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Symmetry Reductions of a Hamilton-Jacobi-Bellman Equation Arising in Financial Mathematics

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Abstract

We determine the solutions of a nonlinear Hamilton-Jacobi-Bellman equation which arises in the modelling of mean-variance hedging subject to a terminal condition. Firstly we establish those forms of the equation which admit the maximal number of Lie point symmetries and then examine each in turn. We show that the Lie method is only suitable for an equation of maximal symmetry. We indicate the applicability of the method to cases in which the parametric function depends also upon the time.

1 Introduction

In the early seventies a revolution pioneered by the ideas of Samuelson, Mogdigliani, Merton, Black and Scholes occurred in economic modelling. The major step was the formal introduction of the concepts of stochastic calculus into Econometrics. Although the original conception was Samuelson's, the initially controversial paper by Black and Scholes [3] firmly embedded this work into financial economics. Subsequently there has been a veritable explosion of work on the subject to an extent that the field has separated to become Financial Mathematics.

The Black-Scholes equation is an evolution equation of the same equivalence class as the Schrödinger equation and the diffusion equation. This is unsurprising as all three are archetypal models dealing with random phenomena in a specific context, finance, particle-motion and dispersion. In each of these cases during the course of the modelling

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procedure the randomness incorporated into the model is made deterministic. The averaging 'method' in the original Black-Scholes paper was to form a risk-free portfolio and thus eliminate the random component of the governing Itô equations. The classical derivation of the heat equation defines a temperature relating to interactive particle dynamics within a physical medium while the Schrödinger equation defines deterministically the nonrelativistic quantum mechanical description of the probability distribution of a particle.

To every Itô equation one associates a corresponding Fokker-Planck or Feynman-Kac partial differential equation[2]. For example the Itô equation describing the motion of a free particle in one spatial dimension under the effect of constant noise is

$$dX_t = \sigma^2 dW_t.$$

This corresponds to the diffusion equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$$

in the case that $\sigma^2 = 1$. In general an Itô equation governing a one point process,

$$dX_t = \mu(x,t)dt + \sigma(x,t)dW_t,$$

has the deterministic counterpart

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} = 0$$

or

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} = r(x,t)u$$

in the Fokker-Planck and Feynman-Kac versions respectively.

The Girsanov Theorem in stochastic calculus implies that under a change of measure and thereby an induced change in the Brownian motion W one can modify the drift term $\mu(x,t)$ of an Itô equation arbitrarily. In the case that one eliminates the drift completely the free particle Itô equation is recovered. The natural generalization would be that one can map classes of Fokker-Planck and Feynmann-Kac partial differential equations to variants of the diffusion equation. Furthermore the relationship to evolution equations in general is placed into context by an interpretation of the generator of an Itô process,

$$\lim_{h\to 0} = \mathbb{E}\left[\frac{f(X_{t+h},t) - f(X_t,t)}{h}\middle| X_t = x\right] = \mathcal{L}_{\mathcal{X}} f,$$

where

$$\mathcal{L}_{\mathcal{X}} \equiv \frac{\partial}{\partial t} + \mu(x,t)\frac{\partial}{\partial x} + \sigma^2(x,t)\frac{\partial^2}{\partial x^2},$$

as an expected rate of change of a function of a random variable. Typically these ideas are exemplified in the Black-Scholes equation.

[2]In this context we note that a recent paper [10] has reported the development of reverse Feynman-Kac formulæ for the same purpose.

The Black-Scholes equation can be derived in three ways, each of which individually is a rich source of partial differential equations in economics and finance, the original hedging approach, the martingale approach and another [11][p 601] which derives the equation as an Hamilton-Jacobi-Bellman solution to a stochastic control problem in an interest-rate model. For Black-Scholes there is a myriad of connections between these approaches, the Feynmann-Kac theorem, the persistence of the Itô generator since the option is a function of the share price and the share price follows geometric Brownian motion, the no arbitrage argument, which is similar to a conservation law in that it says that on average the option value behaves in the same way as a risk-free instrument, and obviously the equation itself.

The class of Hamilton-Jacobi-Bellman equations arises in the sphere of stochastic control theory [12]. A typical problem is to minimize an index,

$$J = \min_{u \in U} \mathbb{E}_t \left\{ \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) \right\} d\tau,$$

with respect to the stochastic constraints,

$$d\mathbf{X} = \mu(\mathbf{X}(t), \mathbf{u}(t))dt + \sigma(\mathbf{X}(t), \mathbf{u}(t))dW,$$

over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and permissible controls $\mathbf{u} \in U$. This has the optimal solution, f, given in the $(1+1)$-dimensional case by the partial differential equation

$$\frac{\partial f}{\partial t} + \min_{u \in U} \left[\mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} + g(x, u) \right] = 0.$$

Minimization over the controls $u \in U$ implies that, if Pontryagin's principle from standard problems of control theory, that on the optimal control the Hamiltonian is minimized with respect to the control variables u, can be invoked [5][p 300-307],

$$H = \left[\mu(x, t) \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x, t) \frac{\partial^2 f}{\partial x^2} + g(x, u) \right] \tag{1.1}$$

has an interpretation as a Hamiltonian, at least heuristically.

