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The Asymptotic Behavior of the Solution of Boundary Value Problems for the sine-Gordon Equation on a Finite Interval

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Abstract

In this article we use the Fokas transform method to analyze boundary value problems for the sine-Gordon equation posed on a finite interval. The representation of the solution of this problem has already been derived using this transform method. We interchange the role of the independent variables to obtain an equivalent representation which can be used to study the asymptotic behavior for large times. We use this analysis to prove that the solution corresponding to constant boundary data is dominated for large times by the underlying similarity solution.

Dedicated to Francesco Calogero in occasion of his 70th birthday

1 Introduction

The sine-Gordon (sG) equation is a significant example of evolution problem belonging to the class of integrable PDEs in one space dimensions.

We consider the hyperbolic form of this equation, which is given by

$$q_{xt} + \sin q = 0, \quad q = q(x,t). \tag{1.1}$$

The initial value problem posed on the full line can be solved by the Inverse Scattering Transform, and the associated spectral problem is the one studied by Zakharov and Shabat [2].

A boundary value problem on the half line $x > 0$ or on any finite interval $(0, L)$, with $t > 0$, is well posed as soon as one initial and one boundary conditions (at $t = 0$ and $x = 0$ respectively) are prescribed, and there are no additional boundary values to be characterized. For this reason, the analysis of this equation is simpler than the analysis of other integrable evolution PDEs, such as NLS or KdV, which can also be solved by the inverse scattering transform.

We are interested in the large time asymptotic behavior of the solution of such problems, and in understanding the generation of solitons and assessing their asymptotic role. It is well known that the solution of the initial value problem on the whole line decomposes asymptotically into a finite train of special solutions, for any given initial waveform carrying sufficient energy [9]. These special solutions are either soliton-kinks, or soliton-breathers. In the classical formulation, these solutions correspond to the poles of the spectral problem lying in the upper (or the lower) half of the complex spectral plane. However, the question of what happens when one considers the initial and boundary value problem posed on the half line $x > 0$ is still open. One can clearly construct boundary conditions such that this problem can be considered a restriction of the whole line problem, and then its solution must be consistent with the corresponding solution of the Cauchy problem and therefore may produce soliton asymptotics. However, generic boundary conditions yield a problem that cannot be considered as the restriction to $x > 0$ of an initial value problem, and the asymptotic behavior can be rather different.

This matter has been unexpectedly controversial. Consider the particular boundary value problem obtained by prescribing the constant boundary data $q(0,t) = \gamma$. The analysis of this equation using the classical Zakharov-Shabat spectral problem, either by the usual inverse scattering transform [11] or by the generalization proposed by Fokas [3, 4], yields a spectral transform with infinitely many zeros in the upper half complex spectral plane. Since in the full line case these zeros are associated with the soliton components of the asymptotics of the equation, one might expect that, as $t \to \infty$, the solution, in the half line case, will decompose into infinitely many solitonic components.

However, the zeros of the Zakharov-Shabat formulation do not appear to be relevant for soliton generation. Leon in [10] has argued that these zeros are spurious and an artifact of the lack of continuity of the spectral functions near the real axis. He associates to the sine-Gordon equation (1.1) a spectral problem, different from the usual one, defined by a weakly commuting Lax pair. Using the spectral data associated to this problem, he showed that, in correspondence with a constant boundary datum and vanishing initial conditions, the relevant spectral function has only a finite number of zeros as $t \to \infty$. However, whether these zeros are indeed related to soliton solutions which are important for the large time asymptotics of the whole solution was not verified. We note that although the example presented in [3] and [10] generates the same spectral function, in the former case the problem is posed on a finite interval, with nonzero initial conditions, while in the latter case, the problem is posed on the half line with a vanishing boundary condition, and a stepwise constant boundary datum. Since this boundary datum is not differentiable, this is somewhat of a spurious example, and one should really pose this problem on a finite interval, with constant, but nonzero, initial condition.

The only explicit asymptotic results we are aware of in the literature for the sine-Gordon equation are the results in [9] on the generation of solitons for the equation on the infinite line, with data that are not necessarily decreasing. We cite also the results in [8] on the same problem for the stimulated Raman scattering (SRS) on the half line. In both cases, the behavior at large times is dominated by asymptotic solitons, generated by the continuous rather than the discrete spectrum. The points in the discrete spectrum are however assumed (and we stress that this is an *assumption*) to be finitely many, and their asymptotic contribution is, as usual, in the form of canonical solitons.

In this paper, we find an alternative representation for the solution of such boundary value problems, and prove that the large time asymptotic behavior of the solution corresponding to constant boundary conditions is dominated by the similarity solution. Our main result is the following.

