A bivariate matrix Padé-type method of the 2-D filters

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Abstract—In this paper, according to the special generating polynomial, a class of bivariate matrix Padé-type approximation (BMPTA) is given by introducing a bivariate matrix-valued linear functional on the scalar polynomial space. An application in state-space realization of the 2-D filters is also given in the end.

Keywords—matrix bivariate Padé-type approximation, linear functional, 2-D filters

I. INTRODUCTION

The problem of matrix Padé approximants and the wide application has been studied for a long time in many fields, such as in scattering physics, multiport network synthesis, model reduction, design of multi-input multi-output digital filters[1,2]. Bose and Basu[3] introduced the classic bivariate matrix valued padé approximant and discussed the existence, uniqueness and recursive computation with inverse matrix. Author Gu[4] defined generalized inverse Thiele-type bivariate matrix Padé-type approximation, linear functional on the scalar polynomial approximation (BMPTA) is given by introducing a bivariate polynomial, a class of bivariate matrix Padé-type approximation (BGMPA) with scalar denominator polynomials. As compared the type bivariate matrix Padé approximant (BGMPA) with matrix-valued linear functional on the scalar polynomial approximation (BMPTA), in the inner product space (BMPTA). In this paper we give a special class of BMPTA by giving a scalar approximation in the inner product space (BMPTA). In this paper we give a special class of BMPTA by giving a scalar approximation in the inner product space (BMPTA).

II. DEFINITION OF BMPTA

Let $\hat{P}$ be the set of scalar polynomials in two real variable whose coefficients belong to the complex field $C$. Consider a bivariate matrix-valued function $f(x,y)$ has formal series in two variable $x$ and $y$

$$f(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} x^i y^j = c_{00} + (c_{10} x + c_{01} y) + (c_{20} x^2 + c_{11} x y + c_{02} y^2) + \ldots$$

where $c_{ij} \in C^{M \times N}$.

Let $\hat{\phi} : \hat{P} \rightarrow C^{M \times N}$ be a generalized linear function $\hat{\phi}(s^t) = c_{ij}$, $i, j = 0, 1, \ldots$

Suppose that $|t x| < 1, |s y| < 1$, we have

$$(1 - tx)^{-1} = 1 + tx + (tx)^2 + \ldots, (1 - sy)^{-1} = 1 + sy + (sy)^2 + \ldots$$

For the given series (1), from (2) and (3) we get

$$\hat{\phi}(s^t) = \left(\frac{1}{(1 - tx)^n} + \frac{1}{(1 - sy)^m}\right)$$

Let $V(x,y)$ be a bivariate scalar polynomial:

$$V(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{ij} x^i y^j$$

And assume $b_{n_1,n_2} \neq 0$. Define the bivariate matrix-valued polynomial $W(x,y)$:

$$W(x,y) = \hat{\phi}(V(x,y)) = V(t,s) - V(t,y) - V(x,s)$$

Where $\hat{\phi}$ acts on $t$ and $s$, and $x, y$ are parameters. Let

$$\tilde{V}(x,y) = x^n y^m V(x^{-1}, y^{-1})$$

$$\tilde{W}(x,y) = x^{n_1-1} y^{m_2-1} W(x^{-1}, y^{-1})$$

Definition 2.1 Let $\tilde{V}(x,y) \neq 0$, $f(x,y)$ be a matrix power series, then the matrix-valued rational function $R_{n_1,n_2}(x,y) = \frac{\tilde{W}(x,y)}{\tilde{V}(x,y)}$ is defined to be a bivariate matrix Padé-type approximation (BMPTA) of degree $(n_1 - 1, n_2 - 1 / n_1, n_2)$, which satisfies the approximation condition:

$$V(x,y)f(x,y) - W(x,y) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} d_{ij} x^i y^j$$

where all the coefficients $d_{ij}$ are $M \times N$ matrices of constants, and $V(x,y)$ is said to be the generating polynomial.