In this paper we wish to investigate the properties of one of these Hamilton-Jacobi-Bellman equations from the viewpoint of the Lie symmetry analysis. The Black-Scholes equation was examined some years ago by Ibragimov and Gazizov [9]. In a sense the symmetry analysis of the Black-Scholes equation may be regarded as trivial since it is related to the standard heat equation by means of an obvious point transformation and the symmetry properties and solutions of the heat equation are the stuff of textbooks [4]. However, one should note that the solution of the Black-Scholes equation requires the conjunction of its terminal condition and terminal conditions do not translate well into boundary and initial conditions under point transformations. Moreover the emphasis in the paper of Ibragimov and Gazizov is not on the calculation of the symmetries but on the construction of the fundamental solution which is not trivial since the calculation of the fundamental solution by transformation of the fundamental solution of the heat equation is not straightforward. Furthermore the paper contains a second part which deals with the Jacobs-Jones two-factor variable model.

The challenge in treating Hamilton-Jacobi-Bellman equations by the methods of the Lie symmetry analysis is to incorporate the conditions to be imposed upon the solutions to the equations which in the case of a linear partial differential equation are infinite in number, have the property of linear superposition and so provide the prospect of a reasonable selection. There are three ways to respond to this challenge.

One method, which is found in the literature, is to include the boundaries as part of the problem in the determination of symmetries.

Another approach is to attempt to use the infinite sets of solutions which we can obtain for some problems – those with a reasonable amount of symmetry – and determine whether they can be used as a basis set.

A third approach is to compute all of the symmetries and then to see which combinations of them are consistent with the boundary and initial/terminal conditions.

All three methods have their own relevance in different contexts.

In the case of the Schrödinger Equation for a confining potential such as that of the simple harmonic oscillator one looks for the infinite sets of solutions to act as the bases since the realisation of a state must be one of these functions. In the construction of the solutions one selects that infinite set which is consistent with the boundary conditions. This is not an arduous business. That is not the case in other areas of application for in them the nature of the boundary/initial/terminal conditions intrudes rather more forcibly.

The Hamilton-Jacobi-Bellman equation we wish to address is not an obvious relative of the Black-Scholes or heat equations. The equation is

$$\frac{\partial J}{\partial t} + a\frac{\partial J}{\partial x} + \tfrac{1}{2}b^2\frac{\partial^2 J}{\partial x^2} - \tfrac{1}{2}\left(\frac{\partial J}{\partial x}\right)^2 + \left(\frac{\mu}{x}\right)^2 = 0 \tag{1.2}$$

with the terminal condition $J(T, x) = 0$ and is presented by Heath *et al* [8] [p 516] as an equation for mean-variance hedging.

Heath *et al* observe that μ in the ultimate term of (1.2) need not be a constant. In fact it is better to write the equation as

$$\frac{\partial J}{\partial t} + a\frac{\partial J}{\partial x} + \tfrac{1}{2}b^2\frac{\partial^2 J}{\partial x^2} - \tfrac{1}{2}\left(\frac{\partial J}{\partial x}\right)^2 + \nu(x) = 0 \tag{1.3}$$

to allow for the possible dependence of μ upon x.

The approach which we adopt for seeking solutions of (1.3) with its terminal condition is that of calculating the Lie point symmetries of (1.3) and then determining if there exists a symmetry, or symmetries, compatible with the terminal condition. In the case that a symmetry exists the similarity solution may be determined in the usual fashion as we demonstrate below.

This paper is structured as follows. In the next section we examine (1.3) for Lie point symmetries and identify those functions, $\nu(x)$, for which there exists additional symmetry. In §3 we examine those cases in turn and determine the existence or not of symmetries compatible with the terminal condition and obtain the solutions in the case of existence. For those cases in which we can find a solution it is of some comfort to know that the solution is unique due to the Fokker-Planck Theorem. We conclude with some comments in §4.

We emphasise that we are looking for solutions through the methods of the Lie group analysis, in particular an analysis based upon the existence of Lie point symmetries. The absence of Lie point symmetries does not imply that (1.3), or any $(1 + 1)$-dimensional evolution equation, is not integrable. However, the absence of symmetry does make the existence of a closed-form solution unlikely.