Theorem 1. *Consider the sine-Gordon equation*

$$q_{xt} + \sin q = 0, \quad 0 < x < L, \ \ t > 0,$$

with given initial and boundary conditions

$$q(x,0) = q_0(x), \quad q(0,t) = \gamma, \quad q_0(0) = \gamma,$$

where $\gamma \neq 0$ is a constant. The leading behavior of this boundary value problem is given by

$$q_t(x,t) = \frac{\xi}{2t}\tilde{q}(\xi), \quad \xi = \sqrt{xt}, \tag{1.2}$$

where

$$\tilde{q}(\xi) = -\frac{2\sin\gamma}{\pi(1+\cos\gamma)}\int_{-\infty}^{\infty}\overline{N_2}(\xi,\lambda)\mathrm{e}^{-i\xi(\lambda+\frac{1}{\lambda})}d\lambda, \tag{1.3}$$

and the functions N_1, N_2 are the unique solution of the system of linear integral equations

$$\begin{aligned} N_1(\xi,\lambda) &= \frac{\sin\gamma}{2\pi i(1+\cos\gamma)}\int_{-\infty}^{\infty}\overline{N_2}(\xi,\lambda')\mathrm{e}^{-i\xi(\lambda'+\frac{1}{\lambda'})}\frac{d\lambda'}{\lambda'-(\lambda+i0)}, \\ N_2(\xi,\lambda) &= 1-\frac{\sin\gamma}{2\pi i(1+\cos\gamma)}\int_{-\infty}^{\infty}\overline{N_1}(\xi,\lambda')\mathrm{e}^{-i\xi(\lambda'+\frac{1}{\lambda'})}\frac{d\lambda'}{\lambda'-(\lambda+i0)}. \end{aligned} \tag{1.4}$$

We are motivated by the results in [5] on a similar boundary value problem for the Transient Stimulated Raman Scattering (SRS). Note that the Riemann-Hilbert problem associated with the solution of SRS has the same form as the one associated with sG. In what follows we combine the results of [5] with the analysis of the sine-Gordon equation on a finite interval given in [3].

We restrict ourselves to data given on a finite space interval. Since this equation is an evolution equation with respect not only to time, but also to space, this restriction is not as significant as for other evolution (in time) PDEs. Indeed, the solution $q(x_0,t_0)$ depends only on the values of (x,t) satisfying $x < x_0$, $t < t_0$. Therefore the value of the solution of the boundary value problem posed on the half line, at any given point x_0, should be identical to the value of the solution, at the same point x_0, obtained by solving the problem posed on the finite interval $(0,x_0)$.

We first briefly review the solution method for the linearized equation. Using the fact that the roles of the two variables x and t are symmetric in this equation, we interchange them in the derivation of the spectral problem, and hence give an expression for the solution which is not the usual one but is equivalent to it. We then use the same idea to analyze the nonlinear equation. Finally, we consider the special case of constant boundary data, and show that in this case, the large time asymptotics are dominated by the similarity solution, while the contribution of the zeros of the spectral functions (if any exist) is exponentially small.

2 The linearized equation

We consider the small q limit of the sine-Gordon equation, given by

$$q_{xt} + q = 0, \quad x \in (0, L), \;\; t \in (, 0, T). \tag{2.1}$$

Our first remark that this is an evolution equation *in both x and t*. This implies, as we show below and is well know [7], that the solution at any point (L, T) depends only on the values of $q(x, t)$ for $x < L$ and $t < T$. We therefore consider this equation in the domain $(0, L) \times (0, T)$, where L and T are arbitrary but finite.

This equation is supplemented with the following initial and boundary conditions, taken to be sufficiently smooth and compatible:

$$q(x, 0) = q_0(x), \; x \in (0, L) \quad q(0, t) = f(t), \; t \in (0, T), \quad f(0) = q_0(0). \tag{2.2}$$

In what follows, we assume that a solution of this boundary value problem exists, and we derive a representation for it.

We introduce the spectral parameter k and rewrite the equation in the form

$$\left(\mathrm{e}^{-ikt-\frac{i}{k}x} q_t\right)_x + \left(\mathrm{e}^{-ikt-\frac{i}{k}x} \frac{i}{k} q\right)_t = 0, \quad k \in \mathbb{C},$$

or equivalently as the compatibility condition of the Lax pair

$$\mu_t - ik\mu = q_t, \quad \mu_x - \frac{i}{k}\mu = -\frac{i}{k}q. \tag{2.3}$$

In view of the analogy with the nonlinear case, we will derive the solution representation using the Lax pair. The solutions μ of (2.3) are of the form

$$\mu(x, t, k) = -\frac{i}{k}\mathrm{e}^{ik(t-t_0)} \int_{x_0}^{x} \mathrm{e}^{(i/k)(x-y)} q(y, t_0)dy + \int_{t_0}^{t} \mathrm{e}^{ik(t-s)} q_t(x, s)ds,$$

where x_0, t_0 are two arbitrary values.