We denote a BMPTA of degree $(n_1 - 1, n_2 - 1 / n_1, n_2)$ by $(n_1 - 1, n_2 - 1 / n_1, n_2)_{f}(x,y)$.
Remark 2.2 In fact, the bivariate matrix power series in (1) can be expanded by the following form:
\[
f(x, y) = \sum_{i=0}^{\infty} c_{i0}x^i + \sum_{j=0}^{\infty} c_{0j}y^j + \sum_{k=1}^{\infty} c_{kj}x^ky^k
\]  
(9)

and the approximation condition in definition 2.1 indicates that 
\((n_1 - 1, n_2 - 1/n_1, n_2)\), \((x, y) \in R_{n_1, n_2}\) is to agree with (9) in all \(n_1n_2 = n_1 + (n_2 - 1) + m\) terms. These \(n_1, n_2\) terms in (9) include:

(i) \(n_1\) terms before \(x^{n_1} = \sum_{i=0}^{n_1} c_{i0}x^i\)

(ii) \(n_2 - 1\) terms before \(y^{n_2 - 1} = \sum_{j=0}^{n_2 - 1} c_{0j}y^j\)

\(\sum_{j=0}^{\infty} d_{ij}x^iy^j = c_{i0}x^i, \ldots, c_{n_1-1,0}x^{n_1-1}\)

Example 2.3 Let \(n_1 = n_2 = 3\) then \((2, 2/3, 3)\), \((x, y)\)

yi is to agree with (9) in \(n_1n_2 = 9\) terms:

(i) \(n_1 = 3\) terms: \(c_{00}, c_{10}x, c_{20}x^2\)

(ii) \(n_2 - 1 = 2\) terms: \(c_{01}y, c_{02}y^2\)

(iii) \(m = n_1n_2 - (n_1 + n_2) + 1 = 4\) terms:

\(c_{11}xy, c_{21}x^2y, c_{12}xy^2, c_{22}x^2y^2\)

III. A CONSTRUCTION OF BMPTA

Now we restrict the generating polynomial \(V_{n_1,n_2}(x, y) = u_{n_1}(x)v_{n_2}(y)\) Where
\(u_{n_1}(x) = (x - t_1)(x - t_2)\ldots(x - t_{n_1}), v_{n_2}(y) = (y - s_1)(y - s_2)\ldots(y - s_{n_2})\)

Obviously, the zeros of \(u_{n_1}\) : \(\{t_1 : i = 1, 2, \ldots, n_1\}\), the zeros of \(v_{n_2}\) : \(\{s_j : j = 1, 2, \ldots, n_2\}\)

From (6)\(\tilde{v}(x, y) = \phi(x^{-1}u(x^{-1}) - u(t)) \cdot y^{-1}v(y^{-1}) - v(s)\)  
(10)

Where \(\tilde{u}_{n_1}(x) = \prod_{i=1}^{n_1}(1 - t_i), \tilde{v}_{n_2}(y) = \prod_{j=1}^{n_2}(1 - s_j)\)

From (10), we define \(T_{n_1}(x) = x^{n_1} - u_{n_1}(x)\)

\(= a_0(x) + a_1(x) + \ldots + a_{n_1}(x)t^{n_1-1}\)

Let \(A = (a_j), B = (b_j) \in C^{M \times N}\), in [10] the matrix

direct inner product is defined by \(\langle A, B \rangle = \sum_{j=1}^{M} \sum_{i=1}^{N} a_{ij}b_{ij}\)

\(T_{n_1}(x)T_{n_2}(y)\) can be written as follows:

\(T_{n_1}(x)T_{n_2}(y) = a_0(x)b_0(y) + a_1(x)b_0(y)s + \ldots + a_{n_1}(x)b_{n_1}(y)s^{n_1-1} + a_0(x)b_0(y)\)

\(\text{Remark 2.2}\) is to agree with (9) in [12] only contains the zeros of:
\(n_1n_2 = n_1 + (n_2 - 1) + m\) terms before \(n_1, n_2\).

\(\phi(T_{n_1}(x)T_{n_2}(y)) = \phi = \langle t^{-1}s^{-1}\rangle\)

\([a_{i-1}(x)b_{j-1}(y)] =\)

\(H_{n_1,n_2} = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0n_2-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{10} & c_{11} & \cdots & c_{1n_2-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n_1-1,0} & c_{n_1-1,1} & \cdots & c_{n_1-1,n_2-1} \end{pmatrix}\)

\(G_{n_1,n_2} = [a_{i-1}(x)b_{j-1}(y)]\)

According to the above analysis, we can easily get the following theorem:

Theorem 3.1 Let \(\tilde{v}(0, 0) \neq 0\), then
\(n_1 - 1, n_2 - 1/n_1, n_2) \in R_{n_1, n_2}\)

\(\langle T_{n_1}(x)T_{n_2}(y) \rangle = \frac{1}{\tilde{u}_{n_1}(x)\tilde{v}_{n_2}(y)}\cdot H_{n_1,n_2} \cdot G_{n_1,n_2} \rangle\)

Remark: \(H_{n_1,n_2}\) in (11) only contains the coefficients of \(\langle f(x, y)\), and \(G_{n_1,n_2}\) in (11) only contains the zeros of.