2 Preliminary analysis

We analyse (1.3) for Lie point symmetries using Program LIE [7, 19]. For general $\nu(x)$ we find that there are the three Lie point symmetries

$$\Gamma_1 = f(t,x)\exp[-J/b^2]\partial_J \qquad (2.1a)$$

$$\Gamma_2 = \partial_J \qquad (2.1b)$$

$$\Gamma_3 = \partial_t, \qquad (2.1c)$$

where $f(t,x)$ is a solution of the linear equation

$$\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \frac{\nu(x)}{b^2}f = 0 \qquad (2.1d)$$

in which the differential part is just the standard generator of an Itô process. The symmetries (2.1b) and (2.1c) are to be expected from the lack of presence of J and t in (1.3). However, the symmetry (2.1a) in conjunction with (2.1d) is unexpected and indicates that (1.3) is linearisable by means of a point transformation.

The linear equation (2.1d) immediately provides the clue for potential candidate functions, $\nu(x)$, for which (1.3) can display a richer structure of symmetry.

We recall [16, 1] that there is a close connection between the Lie point symmetries of a linear parabolic evolution equation and the Noether point symmetries of related Lagrangian systems. The connection is made directly between the Noether point symmetries of the Action Integral and the additional Lie point symmetries of the corresponding Schrödinger equation. However, once the transition to the Schrödinger Equation is made, the connection may be extended to any evolution equation related to the Schrödinger Equation by means of a point transformation. In particular the transformation $t \longrightarrow it$ which renders the Schrödinger Equation into a heat equation maintains the Lie point symmetries and once the transition is made the Black-Scholes equation and others of Financial Mathematics are connected in a single symmetry approach.

In the case of the standard Lagrangian

$$L = \tfrac{1}{2}\dot{x}^2 - V(t,x) \qquad (2.2)$$

and its corresponding time-dependent Schrödinger Equation

$$2i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - 2V(t,x)u = 0 \qquad (2.3)$$

the Noether point symmetries of the former are characterised by the potential as

V	no sym	algebra
$V(t,x)$	0	$-$
$V(x)$	1	A_1
$\omega^2x^2+\dfrac{h^2}{x^2}$	3	$sl(2,R)$
$\mu^2+\dfrac{h^2}{x^2}$	3	$sl(2,R)$
ω^2x^2	5	$sl(2,R)\oplus_s 2A_1$
μ^2x	5	$sl(2,R)\oplus_s 2A_1$
μ^2	5	$sl(2,R)\oplus_s 2A_1$

(2.4)

and the Lie point symmetries of the latter by

V	no sym	algebra
$V(t,x)$	$0+1+\infty$	$A_1\oplus_s\infty A_1$
$V(x)$	$1+1+\infty$	$2A_1\oplus_s\infty A_1$
$\omega^2x^2+\dfrac{h^2}{x^2}$	$3+1+\infty$	$\{A_1\oplus sl(2,R)\}\oplus_s\infty A_1$
$\mu^2+\dfrac{h^2}{x^2}$	$3+1+\infty$	$\{A_1\oplus sl(2,R)\}\oplus_s\infty A_1$
ω^2x^2	$5+1+\infty$	$\{sl(2,R)\oplus_s W\}\oplus_s\infty A_1$
μ^2x	$5+1+\infty$	$\{sl(2,R)\oplus_s W\}\oplus_s\infty A_1$
μ^2	$5+1+\infty$	$\{sl(2,R)\oplus_s W\}\oplus_s\infty A_1$,

(2.5)

where W is the Heisenberg-Weyl algebra with

$$[\Sigma_1,\Sigma_2]_{LB}=0,\ [\Sigma_1,\Sigma_3]_{LB}=0,\ [\Sigma_2,\Sigma_3]_{LB}=\Sigma_1$$

in which it is emphasised that the properties are stated up to an equivalence transformation $x\longrightarrow(x-\alpha(t))/\rho(t)$, $t\longrightarrow\int\rho^{-2}(t)\mathrm{d}t$ in the classical context and the natural related transformation of the Schrödinger Equation [13, 14].

In (1.3), due to (2.1d), $\nu(x)$ plays the role of a potential. Consequently from (2.5) we realise that there are several cases for which the additional symmetry could lead to a similarity solution of (1.3) compatible with the terminal condition. Naturally we should initially consider the most general case in which the Lie point symmetries are those of (2.1). The terminal condition $J(T,x)=0$ is in fact a double condition for, as one recalls, all variables are regarded as independent in the Lie analysis for the purposes of partial differentiation. Thus we have the dual conditions

$$t=T\quad\text{and}\quad J=0 \tag{2.6}$$

to be satisfied. In the case of the latter the former also applies. If we take the general symmetry (which really is *the* Lie point symmetry of the equation; in an analysis we separate an n-parameter symmetry into n one-parameter symmetries since this gives the opening to the consideration of the algebraic structure of the symmetries; if one considers the symmetry as an n-parameter differential operator, it is difficult to construct an algebra of a symmetry with itself),

$$\Gamma = a_1\partial_J + a_2\partial_t, \tag{2.7}$$

the requirements of (2.6) insist that

$$a_2 = 0 \quad \text{and} \quad a_1 = 0. \tag{2.8}$$

This is not a recipe for symmetrical progress!