Choosing $x_0 = L$, $t_0 = 0$, we obtain a solution which is analytic and bounded for $k \in \mathbb{C}^+$:

$$\mu^+(x, t, k) = \frac{i}{k}\mathrm{e}^{ikt} \int_{x}^{L} \mathrm{e}^{(i/k)((x-y)} q(y, 0)dy + \int_{0}^{t} \mathrm{e}^{ik(t-s)} q_t(x, s)ds, \tag{2.4}$$

while choosing $x_0 = 0$, $t_0 = T$, we obtain a solution which is analytic and bounded for $k \in \mathbb{C}^-$:

$$\mu^-(x, t, k) = -\frac{i}{k}\mathrm{e}^{ik(t-T)} \int_{0}^{x} \mathrm{e}^{(i/k)((x-y)} q(y, T)dy - \int_{t}^{T} \mathrm{e}^{ik(t-s)} q_t(x, s)ds. \tag{2.5}$$

It is easy to verify, by integration by parts, that these functions satisfy

$$\begin{aligned} \mu &= \frac{q_t}{ik} + O(\frac{1}{k^2}), \quad k \to \infty, \\ \mu &= q + ikq_x + O(k^2), \quad k \to 0. \end{aligned} \tag{2.6}$$

The difference of these two solutions satisfies the homogeneous Lax pair, hence it is given by

$$\mu^+ - \mu^- = \mathrm{e}^{ikt+\frac{i}{k}x}\rho(k), \quad \rho(k) = \frac{i}{k}\int_0^L \mathrm{e}^{-(i/k)y} q(y,0)dy + \int_0^T \mathrm{e}^{-iks} q_t(0,s)ds. \quad (2.7)$$

Since μ^+ as a function of k, is analytic and bounded in $\mathbb{C}^+$, while μ^- is analytic and bounded in $\mathbb{C}^-$, equation (2.7) and the asymptotic behavior (2.6) determine a Riemann-Hilbert problem for the function $\mu(x,t,k)$. This means that $\mu(x,t,k)$ is the unique function, sectionally analytic in $\mathbb{C}$ and with the decay prescribed by (2.6), that coincides with μ^+ in $\mathbb{C}^+$, with μ^- in $\mathbb{C}^-$, and that satisfies the jump condition (2.7) across $\mathbb{R}$. This function is given by

$$\mu(x,t,k) = \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\lambda t+(i/\lambda)x}\rho(\lambda)}{\lambda - k} d\lambda.$$

The computation of $\mu_t - ik\mu$ (modulo some technicalities regarding the slow decay at infinity of $\rho(k)$, see [3]), and integration with respect to t, yield $q(x,t)$:

$$q(x,t) = \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikt+(i/k)x}\rho(k)}{k} dk. \quad (2.8)$$

It can then be verified that the function defined by (2.8) satisfies the equation and the given initial and boundary conditions, see [3]. Also, as shown in [3], (2.8) is equivalent to the expression

$$q(x,t) = \frac{1}{2}q_0(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ikt+(i/k)x}\left(\int_0^x \mathrm{e}^{-(i/k)y} q_0(y)dy\right)\frac{dk}{k^2} + \\ + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{e}^{ikt+(i/k)x}\left(\int_0^t \mathrm{e}^{iks}\dot{f}(s)ds\right)\frac{dk}{k}.$$

This expression shows that indeed the solution at any point (x,t) depends only on the values of the variables in $(0,x)$ and $(0,t)$.

3 The sine-Gordon equation in light cone coordinates

We now consider an initial boundary value problem for the sine-Gordon equation

$$q_{xt} + \sin q = 0, \quad 0 < x < L, \ \ 0 < t < T, \quad (3.1)$$

with given conditions (2.2). We follow the same steps used to solve the linearized problem. As before, we consider this equation in the domain $(0,L)\times(0,T)$. As we are interested in the large t behavior of this equation, we eventually let $T\to\infty$, still keeping L finite, so that $\lim_{t\to\infty} x/t = 0$.

Our first step is to interchange the role of the x and t variables in the classical formulation, to obtain a Lax pair that is the nonlinear analogue of (2.3). Hence the Lax pair we use is given by:

$$\mu_t + ik\,[\sigma_3,\mu] = Q\mu, \quad \mu_x + \frac{i}{4k}\,[\sigma_3,\mu] = \tilde{Q}\mu, \quad (3.2)$$

where

$$Q = \left(\begin{array}{c} 0 - \frac{1}{2}q_t(x,t) \\ \frac{1}{2}q_t(x,t)0 \end{array} \right), \quad \tilde{Q} = \left(\begin{array}{c} \cos q(x,t) - 1 \sin q(x,t) \\ \sin q(x,t) 1 - \cos q(x,t) \end{array} \right).$$