\(V_{n_1,n_2}(x, y) = u_{n_1}(x)v_{n_2}(y)\)

IV. A PARTIAL REALIZATION OF THE 2-D FILTERS

The realization problem of two-dimensional linear filters is an important problem of model reduction in control.
theory. The input-output behavior of such a system is defined by formal power series in two-variables. The authors of [9] proved that if the power series is rational, the dynamics of the F-M linear filter is described by updating equations on finite-dimensional local state space.

Let $K$ be the complex plane $C$ or real plane $R$. Let the state equation of 2-D linear discrete time-invariable system for Fornasin-Marchesini model (F-M system) in [9] be 
\[
\sum_{i=n}^{m}(A_0, A_1, A_2, B, C) : \\
X(h+k+l)=A_0X(h+k)+A_1X(h+1,k)+A_2X(h,1,k+l)+BU(h,k) \\
Y(h,k)=CK(h,k).
\]

(12)

Where $X(h,k) \in K^q$ is the local state vector, $U(h,k)$ is the input vector, $Y(h,k)$ is the output vector, $A_0, A_1, A_2 \in K^{q \times q}$. Boundary conditions for (12) are given by $X(h,0), \dot{X}(0,k), h,k=0,1,2,\ldots$. By acting bivariate Z-transform on both sides of (12), we obtain the transfer function of F-M system 
\[
G = G(x,y) = C(I - A_0xy - A_1x - A_2y)^{-1}B 
\]

(13)

Assume that the matrix-valued polynomial $(I - A_0xy - A_1x - A_2y)^{-1}$ have the inverse matrix given by
\[
f(x,y) = (I - A_0xy - A_1x - A_2y)^{-1} = \sum_{i,j=0} c_{ij}x^iy^j \]

(14)

Where $c_{ij} \in K^{q \times q}$. Thus the output vector $Y$ can be written as
\[
Y(x,y) = C \sum_{i,j=0} X(h,k)x^iy^j = Cf(x,y)BU
\]

(15)

**Definition 4.1** Let 
\[
(n_1-1, n_2-1, n_1, n_2), (x,y) = \hat{W}(x,y) / \hat{V}(x,y)
\]

be a BMPTA for $f(x,y)$ in (14). Replace $f(x,y)$ by 
\[
(n_1-1, n_2-1, n_1, n_2), (x,y) \quad \text{in (13)(15)}
\]

respectively, such that 
\[
G_{n_1,n_2}(x,y) = C(n_1-1, n_2-1, n_1, n_2), (x,y)B, \\
Y_{n_1,n_2}(x,y) = C(n_1-1, n_2-1, n_1, n_2), (x,y)BU
\]

(16)

Thus 
\[
\sum_{n_1,n_2} = (A_0, A_1, A_2, B, C; (n_1-1, n_2-1, n_1, n_2), (x,y))
\]

is called a 2-D partial realization of type $(n_1-1, n_2-1, n_1, n_2)$ for F-M system.

**Example 4.2** Consider the following bivariate matrix power series:
\[
f(x,y) = (I - A_0xy - A_1x - A_2y)^{-1} = \sum_{i,j=0} c_{ij}x^iy^j
\]

Where

\[
A_0 = 0, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \\
A_2 = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 2 & 1 \end{pmatrix}^T, \quad C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}
\]

Step 1 Compute 
\[
V(x,y) = x^2y^2\hat{V}(x^{-1}, y^{-1})
\]

Step 2 Compute 
\[
W(x,y) = \hat{\phi}(ts + ty + xx + xy - t - x - 3x - 3y + 3)
\]

(14)

Step 3 Compute 
\[
G_{2,2}(x,y) = C(1,1/2,2) \cdot \hat{V}(x,y)B = \frac{1}{V(x,y)} C\hat{W}B
\]

(15)

We can verify that $(1,1/2,2)$, $(x,y)$ is to agree with (12) in $n_1n_2 = 4$ terms: $c_{00}, c_{10}x, c_{01}y, c_{11}xy$. From definition 4.1, $\sum_{2,2} = (A_0, A_1, A_2, B, C; (1,1/2,2))$ is the partial realization of type $(1,1/2,2)$ and 
\[
Y_{2,2}(x,y) = C(1,1/2,2) \cdot BU
\]

**REFERENCES**