We take cognisance of the particular 'potentials' of (2.4) and (2.5). In turn we put $\nu(x)$ to be a particular function of the classes listed in (2.4) and (2.5) to determine the symmetries. Then we investigate the possibility that the particular representations are compatible with the terminal condition. In the cases that is be so, we determine the similarity solution.

3 Symmetries and solutions

We commence with the 'potentials' which allow the greatest amount of symmetry.

3.1 The case $\nu(x) = \mu^2$

The symmetries are

$$\begin{aligned}
\Gamma_1 &= f(t,x)\exp[J/b^2]\partial_J \\
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= t\partial_t + \tfrac{1}{2}(x+at)\partial_x + \mu^2 t\partial_J \\
\Gamma_5 &= t^2\partial_t + tx\partial_x + \tfrac{1}{2}\left(b^2t - 2\mu^2tx - (x-at)^2\right)\partial_J \\
\Gamma_6 &= \partial_x \\
\Gamma_7 &= t\partial_x - (x-at)\partial_J,
\end{aligned} \tag{3.1}$$

where $f(t,x)$ is a solution of

$$\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \frac{\mu^2}{b^2}f = 0. \tag{3.2}$$

The general Lie point symmetry of (1.3) with $\nu(x) = \mu^2$ is

$$\Gamma = \sum_{i=2}^{7} a_i\Gamma_i, \tag{3.3}$$

where the a_i, $i = 2, 7$, are constants to be determined.

The application of Γ to the terminal, *ie* $t = T$, gives

$$a_3 + a_4 T + a_5 T^2 = 0 \tag{3.4}$$

and to the terminal condition itself gives

$$a_2 + a_4\mu^2 T + \tfrac{1}{2}a_5\left(b^2T - 2\mu^2 T x - (x - aT)^2\right) - a_7(x - aT) = 0. \tag{3.5}$$

From (3.5) we obtain three relationships, *videlicet*

$$\begin{array}{ll} x^2 & a_5 = 0 \\ x & a_7 = 0 \\ - & a_2 + a_4\mu^2 T = 0 \end{array}$$

and from (3.4) we have

$$a_3 + a_4 T = 0$$

whence

$$a_2 = -a_4\mu^2 T, \quad a_3 = -a_4 T \tag{3.6}$$

so that we find the system of equation and terminal condition are invariant under the two Lie point symmetries

$$\begin{aligned} \Lambda_1 &= \partial_x \\ \Lambda_2 &= (T - t)\partial_t + \mu^2 (T - t)\partial_J. \end{aligned} \tag{3.7}$$

We can now seek a similarity solution of (1.3) with $\nu(x) = \mu^2$ which is compatible with the given terminal condition.

Invariance under Λ_1 means that $J = J(t)$ only. The associated Lagrange's system of Λ_2 is

$$\frac{\mathrm{d}t}{T - t} = \frac{\mathrm{d}J}{\mu^2 (T - t)},$$

ie the characteristic is $\omega = J - \mu^2 t$. From (1.3) with $\nu(x) = \mu^2$ we obtain the similarity solution

$$J = K - \mu^2 t$$

so that the solution which is consistent with the terminal condition is

$$J = \mu^2 (T - t). \tag{3.8}$$

3.2 The case $\nu(x) = \mu^2 x$

The same number of symmetries is obtained. They have the expressions

$$
\begin{aligned}
\Gamma_1 &= f(t,x)\exp[J/b^2]\partial_J \\
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= \partial_x - \mu^2 t\partial_J \\
\Gamma_5 &= t\partial_x + \left(at - \tfrac{1}{2}\mu^2 t^2 - x\right)\partial_J \\
\Gamma_6 &= 2t\partial_t + \left(at + x + \tfrac{3}{2}\mu^2 t^2\right)\partial_x + \mu^2\left(at^2 - 3tx - \tfrac{1}{2}\mu^2 t^3\right)\partial_J \\
\Gamma_7 &= 2t^2\partial_t + \left(2tx + \mu^2 t^3\right)\partial_x + \left(b^2 t + \mu^2 a t^3 - 3\mu^2 t^2 x - (x-at)^2 - \tfrac{1}{4}\mu^4 t^4\right)\partial_J,
\end{aligned}
\tag{3.9}
$$

where $f(t,x)$ is a solution of

$$
\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \frac{\mu^2}{b^2}xf = 0. \tag{3.10}
$$