The solutions of this Lax pair are of the form

$$\mu(x,t,k) = I - \frac{i}{4k} \mathrm{e}^{-ik(t-t_0\hat{\sigma}_3} \int_{x_0}^{x} \mathrm{e}^{-(i/4k)(x-y)\hat{\sigma_3}} \tilde{Q}\mu(y,0,k)dy + \\ + \int_{t_0}^{t} \mathrm{e}^{-ik(t-s)\hat{\sigma}_3} Q\mu(x,s,k)ds. \tag{3.3}$$

Choosing $x_0 = L$, $t_0 = 0$ and $x_0 = 0$, $t_0 = T$ we obtain two particular solutions of the Lax pair (3.2):

$$\Phi(x,t,k) = I + \frac{i}{4k} \mathrm{e}^{-ikt\hat{\sigma}_3} \int_{x}^{L} \mathrm{e}^{-\frac{i}{4k}(x-y)\hat{\sigma}_3} \tilde{Q}\Phi(y,0,k)dy + \\ + \int_{0}^{t} \mathrm{e}^{-ik(t-s)\hat{\sigma}_3} Q\Phi(x,s,k)ds, \tag{3.4}$$

$$\Psi(x,t,k) = I - \frac{i}{4k} \mathrm{e}^{-ik(t-T)\hat{\sigma}_3} \int_{0}^{x} \mathrm{e}^{-\frac{i}{4k}(x-y)\hat{\sigma}_3} \tilde{Q}\Psi(y,T,k)dy - \\ - \int_{t}^{T} \mathrm{e}^{-ik(t-s)\hat{\sigma}_3} Q\Psi(x,s,k)ds \tag{3.5}$$

We write $\Phi = (\Phi^+, \Phi^-)$ where $\Phi^\pm$ are column vectors. It is easy to verify that these vectors are bounded and analytic in $\mathbb{C}^\pm$, respectively. Their components are given explicitly by

$$\left(\begin{array}{c} \Phi_1^+ \\ \Phi_2^+ \end{array} \right) = \left(\begin{array}{c} 1 \\ 0 \end{array} \right) + \int_0^t \left(\begin{array}{c} -\frac{1}{2}q_t(x,s)\Phi_2^+(x,s) \\ \frac{1}{2}q_t(x,s)\Phi_1^+(x,s)\mathrm{e}^{2ik(t-s)} \end{array} \right) ds +$$

$$+ \frac{i}{4k} \int_x^L \left(\begin{array}{c} (\cos q(y,0) - 1)\Phi_1^+(y,0) + \sin q(y,0)\Phi_2^+(y,0) \\ \left[\sin q(y,0)\Phi_1^+(y,0) + (1 - \cos q(y,0))\Phi_2^+(y,0)\right] \mathrm{e}^{2ikt + (i/2k)(x-y)} \end{array} \right) dy$$

and similarly

$$\left(\begin{array}{c} \Phi_1^- \\ \Phi_2^- \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) + \int_0^t \left(\begin{array}{c} -\frac{1}{2}q_t(x,s)\Phi_2^-(x,s)\mathrm{e}^{-2ik(t-s)} \\ \frac{1}{2}q_t(x,s)\Phi_1^-(x,s) \end{array} \right) ds +$$

$$+ \frac{i}{4k} \int_x^L \left(\begin{array}{c} \left[(\cos q(y,0) - 1)\Phi_1^-(y,0) + \sin q(y,0)\Phi_2^-(y,0)\right] \mathrm{e}^{-2ikt - (i/2k)(x-y)} \\ \sin q(y,0)\Phi_1^-(y,0) + (1 - \cos q(y,0))\Phi_2^-(y,0) \end{array} \right) dy$$

In the same way, $\Psi = (\Psi^-, \Psi^+)$, where $\Psi^\mp$ are analytic and bounded in $\mathbb{C}^\mp$, and satisfy equations similar to the ones for Φ, with $\int_0^t$ replaced by $-\int_t^T$ and $\int_x^L$ replaced by $-\int_0^x$.

It is immediate to verify that the matrices above satisfy special symmetry relations, given by

$$\Phi_{22}(k) = \Phi_{11}(-k), \Phi_{12}(k) = -\Phi_{21}(-k), \\ \Psi_{22}(k) = -\Psi_{11}(-k), \Psi_{12}(k) = \Psi_{21}(-k). \tag{3.6}$$

and in addition, for real q,

$$\Phi_{22}(k) = \overline{\Phi_{11}(\bar{k})}, \quad \Phi_{12}(k) = -\overline{\Phi_{21}(\bar{k})}, \quad \Psi_{22}(k) = \overline{\Psi_{11}(\bar{k})}, \quad \Psi_{12}(k) = -\overline{\Psi_{21}(\bar{k})}. \tag{3.7}$$

For $k \in \mathbb{R}$, the matrices Ψ and Φ are related as follows:

$$\Phi(x,t,k) = \Psi(x,t,k) \mathrm{e}^{-ikt\hat{\sigma}_3 - (i/4k)x\hat{\sigma}_3} \rho(k).$$

Indeed, since they both satisfy the x-equation, then there exists a function $\mathcal{T}(t,k)$ such that $\Phi = \Psi \mathcal{T}(t,k)$, and since they both satisfy the t-equation, then there exists a function $\mathcal{X}(x,k)$ such that$\Phi = \Psi \mathcal{X}(x,k)$. It follows that these two functions are related through a function of k only. We call this function $\rho(k)$.

