to the Sobolev imbeddings, $u_x(\cdot,t)$ is bounded in $\mathbb{R}$ for all $t \in [0,T)$, we deduce that

$$0 < C_1(t) \leq \psi_x(t;x) \leq C_2(t) < +\infty, \qquad x \in \mathbb{R},\ t \in [0,T), \tag{2.5}$$

which allows us to conclude that $\varphi(.,t)$, $t \in [0,T)$, are all diffeomorphisms of $\mathbb{R}$. Also, from (2.2) we get $\varphi_{xt} = u_x\varphi_x$. Further, let us multiply the equation (2.1) by φ_x^3. Then, if we take $\varphi(x,t)$ and t as arguments of u, m instead of the usual x, t, the result of multiplication reads as

$$\begin{aligned} 0 &= (m_t + um_x + 3u_x m)\,\varphi_x^3 = m_t\varphi_x^3 + 3m\,(u_x\varphi_x)\,\varphi_x^2 + um_x\varphi_x^3 \\ &= m_t\varphi_x^3 + 3m\varphi_{xt}\varphi_x^2 + \varphi_t m_x\varphi_x^3 = \frac{d}{dt}\left(m\varphi_x^3\right). \end{aligned}$$

Via an integration,

$$m(\varphi(x,t),t)\varphi_x^3(x,t) = m(\varphi(x,0),0)\varphi_x^3(x,0) = m(x,0) = m_0(x).$$

Finally, due to (2.5), if the support of m_0 is included in $[a,b]$ then the support of $m(.,t)$ will be included in $[\varphi(a,t),\varphi(b,t)]$. The proof is complete. ■

Theorem 2. *Let $u_0 : \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support. If the solution $u(x,t)$ with initial data $u_0(x)$ of (1.1) exists on some time interval $[0,\epsilon)$ with $\epsilon > 0$ and, at every instant $t \in [0,\epsilon)$, the function $u(.,t)$ has compact support, then u is identically zero.*

Proof. According to the preceding theorem, $m(.,t)$ has compact support at every moment $t \in [0,\epsilon)$. We recall that, by the Paley-Wiener theorem [21], an entire (analytic) function $g(\xi)$, where $\xi = \eta + i\zeta$ and η, $\zeta \in \mathbb{R}$, is the Fourier transform of a smooth function $f : \mathbb{R} \to \mathbb{R}$ with compact support in $[-a,a]$ (for $a > 0$), namely

$$g(\xi) = \mathcal{F}_f(\xi) = \int_{\mathbb{R}} f(q)e^{-i\xi q}dq,$$

if and only if for every integer $n \geq 0$ there exists $c_n > 0$ such that

$$|g(\xi)| \leq \frac{c_n e^{a|\zeta|}}{(1+|\xi|^n)}, \qquad \xi \in \mathbb{C}.$$

Since

$$\mathcal{F}_{m(.,t)}(\xi) = (1+\xi^2)\mathcal{F}_{u(.,t)}(\xi), \qquad \xi \in \mathbb{C},\ t \in [0,\epsilon),$$

it is obvious that the analiticity of $\mathcal{F}_{u(.,t)}$, if assumed, will imply the analiticity of $\mathcal{F}_{m(.,t)}(\xi)$. In such a case, the function $\mathcal{F}_{m(.,t)}$ has value zero at i, $-i$ for all $t \in [0,\epsilon)$, yielding

$$\int_{\mathbb{R}} e^x m(x,t)dx = \int_{\mathbb{R}} e^{-x} m(x,t)dx = 0, \qquad t \in [0,\epsilon).$$

Since both u and m have compact support, we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}} e^x m(x,t)dx = \int_{\mathbb{R}} e^x m_t dx = -3\int_{\mathbb{R}} e^x u_x m dx - \int_{\mathbb{R}} e^x m_x u dx$$

$$= -3\int_{\mathbb{R}} e^x u_x m dx + \int_{\mathbb{R}} e^x u_x m dx + \int_{\mathbb{R}} e^x m u dx = -2\int_{\mathbb{R}} e^x u_x m dx + \int_{\mathbb{R}} e^x m u dx$$

$$= -2\int_{\mathbb{R}} e^x u_x u dx + 2\int_{\mathbb{R}} e^x u_x u_{xx} dx + \int_{\mathbb{R}} e^x u^2 dx - \int_{\mathbb{R}} e^x u_{xx} u dx$$

$$= \Big(-\int_{\mathbb{R}} e^x u_x u dx + \frac{1}{2}\int_{\mathbb{R}} e^x u^2 dx\Big) - \int_{\mathbb{R}} e^x u_x^2 dx + \int_{\mathbb{R}} e^x u^2 dx$$

$$+\Big(\int_{\mathbb{R}} e^x u_x^2 dx + \int_{\mathbb{R}} e^x u_x u dx\Big) = \frac{3}{2}\int_{\mathbb{R}} e^x u^2 dx = 0,$$

which implies that $u(.,t) \equiv 0$ for all t. The proof is complete. ■

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