We take the general combination of symmetries of (3.3) and apply it to the terminal condition (2.6). The first of (2.6) gives

$$
a_3 + 2Ta_6 + 2T^2 a_7 = 0 \tag{3.11}
$$

and the second

$$
\begin{aligned}
&a_2 - \mu^2 T a_4 + \left(aT - \tfrac{1}{2}\mu^2 T^2 - x\right)a_5 + \left(\mu^2 a T^2 - 3\mu^2 T x - \tfrac{1}{2}\mu^4 T^3\right)a_6 \\
&+ \left(2aTx + \mu^2 a T^3 - 3\mu^2 T^2 x - a^2 T^2 + b^2 T - x^2 - \tfrac{1}{4}\mu^4 T^4\right)a_7 = 0.
\end{aligned}
\tag{3.12}
$$

In (3.12) x is a variable, coefficients of powers of x are separately zero and we obtain the three conditions

$$
\begin{aligned}
x^2 &: \quad a_7 = 0 \\
x &: \quad a_5 = -3\mu^2 T a_6 \\
- &: \quad a_2 = \mu^2 T a_4 + \left(2\mu^2 a T^2 - \mu^4 T^3\right)a_6
\end{aligned}
\tag{3.13}
$$

so that we have a two-parameter (a_4 and a_6 are the parameters) symmetry which is consistent with the terminal condition (2.6). We write this symmetry as the two one-parameter symmetries

$$
\Sigma_1 = \partial_x - \mu^2(t-T)\partial_J \tag{3.14a}
$$

$$
\begin{aligned}
\Sigma_2 &= 2(t-T)_t + \left(at + x + \tfrac{3}{2}\mu^2 t^2 - 3\mu^2 tT\right)\partial_x \\
&\quad + \mu^2(t-T)\left\{a(t-2T) - \tfrac{1}{2}\mu^2\left(t^2 - 2tT - 2T^2\right) - 3x\right\}\partial_J.
\end{aligned}
\tag{3.14b}
$$

We note that $[\Sigma_1, \Sigma_2]_{LB} = \Sigma_1$, *ie* Σ_1 is the normal subgroup which reinforces the attraction of the simplicity of Σ_1 for use to determine the similarity solution.

The invariants of (3.14a) are determined by the solution of the associated Lagrange's system

$$
\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{-1} = \frac{\mathrm{d}T}{\mu^2(t-T)} \tag{3.15}
$$

and are t and $J + \mu^2(t-T)x$ so that we see a similarity solution of the form

$$J(t,x) = f(t) - \mu^2(t-T)x. \tag{3.16}$$

We substitute this into

$$\frac{\partial J}{\partial t} + a\frac{\partial J}{\partial x} + \tfrac{1}{2}b^2\frac{\partial^2 J}{\partial x^2} - \tfrac{1}{2}\left(\frac{\partial J}{\partial x}\right)^2 + \mu^2 x = 0 \tag{3.17}$$

to find that

$$f(t) = K + \tfrac{1}{2}a\mu^2(t-T)^2 + \tfrac{1}{6}\mu^4(t-T)^3, \tag{3.18}$$

where K is a constant of integration which is zero when the terminal condition is applied. Thus the similarity solution corresponding to Σ_1 is

$$J(t,x) = \tfrac{1}{2}a\mu^2(t-T)^2 + \tfrac{1}{6}\mu^4(t-T)^3 - \mu^2(t-T)x. \tag{3.19}$$

The application of Σ_2 to (3.19) is an identity, *ie* (3.19) is a similarity solution common to both symmetries which satisfy the terminal condition (2.6).

3.3 The case $\nu(x) = \frac{1}{2}\omega^2 x^2$

LIE returns the symmetries

$$\begin{aligned}
\Gamma_1 &= f(t,x)\exp[J/b^2]\partial_J \\
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= \exp[\omega t]\left[\partial_x + (a-\omega x)\partial_J\right] \\
\Gamma_5 &= \exp[-\omega t]\left[\partial_x + (a+\omega x)\partial_J\right] \\
\Gamma_6 &= \exp[2\omega t]\left[\partial_t + \omega x\partial_x + \left(a\omega x - \tfrac{1}{2}a^2 + \tfrac{1}{2}\omega b^2 - \omega^2 x^2\right)\partial_J\right] \\
\Gamma_7 &= \exp[-2\omega t]\left[\partial_t - \omega x\partial_x - \left(a\omega x + \tfrac{1}{2}a^2 + \tfrac{1}{2}\omega b^2 + \omega^2 x^2\right)\partial_J\right],
\end{aligned} \tag{3.20}$$

where $f(t,x)$ is a solution of

$$\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \tfrac{1}{2}\frac{\omega^2}{b^2}x^2 f = 0. \tag{3.21}$$