At $x = 0$, $t = T$ we have $\Psi(0,T,k) = I$, therefore

$$\rho(k) = \mathrm{e}^{ikT\hat{\sigma}_3} \Phi(0,T,k) = I + \frac{i}{4k} \int_0^L \mathrm{e}^{\frac{i}{4k} y\hat{\sigma}_3} \tilde{Q} \Phi(y,0,k) dy + \\ + \int_0^T \mathrm{e}^{iks\hat{\sigma}_3} Q \Phi(0,s,k) ds. \tag{3.8}$$

Any solution $\mu(x,t,k)$ of the Lax pair (3.2) has constant determinant equal to 1, and the following asymptotic behavior as $k \to \infty$ and $k \to 0$:

$$\mu(x,t,k) = I + \frac{\mu_1(x,t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \\ \mu(x,t,k) = \mu_0(x,t) + O(k), \quad k \to 0. \tag{3.9}$$

Equation (3.9(a)) is obtained by integration by parts of (3.3). ¿From (3.9(a)), using the fact that μ satisfies the first equation of the Lax pair, we obtain that $i[\sigma_3, \mu_1] = Q$, hence

$$q_t = -4i(\mu_1)_{12} = -4i \lim_{k \to \infty} k(\mu - I)_{12}. \tag{3.10}$$

To verify that the behavior at $k \to 0$ is indeed as given by equation (3.9(b)), assume that μ has this form. Then substituting into the two equations of the Lax pair, we find

$$Q\mu_0 = (\mu_0)_t, \quad \tilde{Q}\mu_0 = [\mu_0, \sigma_3].$$

The second equation can be rewritten as

$$\mu_0 \sigma_3 \mu_0^{-1} = \tilde{Q} + \sigma_3 = \begin{pmatrix} \cos q & \sin q \\ \sin q & -\cos q \end{pmatrix}.$$

Since both left and right hand side have determinant equal to -1, this equation can be solved in terms of one of the matrix elements. For example, we obtain

$$(\mu_0)_{12} = \frac{\cos q - 1}{(\mu_0)_{21}}, \quad (\mu_0)_{11} = -\frac{\sin q}{2(\cos q - 1)} (\mu_0)_{21}, \quad (\mu_0)_{22} = \frac{\sin q}{2(\mu_0)_{21}}.$$

The data above define a Riemann-Hilbert problem. This problem, modulo interchanging the role of x and t, is studied in detail in [3]. It can be written in a canonical form as follows:

$$\left(\Psi^+, \frac{\Phi^+}{\rho_{11}}\right) = \left(\frac{\Phi^-}{\rho_{22}}, \Psi^-\right) G, \quad G = \begin{pmatrix} 1 & \frac{\rho_{21} E}{\rho_{11}} \\ -\frac{\rho_{12}\bar{E}}{\rho_{22}} & \frac{1}{\rho_{11}\rho_{22}} \end{pmatrix}, \; k \in \mathbb{R}, \tag{3.11}$$

where $E = \mathrm{e}^{2ikt+(i/2k)x}$. Note that

$$\lim_{k\to\infty}\left(\Psi^+, \frac{\Phi^+}{\rho_{11}}\right) = \lim_{k\to\infty}\left(\frac{\Phi^-}{\rho_{22}}, \Psi^-\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Any matrix Riemann-Hilbert problem of the form $\mathcal{T}^+ = \mathcal{T}^- G$, where $\lim_{k\to\infty}\mathcal{T}^\pm = J$ is a constant matrix, and G satisfies a certain definiteness condition, is uniquely solvable [1]. The solution is given by the formula

$$\mathcal{T}^-(k) = J + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathcal{T}^-(k')\mathcal{G}(k')\frac{dk'}{k' - (k - i0)}, \quad \mathcal{T}^+ = \mathcal{T}^- G, \tag{3.12}$$

where $\mathcal{G} = G - I$, I the identity matrix.