When we examine (1.3) with $\nu(x) = \frac{1}{2}\omega^2 x^2$, we realise that the first case ($\nu(x) = \mu^2$) considered was somewhat specific for the analysis of the present case resembles Case 2 in its structure if not its details. Consequently we can simply highlight the main steps. The application of the general symmetry (3.3) to the terminal condition (2.6) leads to the two symmetries

$$\Sigma_1 = \cosh\omega(t-T)\partial_x + \left[a(\cosh\omega(t-T) - 1) - \omega x\sinh\omega(t-T)\right]\partial_J \tag{3.22a}$$

$$\begin{aligned}
\Sigma_2 = {} & \sinh 2\omega(t-T)\partial_t + \left[a\exp[\omega(t-T)] + \omega x\cosh 2\omega(t-T)\right]\partial_x \\
& + \left[a(a-\omega x)\exp[\omega(t-T)] - a^2 - \tfrac{1}{2}\omega b^2\right. \\
& \left. + \left(a\omega x + \tfrac{1}{2}\omega b^2\right)\cosh 2\omega(t-T) - \left(\tfrac{1}{2}a^2 + \omega^2 x^2\right)\sinh 2\omega(t-T)\right]_J .
\end{aligned} \tag{3.22b}$$

Again we are favoured by fortune in that the Lie Bracket of Σ_1 and Σ_2 gives $\omega\Sigma_1$ and we use Σ_1 to determine the similarity solution which is

$$\begin{aligned} J(t,x) &= ax\left[1-\operatorname{sech}\omega(t-T)\right]+\tfrac{1}{2\omega}\left(a^2-\omega^2x^2\right)\tanh\omega(t-T) \\ &\quad +\tfrac{1}{2}b^2\log\cosh\omega(t-T)-\tfrac{1}{2}a^2(t-T). \end{aligned} \tag{3.23}$$

After not a little calculation one finds that Σ_2 annihilates the solution (3.23) and again we have just the single solution.

In all three cases the single solution is the unique solution as predicted by the Fokker-Planck Theorem. It is interesting how two disparate approaches to the same problem lead to the same uniqueness.

We now turn to the functions $\nu(x)$ which give lesser symmetry.

3.4 The case $\nu(x) = \frac{1}{2}\frac{\mu^2}{x^2}$

This 'potential' has a rich history dating back to the time of Newton and his young colleague Cotes and occurring in more recent aspects as the Ermakov system [6] and the Ermakov-Pinney equation [18]. The relationship of the latter to the former was recently highlighted [15].

The equation

$$\frac{\partial J}{\partial t}+a\frac{\partial J}{\partial x}+\tfrac{1}{2}b^2\frac{\partial^2 J}{\partial x^2}-\tfrac{1}{2}\left(\frac{\partial J}{\partial x}\right)^2+\tfrac{1}{2}\frac{\mu^2}{x^2}=0 \tag{3.24}$$

possesses the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= f(t,x)\exp[-J/b^2]\partial_J \\ \Gamma_2 &= \partial_J \\ \Gamma_3 &= \partial_t \\ \Gamma_4 &= 2t\partial_t+x\partial_x+\left(ax-a^2t\right)\partial_u \\ \Gamma_5 &= 2t^2\partial_t+2tx\partial_x+\left(b^2t-(x-at)^2\right)\partial_J, \end{aligned} \tag{3.25}$$

where $f(t,x)$ is a solution of

$$\frac{\partial f}{\partial t}+\tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2}+a\frac{\partial f}{\partial x}-\tfrac{1}{2}\frac{\mu^2}{b^2x^2}f=0. \tag{3.26}$$

The terminal condition (2.6) produces the two conditions

$$a_3+2Ta_4+2T^2a_5=0 \tag{3.27a}$$

$$a_2+(ax-a^2T)a_4+\left(2aTx-a^2T^2+b^2T-x^2\right)a_5=0 \tag{3.27b}$$

and (3.27b) separates into three separate conditions so that the four parameters $a_2 - a_5$ satisfy a four-dimensional homogeneous system based on four linearly independent symmetries. Not surprisingly only the trivial solution exists so that there is no similarity solution for (1.2) when μ is a constant.