In the present case, $\mathcal{T}^- = \left(\frac{\Phi^-}{\rho_{22}}, \Psi^-\right)$. Considering the second column vector, and using the symmetry conditions relating the components of Ψ^+ and Ψ^-, the expression (3.12) yields for the vectors Ψ^+, Ψ^- the integral equation

$$\Psi^+(x,t,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho_{12}(k')}{\rho_{22}(k')}\Psi^-(k')\mathrm{e}^{-2ik't-(i/2k')x}\frac{dk'}{k' - (k + i0)}. \tag{3.13}$$

Finally, using (3.10) (with $\mu = \Psi$), we find that the solution $q(x,t)$ of the given boundary value problem is characterized by

$$q_t(x,t) = \frac{2}{\pi}\int_{-\infty}^{\infty} \Psi_1^-(x,t,k)\frac{\rho_{12}(k)}{\rho_{22}(k)}\mathrm{e}^{-2ikt-(i/2k)x}dk. \tag{3.14}$$

Indeed, choosing $\mu = \Psi$ in equation (3.10), since $\Psi_{12} = \Psi_1^+$, we have

$$q_t = -4i\lim_{k\to\infty} k\Psi_1^+,$$

which yields equation (3.14). The use of equation (1.1) yields an expression for $q(x,t)$.

We are only interested here in the asymptotic behavior of the solution as $t \to \infty$. We now show that to analyze this behavior, we need to evaluate the limit of the quotient $\rho_{12}(k)/\rho_{22}(k)$ as $k \to 0$. Since the matrix $\rho(k)$ can be characterized via the values of the solution $\Phi(x,t,k)$ of the Lax pair at $x = 0$, $t = T$, we start by analyzing the behavior of the matrix defined by (3.4).

3.1 Small k behavior of $\Phi(x, 0, k)$

We start by remarking how, because of the symmetry properties (3.6) of these matrices, we only need to analyze the asymptotic behavior of one of the columns of Φ. By definition, we have

$$\Phi_{12}(x,0,k) = \frac{i}{4k}\int_x^L \left[(\cos q_0(y) - 1)\Phi_{12}(y,0,k) + \sin q_0(y)\Phi_{22}(y,0,k)\right]\mathrm{e}^{-(\frac{i}{2k})(x-y)}dy,$$

$$\Phi_{22}(x,0,k) = 1 + \frac{i}{4k}\int_x^L \left[\sin q_0(y)\Phi_{12}(y,0,k) + (1 - \cos q_0(y))\Phi_{22}(y,0,k)\right]dy. \tag{3.15}$$

Integrating by parts the expression for $\Phi_{12}(x,0,k)$, and using the fact that $\Phi_{12}(L,0,k)=0$ and $\Phi_{22}(L,0,k)=1$, we find

$$\Phi_{12}(x,0,k) = -\frac{\sin q_0(x)}{1+\cos q_0(x)}\Phi_{22}(x,0,k) + \frac{\sin q_0(L)}{1+\cos q_0(L)}\mathrm{e}^{-(i/2k)(x-L)} - \\ -\frac{1}{2}\int_x^L \frac{d}{dy}\left[(\cos q_0(y)-1)\Phi_{12}(y,0,k) + \sin q_0(y)\Phi_{22}(y,0,k)\right]\mathrm{e}^{-(i/2k)(x-y)}dy.$$

Write

$$\Phi = \Phi^0(x,t) + k\Phi^1(x,t) + O(k^2), \quad k \to 0.$$

Substituting and integrating by parts the first of the above equations, we find

$$\Phi^0_{12}(x,0) = -\frac{\sin q_0(x)}{1+\cos q_0(x)}\Phi^0_{22}(x,0) + \frac{\sin q_0(L)}{1+\cos q_0(L)}\mathrm{e}^{-(i/2k)(x-L)}. \tag{3.16}$$

These equations suggest that we assume for $\Phi_{12}(x,0,k)$, $\Phi_{22}(x,0,k)$ the following behavior:

$$\Phi_{12}(x,0,k) = \alpha_1(x) + \beta_1(x)\mathrm{e}^{-(i/2k)(x-L)} + O(k),$$
$$\Phi_{22}(x,0,k) = \alpha_2(x) + \beta_2(x)\mathrm{e}^{-(i/2k)(x-L)} + O(k).$$

Substituting these expressions in equation (3.16), we find

$$\alpha_1(x) + \beta_1(x)\mathrm{e}^{-(i/2k)(x-L)} = -\frac{\sin q_0(x)}{1+\cos q_0(x)}\left(\alpha_2(x) + \beta_2(x)\mathrm{e}^{-(i/2k)(x-L)}\right) + \\ + \frac{\sin q_0(L)}{1+\cos q_0(L)}\mathrm{e}^{-(i/2k)(x-L)}.$$

It follows that

$$\alpha_1(x) = -\frac{\sin q_0(x)}{1+\cos q_0(x)}\alpha_2(x), \tag{3.17}$$
$$\beta_1(x) = -\frac{\sin q_0(x)}{1+\cos q_0(x)}\beta_2(x) + \frac{\sin q_0(L)}{1+\cos q_0(L)}.$$

3.2 Small k behavior of $\rho(k)$

We recall that $\rho(k) = \mathrm{e}^{ikT\hat{\sigma}_3}\Phi(0,T,k)$. Note that we can write

$$\Phi(x,t,k) = \mathrm{e}^{-ikt\hat{\sigma}_3}\Phi(x,0,k) + \int_0^t \mathrm{e}^{-ik(t-s)\hat{\sigma}_3}Q\Phi(x,s,k)ds.$$

Therefore,

$$\mathrm{e}^{ikT\hat{\sigma}_3}\Phi(0,T,k) = \Phi(0,0,k) + \int_0^T \mathrm{e}^{iks\hat{\sigma}_3}Q\Phi(0,s,k)ds,$$

and, as $k \to 0$, we find

$$\frac{\rho_{12}(k)}{\rho_{22}(k)} \sim \frac{\alpha_1(0) + \beta_1(0)\mathrm{e}^{(i/2k)L} - \frac{1}{2}\int_0^T \mathrm{e}^{2iks}q_t(0,s)\Phi_{22}(0,s,k)ds}{\alpha_2(0) + \beta_2(0)\mathrm{e}^{(i/2k)L} + \frac{1}{2}\int_0^T q_t(0,s)\Phi_{12}(0,s,k)ds}. \tag{3.18}$$

4 The case of constant boundary data - proof of Theorem 1

We now consider in detail the boundary value problem obtained when $q(0,t) = \gamma$ is a constant, $\gamma \neq 0$. For consistency, it must be $\gamma = q_0(0)$, where $q_0(x)$ is the given initial condition. We want to show that in this case, the asymptotic behavior of the solution for large t is dominated by the similarity solution, while the zeros of the spectral function $\frac{\rho_{12}(k)}{\rho_{22}(k)}$, when they exist, do not play a role in the leading asymptotic behavior of the solution of these boundary value problems.

We introduce the similarity variable $\xi = \sqrt{xt}$, and write $k = \frac{1}{2}\sqrt{\frac{x}{t}}\lambda$. Then

$$2ikt + (i/2k)x = i\xi\left(\lambda + \frac{1}{\lambda}\right).$$

The Riemann-Hilbert problem with respect to these variables is identical to the Riemann-Hilbert problems characterizing the solution of the so-called Painlevé III ODE, which is analyzed in [6].

With respect to the variable λ, the kernel of equation (3.13) is

$$\frac{\rho_{12}(\frac{1}{2}\sqrt{\frac{x}{t}}\lambda')}{\rho_{22}(\frac{1}{2}\sqrt{\frac{x}{t}}\lambda')} \mathrm{e}^{-i\xi(\lambda'+\frac{1}{\lambda'})} \frac{d\lambda'}{\lambda' - (\lambda + i0)}. \tag{4.1}$$

If we let $t \to \infty$, since x is finite, then $k \to 0$. Using the method of the stationary phase [1], we find that the leading order behavior of equation (3.13) when $t \to \infty$ depends on $\lim_{k\to 0} \frac{\rho_{12}(k)}{\rho_{22}(k)}$.

In the particular case that $q(0,t) = \gamma$ is constant, since $q_t(0,t) = 0$, the integral terms in the expression (3.18) vanish. Hence the leading term of the ratio ρ_{12}/ρ_{22}, for k small, is given by

$$\frac{\rho_{12}(k)}{\rho_{22}(k)} \sim \frac{\alpha_1(0) + \beta_1(0)\mathrm{e}^{(i/2k)L}}{\alpha_2(0) + \beta_2(0)\mathrm{e}^{(i/2k)L}} \tag{4.2}$$

We consider first the case that $\rho_{22}(k)$ has no zeros in $\mathbb{C}^-$.

The terms containing $\mathrm{e}^{(i/2k)L}$ do not contribute to the integral in (3.13). Indeed, multiplying the expression in (4.2) by the exponential $\mathrm{e}^{-2ikt-(i/2k)x}$, and writing $\alpha_i = \alpha_i(0)$, $\beta_i = \beta_i(0)$, $i = 1, 2$, we find

$$\frac{\rho_{12}(k)}{\rho_{22}(k)}\mathrm{e}^{-2ikt-(i/2k)x} \sim \frac{\alpha_1}{\alpha_2}\left[\mathrm{e}^{-2ikt-(i/2k)x} + \frac{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)\mathrm{e}^{-2ikt-(i/2k)(x-L)}}{1 + \frac{\beta_2}{\alpha_2}\mathrm{e}^{(i/2k)L}}\right]$$