3.5 The case $\nu(x) = \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\frac{\mu^2}{x^2}$

In Mechanics this is the classic Ermakov potential in which the ω is $\omega(t)$ and about which there is a very rich literature. In the present context we find the Lie point symmetries

$$\begin{aligned}
\Gamma_1 &= f(t,x)\exp[J/b^2]\partial_J \\
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= \exp[2\omega t]\left[\partial_t + \omega x\partial_x + \left(a\omega x - \tfrac{1}{2}a^2 + \tfrac{1}{2}\omega b^2 - \omega^2 x^2\right)\partial_J\right] \\
\Gamma_5 &= \exp[-2\omega t]\left[\partial_t - \omega x\partial_x - \left(a\omega x + \tfrac{1}{2}a^2 + \tfrac{1}{2}\omega b^2 + \omega^2 x^2\right)\partial_J\right],
\end{aligned} \tag{3.28}$$

where $f(t,x)$ is a solution of

$$\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \tfrac{1}{2}\left(\frac{\omega^2}{b^2}x^2 + \frac{\mu^2}{b^2x^2}\right)f = 0. \tag{3.29}$$

As in the previous case there is no symmetry which is compatible with the terminal condition, (2.6).

3.6 The case $\nu(x) = \omega^2 + \frac{1}{2}\frac{\mu^2}{x^2}$

The Lie point symmetries are

$$\begin{aligned}
\Gamma_1 &= f(t,x)\exp[-J/b^2]\partial_J \\
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= 2t\partial_t + x\partial_x + \left(ax - a^2t2\omega^2 t\right)\partial_u \\
\Gamma_5 &= 2t^2\partial_t + 2tx\partial_x + \left(b^2t - (x-at)^2\right)\partial_J.
\end{aligned} \tag{3.30}$$

It is perhaps unsurprising that the symmetries are also not compatible with the terminal condition (2.6).

3.7 The case $\nu(x) = \omega^2 x + \frac{1}{2}\frac{\mu^2}{x^2}$

This case is even worse than the previous three as only the three general symmetries of (2.1) are obtained and we have already seen that those symmetries are incompatible with the terminal condition (2.6).

4 Discussion

We have examined the equation for mean-variance hedging proposed by Heath *et al* for the existence of similarity solutions based on symmetries of the nonlinear evolution equation of Hamilton-Jacobi-Bellman type modelling the process which are consistent with the terminal condition for the model. There are three cases for which we can obtain a unique solution to the problem of

$$\begin{aligned}&\frac{\partial J}{\partial t}+a\frac{\partial J}{\partial x}+\tfrac{1}{2}b^2\frac{\partial^2 J}{\partial x^2}-\tfrac{1}{2}\left(\frac{\partial J}{\partial x}\right)^2+\nu(x)=0\\&J(T,x)=0\end{aligned}\tag{4.1}$$

by means of the techniques of Lie symmetry analysis and they are when $\nu(x)$ has one of the forms μ^2, $\mu^2 x$ or $\frac{1}{2}\omega^2x^2$. In the standard representation of (4.1) $\nu(x)$ is written as μ^2/x^2 and we have seen that for constant $\mu\neq 0$ in this form there cannot be a similarity solution since the existing Lie point symmetries of the equation cannot fit with the terminal condition.

The forms of (1.3) with (2.6) which were amenable to solution by means of technique of Lie's symmetry analysis were all of maximal point symmetry for a $(1+1)$ evolution equation. Although something akin to chaos has been speculated for (1+several) evolution equations [1], one does not expect to find such irregularity in the $(1+1)$ equation, *ie* one expects the equation to be integrable even if it be not possible to find a solution in something approaching closed-form simply due to the absence of sufficient symmetry. This then returns us to the whole question of what symmetries – potential, nonclassical, what have you – should be considered. Here we have restricted our attention to Lie point symmetries for this enables one to make ready connection with the traditional areas of Classical Mechanics and Quantum Mechanics. Evidently there is room for further exploration of this topic through more general forms of symmetry.

The Lie symmetry analysis of (1.3) for general $\nu(x)$ was quite revealing for it showed that there exists an infinite number of 'solution' symmetries and this is indicative of a linearising transformation. Indeed in this case, if we look at the infinite class of Lie point symmetries, we find the clue to the linearising transformation since

$$\begin{aligned}\Gamma_1&=\exp\left[J(t,x)/b^2\right]f(t,x)\partial_J\\&=-bf(t,x)\partial_{\left[\exp\left[-J(t,x)/b^2\right]\right]}\end{aligned}\tag{4.2}$$

and so one can see that the transformation $u=\exp\left[-J(t,x)/b^2\right]$ brings Γ_1 into the standard form of a solution symmetry. In the process it makes Γ_2 the homogeneity symmetry characteristic of linear equations of all provenances.

The type of 'potential', $\nu(x)$, which we have considered in the various cases has been autonomous. In Classical Mechanics it is well-known that some nonautonomous potentials may be equally treated without loss of generality. The successful 'potentials' of §3 were μ^2, $\mu^2 x$ and $\frac{1}{2}\omega^2x^2$. Classically the first has no impact as a constant added to a Lagrangian makes no difference to the Euler-Lagrange equations of motion. The second represents motion under the influence of an uniform gravitational field à la Galileo and the third is the simple harmonic oscillator. One would not expect a replacement of μ by $\mu(t)$ or ω by $\omega(t)$ to make a theoretical difference to the degree of difficulty of the solution of the system although one knows that the practical complications have produced a generous literature. However, would such a generalisation impact upon the terminal condition?