The exponential terms $\mathrm{e}^{-2ikt-(i/2k)(x-L)}$ and $\mathrm{e}^{(i/2k)L}$ are analytic and bounded in $\mathbb{C}^-$. The same is true of the other terms appearing in the kernel (3.13), namely $\Psi^-(k)$ and $1/(k' - (k + i0))$. It follows that the integral over $\mathbb{R}$ of the product of these terms vanishes. Using (3.17), we therefore find that the only contribution to the leading behavior of the integral (3.13), for $t \to \infty$, is given by

$$\frac{\rho_{12}(k)}{\rho_{22}(k)} \sim \frac{\alpha_1}{\alpha_2} = -\frac{\sin q_0(0)}{1 + \cos q_0(0)}. \tag{4.3}$$

It is easy to see, as in [5], that if $\rho_{22}(k)$ has zeros in $\mathbb{C}^-$, then the extra terms arising from the contribution of the corresponding poles are of the form

$$\frac{\rho_{12}(k_0)\Psi^-(k_0)\mathrm{e}^{-2ik_0t-(i/2k_0)x}}{(d\rho_{22}/dk)(k_0)(k-k_0)}, \quad \rho_{22}(k_0)=0,\ k_0\in\mathbb{C}^-.$$

Since $|\mathrm{e}^{-2ikt}|$ decreases exponentially with t for $k\in\mathbb{C}^-$, these terms give an exponentially small contribution as $t\to\infty$, and can therefore be ignored when seeking the leading order behavior of the solution for large time.

We note that if the given initial condition is also constant, $q_0(x)=\gamma$, then it is possible to compute explicitly $\rho_{12}(k)$ and $\rho_{22}(k)$, and find

$$\rho_{12}(k)=-\frac{1}{2}\sin\gamma+\frac{1}{2}\sin\gamma\mathrm{e}^{(i/2k)L}, \quad \rho_{22}(k)=\frac{1}{2}(1+\cos\gamma)+\frac{1}{2}(1-\cos\gamma)\mathrm{e}^{(i/2k)L}.$$

In this case, the effective part of the ratio ρ_{12}/ρ_{22} is precisely given by the right hand side of (4.3).

¿From the previous discussion it follows that the leading behavior of equation (3.13) as $t\to\infty$, is determined by

$$\Psi^+(x,t,\lambda)=\begin{pmatrix}0\\1\end{pmatrix}-\frac{1}{2\pi i}\frac{\alpha_1}{\alpha_2}\int_{-\infty}^{\infty}\Psi^-(\xi,\lambda')\mathrm{e}^{-i\xi(\lambda'+\frac{1}{\lambda'})}\frac{d\lambda'}{\lambda'-(\lambda+i0)}.$$

Hence for $\gamma\neq n\pi$, $n\in\mathbb{Z}$, we finally find

$$\Psi^+(\xi,\lambda)=\begin{pmatrix}0\\1\end{pmatrix}-\frac{1}{2\pi i}\frac{\sin\gamma}{1+\cos\gamma}\int_{-\infty}^{\infty}\Psi^-(\lambda')\mathrm{e}^{-i\xi(\lambda+(1/\lambda)}\frac{d\lambda'}{\lambda'-(\lambda+i0)}. \tag{4.4}$$

Setting $N_1(\xi,\lambda)=\Psi_1^+(\xi,\lambda)$, $N_2(\xi,\lambda)=\Psi_2^+(\xi,\lambda)$, and using the symmetry relations (3.7), we find that N_1, N_2 satisfy the system of integral equations (1.4). Defining

$$\tilde{q}(\xi)=\lim_{\lambda\to\infty}4i\lambda N_1(\lambda,\xi),$$

we find the similarity solution (1.3). Finally, we obtain from (3.14) that the leading order behavior of the solution of this boundary value problem, for $t\to\infty$, is given by (1.2).

5 Conclusions

We have given a representation of the solution of boundary value problems for the sine-Gordon equation posed on a finite interval $[0,L]$, with L a finite but otherwise arbitrary positive constant. The interesting property of this equation is that it is an evolution equation with respect to both the space and the time variable. This entails that the value $q(x_0,t_0)$ of the solution at any given point (x_0,t_0) depends only on the values of $q(x,t)$ for $x<x_0$, $t<t_0$.

We have shown that in the particular case that the given boundary conditions are constant, the large asymptotics of the solution are dominated by the similarity solution. Indeed, the large time asymptotic behavior is associated with a 2×2 Riemann-Hilbert problem which is a particular case of a general RH problem associated with an integrable

ODE, the so-called Painlevé III [6]. This problem coincides with the spectral problem arising in connection with the phenomena of transient stimulated Raman scattering, and was already analyzed in the literature [5]. The only new feature of the present analysis is the interchange of the roles that the two variables x and t play in the classical inverse scattering approach for the sine-Gordon equation.

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