We consider the case $\nu(t,x)=\mu^2(t)$ to give a flavour of the situation. The Lie point symmetries are

$$\Gamma_1=f(t,x)\exp[J/b^2]\partial_J$$

$$\begin{aligned}
\Gamma_2 &= \partial_J \\
\Gamma_3 &= \partial_x \\
\Gamma_4 &= \partial_t - \mu^2(t)\partial_J \\
\Gamma_5 &= t\partial_x + (at - x)\partial_J \\
\Gamma_6 &= 2t\partial_t + (at + x)\partial_x - 2t\mu^2(t)\partial_J \\
\Gamma_7 &= 2t^2\partial_t + 2tx\partial_x + \left(b^2 t - 2t^2\mu^2(t) - (x - at)^2\right)\partial_J,
\end{aligned} \tag{4.3}$$

where $f(t,x)$ is a solution of

$$\frac{\partial f}{\partial t} + \tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - \frac{\mu^2(t)}{b^2}f = 0. \tag{4.4}$$

We note that (4.4) is essentially the same as (3.2), but that there have been some changes in the symmetries from those given in (3.1).

The terminal condition imposes the constraints

$$a_4 + 2Ta_6 + 2Ta_7 = 0 \tag{4.5a}$$

$$\begin{aligned}
&a_2 - \mu^2(T)a_4 + (aT - x)a_5 - 2T\mu^2(T)a_6 \\
&+(2aTx - a^2T^2 + b^2T - x^2 - 2T^2\mu^2(T))a_7 = 0
\end{aligned} \tag{4.5b}$$

on the coefficients of the general symmetry (3.3). In (4.5b) the x^2 in the coefficient of a_7 imposes that $a_7 = 0$. It then follows that $a_5 = 0$. When we substitute for a_4 from (4.5a) into what is left of (4.5b), we find that $a_2 = 0$. However, we still have two arbitrary constants and the two symmetries compatible with the terminal condition are

$$\Sigma_1 = \partial_x \tag{4.6a}$$

$$\Sigma_2 = 2(t - T)\left(\partial_t - \mu^2(t)_J\right). \tag{4.6b}$$

From (4.6a) it is evident that $J = f(t)$ and the integration of (1.3) with $\nu(t) = \mu^2(t)$ is trivial. We have

$$\begin{aligned}
J &= K - \int_{t_0}^{t} \mu(s)^2 \mathrm{d}s \\
&= \int_{t_0}^{T} \mu(s)^2 \mathrm{d}s - \int_{t_0}^{t} \mu(s)^2 \mathrm{d}s
\end{aligned} \tag{4.7}$$

when the constant is evaluated using the terminal condition. The action of the second symmetry gives an identity. It is amusing to note that Σ_2 vanishes at the terminal time.

Thus we see that the analysis can be successfully extended to an explicitly time-dependent situation.

In this paper we have examined from the viewpoint of the Lie point symmetry analysis an equation for mean-variance hedging containing a parametric function depending upon the price of the underlying object. For three versions of the parametric function we were able to obtain the unique solution to the equation plus terminal condition. The three functions which admitted a solution were those which provided the maximal set of Lie point symmetries for the nonlinear evolution equation used to model the phenomenon.

In the process the Lie analysis revealed that the equation was linearisable due to the existence of an infinite number of 'solution' symmetries, so-called because the coefficient functions of the symmetries are solutions of a differential equation. In the case of linear equations the solution symmetries come from solutions of a related linear equation, often the equation itself. In this case of a nonlinear evolution equation the solution symmetries come from a linear evolution equation to which the nonlinear equation can be transformed by means of a point transformation. In the case of the Hamilton-Jacobi-Bellman equations of less than maximal symmetry considered here the number of Lie point symmetries was insufficient to be compatible with the terminal condition. One would expect that this would generally be the case. To our knowledge the first application of Lie symmetry analysis to an Hamilton-Jacobi-Bellman equation in Financial Mathematics was the study made by Ibragimov and Gazizov of the Black-Scholes equation [9] which happens to be an equation of maximal symmetry in the usual cases for which the variance and risk-free interest rate are independent of the price of the stock. Had the equation contained a symmetry-destroying term, such as we have seen in several of the cases considered in this paper, there probably would never have been an history of the interaction between Lie symmetry analysis and the Hamilton-Jacobi-Bellman equations of Financial Mathematics.

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